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Temperature Rise Associated With A Joule Heating Effect Of A Disturbed Current Density In A Plate With A Through Crack During Resistance Spot Welding

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Abstract:
Background: *Resistance spot welding is a type of the electric resistance welding, through a process in which contacted metal surfaces are joined by the heat obtained from the Joule heating effect. If there are some defects in the material, such as cracks, the current density will be disturbed by the presence of the crack, and an unexpected temperature rise will be caused by the disturbed current density. This unexpected temperature distribution may cause some troubles in the welding process.,*
Materials and Methods: *The temperature distribution in a cracked plate during resistance spot welding under a uniform current density is considered. In the calculations, the crack faces are supposed to be electrically and thermally insulated. By using the Fourier integral transform method, the electric problem is reduced to a singular integral equation, which is solved numerically. Using the obtained solution, the disturbed components of the electric potential and the current density are computed. Furthermore, the temperature field induced by the disturbed current density components is also calculated. The results for the electric and thermal fields are presented for the various values of the geometric parameters.*
Results and Conclusion: *1. The temperatures along the crack surfaces should not depend on the geometric parameters. 2. The normalized maximum values of the crack surface temperatures would be 0.5. 3. As the crack approaches to the upper or lower plane, the temperature distributions tend to decrease except for the crack surface temperatures. 4. If the thickness of the plate is infinite, the temperatures along the crack surfaces can be obtained as closed form solution.*
Key Word: *Resistance spot welding; Disturbed current density; Joule heating effect; Temperature rise; Fourier transform; Singular integral equation.*

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I. Introduction

Resistance spot welding is a type of the electric resistance welding, through a process in which contacted metal surfaces are joined by the heat obtained from the Joule heating effect. If there are some defects in the material, such as cracks, the current density will be disturbed by the presence of the crack, and an unexpected temperature rise will be caused by the disturbed current density. This unexpected temperature distribution may cause some troubles in the welding process. However, except for Liu's study[1] about the electro-thermo-structural coupled-field simulation of electric connector with edge crack, very few investigations on these phenomena have been made.

In this study, the electric and thermal problems for a cracked plate during resistance spot welding under a uniform current density is considered. First, the Fourier integral transform technique is used to reduce the electric problem to the solution of a singular integral equation[2], which are solved numerically[3]. Next, using the solution of the singular integral equation, the disturbed electric field can be calculated. Furthermore, by solving the Poisson's equation with a heat generation due to the Joule heating effect induced by the disturbed current density components, we can get the temperature field. Numerical calculations are carried out, and detailed results are presented to illustrate the influence of the crack location on the temperature field. The results for the electric field are also included. For the case of the special case, the closed-form solution of the temperature field can be obtained.

II. Formulation Of The Problem

The problem under consideration is an infinite long plate of thickness $h = h_1 + h_2$ containing a through crack with its length being $2c$ parallel to the boundaries, as shown in Figure no1. The system of rectangular Cartesian coordinates (x, y, z) is introduced in the material in such a way that the crack is located along the x -

axis. It is assumed that the current density components in the y -direction $J_{yi}(x,y)\,(i=1,2)$ at the top surface $(y=h_1)$ and the bottom surface $(y=-h_2)$ are maintained at $J_{y1}(x,h_1)=J_{y2}(x,-h_2)=-J_0 \quad (0\le|x|<\infty)$, where J_0 indicates the constant current density. The temperatures $T_i(x,y)\,(i=1,2)$, which are induced by the disturbed current density components, at the surfaces are fixed to $T_1(x,h_1)=T_2(x,-h_2)=0 \quad (0\le|x|<\infty)$, respectively. The crack faces are supposed to be insulated electrically and thermally. In the following, the subscript $i=1,2$ denotes the electric and thermal fields in $0\le y\le h_1$ and $-h_2\le y\le 0$, and the subscripts x,y will be used to refer to the direction of coordinates. The electric conductivity and the thermal conductivity are denoted by κ and λ.

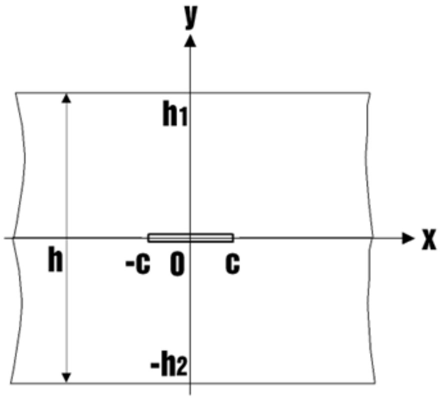


Figure no1: A through crack in a conducting plate.

III. Electric Fields

For the problem considered here, it is convenient to represent the electric potentials $\Phi_i(x,y)\,(i=1,2)$ as the sums of two functions.

$$\Phi_i(x,y)=\Phi^{(1)}(y)+\Phi_i^{(2)}(x,y) \quad (i=1,2) \tag{1}$$

where $\Phi^{(1)}(y)$ and $\Phi_i^{(2)}(x,y)$ $(i=1,2)$ indicate the non-disturbed and disturbed components. The governing equation and the boundary conditions for $\Phi^{(1)}(y)$ are

$$\frac{d^2\Phi^{(1)}(y)}{dy^2}=0 \tag{2}$$

$$\frac{d}{dy}\Phi^{(1)}(y)=\frac{J_0}{\kappa}, \quad \Phi^{(1)}(-h_2)=0 \tag{3}$$

The governing equations and the boundary conditions for $\Phi_i^{(2)}(x,y)$ $(i=1,2)$ are also given by

$$\frac{\partial^2}{\partial x^2}\Phi_i^{(2)}(x,y)+\frac{\partial^2}{\partial y^2}\Phi_i^{(2)}(x,y)=0 \quad (i=1,2) \tag{4}$$

$$\left.\begin{aligned}&\frac{\partial}{\partial y}\Phi_1^{(2)}(x,0^+)=-\frac{J_0}{\kappa} && (0\le|x|<c)\\ &\Phi_1^{(2)}(x,0^+)=\Phi_2^{(2)}(x,0^-) && (c\le|x|<\infty)\end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned}&\frac{\partial}{\partial y}\Phi_1^{(2)}(x,0^+)=\frac{\partial}{\partial y}\Phi_2^{(2)}(x,0^-)\\ &\frac{\partial}{\partial y}\Phi_1^{(2)}(x,h_1)=\frac{\partial}{\partial y}\Phi_2^{(2)}(x,-h_2)=0\end{aligned}\right\} \quad (0\le|x|<\infty) \tag{6}$$

It is easy to find from the equations (2) and (3) that $\Phi^{(1)}(y)$ is

$$\Phi^{(1)}(y)=\frac{J_0}{\kappa}\left(y+h_2\right) \tag{7}$$

The non-disturbed current density components $J_x^{(1)}, J_y^{(1)}$ induced by the non-disturbed electric potential component $\Phi^{(1)}(y)$ are

$$J_x^{(1)} = -\kappa \frac{d}{dx}\Phi^{(1)}(y) = 0, \quad J_y^{(1)} = -\kappa \frac{d}{dy}\Phi^{(1)}(y) = -J_0 \tag{8}$$

Applying the Fourier integral transform method, the general solutions of the equations (4) can be obtained as follows:

$$\Phi_i^{(2)}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{A_{i1}(s)\exp(-|s|y) + A_{i2}(s)\exp(|s|y)\right\}\exp(-\mathrm{i}sx)ds \quad (i=1,2) \tag{9}$$

where 'i' indicates the imaginary unit, and $A_{ij}(s)\ (i,j=1,2)$ are the unknown functions to be solved. We define the following new unknown function[2] $G_E(x)$:

$$G_E(x) = \begin{cases} \dfrac{\partial}{\partial x}\left[\Phi_1^{(2)}(x,0^+) - \Phi_2^{(2)}(x,0^-)\right] & (0 \le |x| < c) \\ 0 & (c \le |x| < \infty) \end{cases} \tag{10}$$

Using the boundary conditions (6), the relationships between $A_{ij}(s)\ (i,j=1,2)$ and $G_E(\xi)$ can be obtained as follows:

$$\left.\begin{aligned}
A_{11}(s) &= -\frac{1}{2\mathrm{i}s\left\{1-\exp(-2|s|h)\right\}}\left\{1-\exp(-2|s|h_2)\right\}\int_{-c}^{c}G_E(\xi)\exp(-\mathrm{i}s\xi)d\xi \\
A_{12}(s) &= -\frac{1}{2\mathrm{i}s\left\{1-\exp(-2|s|h)\right\}}\left\{\exp(-2|s|h_1)-\exp(-2|s|h)\right\}\int_{-c}^{c}G_E(\xi)\exp(-\mathrm{i}s\xi)d\xi \\
A_{21}(s) &= \frac{1}{2\mathrm{i}s\left\{1-\exp(-2|s|h)\right\}}\left\{\exp(-2|s|h_2)-\exp(-2|s|h)\right\}\int_{-c}^{c}G_E(\xi)\exp(-\mathrm{i}s\xi)d\xi \\
A_{22}(s) &= \frac{1}{2\mathrm{i}s\left\{1-\exp(-2|s|h)\right\}}\left\{1-\exp(-2|s|h_1)\right\}\int_{-c}^{c}G_E(\xi)\exp(-\mathrm{i}s\xi)d\xi
\end{aligned}\right\} \tag{11}$$

Making use of the first boundary condition (5) with the equations (11), we have the following singular integral equation for the determination of the unknown function[2] $G_E(\xi)$:

$$\frac{1}{2\pi}\int_{-c}^{c}\left[\frac{1}{\xi - x} + M_E(\xi,x)\right]G_E(\xi)d\xi = -\frac{J_0}{\kappa} \quad (0 \le |x| < c) \tag{12}$$

where the integral kernel $M_E(\xi,x)$ is given by

$$M_E(\xi,x) = \int_0^{\infty}\frac{1}{1-\exp(-2sh)}\left\{2\exp(-2sh) - \exp(-2sh_1) - \exp(-2sh_2)\right\}\sin\left\{s(\xi - x)\right\}ds \tag{13}$$

The singular integral equation (12) is to be solved with the following subsidiary condition obtained from the second boundary condition (5).

$$\int_{-c}^{c}G_E(\xi)d\xi = 0 \tag{14}$$

We solve the singular integral equation (12) and the additional equation (14) by using the Gauss-Chebyshev integration formula[3], and we can get the solution $G_E(\xi)$. Substituting the equations (11) to the equations (9) with the solution $G_E(\xi)$, the general solutions of the equations (4) can be obtained.

For the case of $|y| > 0$, the disturbed components $\Phi_i^{(2)}(x,y)\ (i=1,2)$ can be obtained as follows:

$$\begin{aligned}
\Phi_i^{(2)}(x,y) &= \frac{(-1)^i}{2\pi}\int_0^{\infty}\exp\left(-s|y|\right)L_E(s,x)ds \\
&+ \frac{1}{2\pi}\int_0^{\infty}\frac{1}{1-\exp(-2sh)}\begin{bmatrix}\exp\left\{-s(2h-y)\right\} - \exp\left\{-s(2h+y)\right\} \\ +\exp\left\{-s(2h_1-y)\right\} - \exp\left\{-s(2h_2+y)\right\}\end{bmatrix}L_E(s,x)ds \quad (i=1,2)
\end{aligned} \tag{15}$$

where

$$L_E(s,x) = \frac{1}{s}\int_{-c}^{c}G_E(\xi)\sin\left\{s(\xi - x)\right\}d\xi \tag{16}$$

The first terms of the equations (15) may be

$$\frac{(-1)^i}{2\pi}\int_0^\infty \exp\left(-s\left|y\right|\right)L_E(s,x)ds = \frac{(-1)^i}{2\pi}\int_{-c}^{c}\arctan\left(\frac{\xi - x}{\left|y\right|}\right)G_E(\xi)d\xi \quad (i=1,2) \tag{17}$$

And for the case of $y \to 0^\pm$, the equations (15) will be

$$\Phi_i^{(2)}(x,0^\pm) = \frac{(-1)^i}{2\pi}\int_0^\infty L_E(s,x)ds + \frac{1}{2\pi}\int_0^\infty \frac{1}{1-\exp(-2sh)}\left[\exp(-2sh_1) - \exp(-2sh_2)\right]L_E(s,x)ds \quad (i=1,2) \tag{18}$$

The first terms of the equations (18) are

$$\frac{(-1)^i}{2\pi}\int_0^\infty L_E(s,x)ds = \begin{Bmatrix} \dfrac{(-1)^i}{4}\displaystyle\int_{-c}^{c}\frac{\xi - x}{\left|\xi - x\right|}G_E(\xi)d\xi & (0 \le \left|x\right| < c) \\ 0 & (c \le \left|x\right| < \infty) \end{Bmatrix} \quad (i=1,2) \tag{19}$$

For the case of $\left|y\right| > 0$, the disturbed current density components $J_{xi}^{(2)}(x,y)$, $J_{yi}^{(2)}(x,y)$ $(i=1,2)$ are given by

$$\begin{aligned} J_{xi}^{(2)}(x,y) &= -\kappa\frac{\partial}{\partial x}\Phi_i^{(2)}(x,y) \\ &= -\frac{(-1)^i\kappa}{2\pi}\int_0^\infty \exp\left(-s\left|y\right|\right)\frac{\partial}{\partial x}L_E(s,x)ds \\ &-\frac{\kappa}{2\pi}\int_0^\infty \frac{1}{1-\exp(-2sh)}\begin{bmatrix} \exp\left\{-s(2h-y)\right\} - \exp\left\{-s(2h+y)\right\} \\ +\exp\left\{-s(2h_1-y)\right\} - \exp\left\{-s(2h_2+y)\right\} \end{bmatrix}\frac{\partial}{\partial x}L_E(s,x)ds \quad (i=1,2) \end{aligned} \tag{20}$$

$$\begin{aligned} J_{yi}^{(2)}(x,y) &= -\kappa\frac{\partial}{\partial y}\Phi_i^{(2)}(x,y) \\ &= -\frac{\kappa}{2\pi}\int_0^\infty s\exp\left(-s\left|y\right|\right)L_E(s,x)ds \\ &-\frac{\kappa}{2\pi}\int_0^\infty \frac{s}{1-\exp(-2sh)}\begin{bmatrix} \exp\left\{-s(2h-y)\right\} + \exp\left\{-s(2h+y)\right\} \\ +\exp\left\{-s(2h_1-y)\right\} + \exp\left\{-s(2h_2+y)\right\} \end{bmatrix}L_E(s,x)ds \quad (i=1,2) \end{aligned} \tag{21}$$

where

$$\frac{\partial}{\partial x}L_E(s,x) = -\int_{-c}^{c}G_E(\xi)\cos\left\{s(\xi - x)\right\}d\xi \tag{22}$$

The first terms of the equations (20) and (21) are

$$\left.\begin{aligned} -\frac{(-1)^i\kappa}{2\pi}\int_0^\infty \exp\left(-s\left|y\right|\right)\frac{\partial}{\partial x}L_E(s,x)ds &= -\frac{(-1)^i\kappa}{2\pi}\int_{-c}^{c}\frac{\left|y\right|}{y^2+\left(\xi - x\right)^2}G_E(\xi)d\xi \quad (i=1,2) \\ -\frac{\kappa}{2\pi}\int_0^\infty s\exp\left(-s\left|y\right|\right)L_E(s,x)ds &= -\frac{\kappa}{2\pi}\int_{-c}^{c}\frac{\xi - x}{y^2+\left(\xi - x\right)^2}G_E(\xi)d\xi \end{aligned}\right\} \tag{23}$$

Taking the non-continuity of the equations (19) into consideration, the equations (20) and (21) for $y \to 0^\pm$ become

$$\begin{aligned} J_{xi}^{(2)}(x,0^\pm) &= \frac{(-1)^i\kappa}{2}\begin{Bmatrix} G_E(x) & (0 \le \left|x\right| < c) \\ 0 & (c \le \left|x\right| < \infty) \end{Bmatrix} \\ &+\frac{\kappa}{2\pi}\int_0^\infty \frac{1}{1-\exp(-2sh)}\left[\exp(-2sh_1) - \exp(-2sh_2)\right]ds\int_{-c}^{c}G_E(\xi)\cos\left\{s(\xi - x)\right\}d\xi \quad (i=1,2) \end{aligned} \tag{24}$$

$$J_{y1}^{(2)}(x,0^+) = J_{y2}^{(2)}(x,0^-) = \begin{Bmatrix} J_0 & (0 \le \left|x\right| < c) \\ -\dfrac{\kappa}{2\pi}\displaystyle\int_{-c}^{c}\left[\frac{1}{\xi - x} + M_E(\xi,x)\right]G_E(\xi)d\xi & (c \le \left|x\right| < \infty) \end{Bmatrix} \tag{25}$$

IV. Temperature Fields

The heat generation J_0^2/κ due to only the non-disturbed current density component $J_y^{(1)}=-J_0$ will be ignored. The Poisson's equations with heat generations due to the Joule heating effect induced by the non-disturbed current density component $J_y^{(1)}=-J_0$ and the disturbed current density components $J_{xi}^{(2)}(x,y)$, $J_{yi}^{(2)}(x,y)$ $(i=1,2)$ are

$$\lambda\left\{\frac{\partial^2}{\partial x^2}T_i(x,y)+\frac{\partial^2}{\partial y^2}T_i(x,y)\right\}+Q_i(x,y)=0 \quad (i=1,2) \tag{26}$$

where

$$Q_i(x,y)=\frac{1}{\kappa}\left[\left\{J_{xi}^{(2)}(x,y)\right\}^2+\left\{J_{yi}^{(2)}(x,y)-J_0\right\}^2-J_0^2\right] \quad (i=1,2) \tag{27}$$

The boundary conditions are

$$\left.\begin{array}{ll}\dfrac{\partial}{\partial y}T_1(x,0^+)=0 & (0\le|x|<c)\\ T_1(x,0^+)=T_2(x,0^-) & (c\le|x|<\infty)\end{array}\right\} \tag{28}$$

$$\left.\begin{array}{l}\dfrac{\partial}{\partial y}T_1(x,0^+)=\dfrac{\partial}{\partial y}T_2(x,0^-)\\ T_1(x,h_1)=T_2(x,-h_2)=0\end{array}\right\} \quad (0\le|x|<\infty) \tag{29}$$

Because the Poisson's equations (26) are nonhomogeneous, the temperature fields $T_i(x,y)\,(i=1,2)$ are given as the sums of the particular integrals $T_i^{(1)}(x,y)$ and the complementary functions $T_i^{(2)}(x,y)$ $(i=1,2)$.

$$T_i(x,y)=T_i^{(1)}(x,y)+T_i^{(2)}(x,y) \qquad (i=1,2) \tag{30}$$

We can get the particular integrals $T_i^{(1)}(x,y)$ $(i=1,2)$ after repeated trial and error as follows:

$$T_i^{(1)}(x,y)=-\frac{1}{\lambda\kappa}\left[\frac{1}{2}\{\Gamma_i(x,y)\}^2+J_0y\Gamma_i(x,y)\right] \qquad (i=1,2) \tag{31}$$

where $\Gamma_i(x,y)$ $(i=1,2)$ are

$$\begin{aligned}\Gamma_i(x,y)=&\frac{(-1)^i\kappa}{2\pi}\int_0^\infty\frac{1}{s}\exp(-s|y|)\sin\{s(\xi-x)\}\,ds\int_{-c}^{c}G_E(\xi)d\xi\\ &-\frac{\kappa}{2\pi}\int_0^\infty\frac{1}{s}\left[F_2(s)\exp\{-s(2h_1-y)\}-F_1(s)\exp\{-s(2h_2+y)\}\right]\sin\{s(\xi-x)\}\,ds\int_{-c}^{c}G_E(\xi)d\xi \quad (i=1,2)\end{aligned} \tag{32}$$

In the above equations, $F_j(s)\,(j=1,2)$ are

$$F_j(s)=\frac{1}{1-\exp(-2sh)}\left\{1-\exp\left(-2sh_j\right)\right\} \quad (j=1,2) \tag{33}$$

For the case of $|y|>0$, the first terms of the equations (32) are

$$\frac{(-1)^i\kappa}{2\pi}\int_0^\infty\frac{1}{s}\exp(-s|y|)\sin\{s(\xi-x)\}\,ds\int_{-c}^{c}G_E(\xi)d\xi=\frac{(-1)^i\kappa}{2\pi}\int_{-c}^{c}\arctan\left(\frac{\xi-x}{|y|}\right)G_E(\xi)d\xi \quad (i=1,2) \tag{34}$$

And for the case of $y\to 0^\pm$, the equations (32) become

$$\begin{aligned}\Gamma_i(x,0^\pm)=&\left\{\begin{array}{ll}\dfrac{(-1)^i\kappa}{4}\displaystyle\int_{-c}^{c}\frac{\xi-x}{|\xi-x|}G_E(\xi)d\xi & (0\le|x|<c)\\ 0 & (c\le|x|<\infty)\end{array}\right\}\\ &+\frac{\kappa}{2\pi}\int_0^\infty\frac{1}{s\{1-\exp(-2sh)\}}\{\exp(-2sh_1)-\exp(-2sh_2)\}\sin\{s(\xi-x)\}\,ds\int_{-c}^{c}G_E(\xi)d\xi \quad (i=1,2)\end{aligned} \tag{35}$$

The temperature gradients $q_{yi}^{(1)}(x,y)$ $(i=1,2)$ in the y-direction are

$$q_{yi}^{(1)}(x,y)=\frac{\partial}{\partial y}T_i^{(1)}(x,y)=-\frac{1}{\lambda\kappa}\left[\Gamma_i(x,y)\frac{\partial}{\partial y}\Gamma_i(x,y)+J_0\Gamma_i(x,y)+J_0y\frac{\partial}{\partial y}\Gamma_i(x,y)\right] \quad (i=1,2) \tag{36}$$

where

$$\frac{\partial}{\partial y}\Gamma_1(x,y)=\frac{\partial}{\partial y}\Gamma_2(x,y)=\frac{\kappa}{2\pi}\int_{-c}^{c}\frac{\xi-x}{y^2+(\xi-x)^2}G_E(\xi)d\xi$$

$$+\frac{\kappa}{2\pi}\int_0^\infty\left[F_2(s)\exp\{-s(2h_1-y)\}+F_1(s)\exp\{-s(2h_2+y)\}\right]\sin\{s(\xi-x)\}ds\int_{-c}^{c}G_E(\xi)d\xi \qquad (37)$$

Especially, for the case of $y\to 0^\pm$, the equations (37) become

$$\frac{\partial}{\partial y}\Gamma_1(x,0^+)=\frac{\partial}{\partial y}\Gamma_2(x,0^-)=\begin{cases}-J_0 & (0\le|x|<c)\\ \dfrac{\kappa}{2\pi}\displaystyle\int_{-c}^{c}\left\{\frac{1}{\xi-x}+M_E(\xi,x)\right\}G_E(\xi)d\xi & (c\le|x|<\infty)\end{cases} \qquad (38)$$

Therefore, the temperatures and the temperature gradients shown in the equations (31) and (36) are

$$\left.\begin{aligned}&\frac{\partial}{\partial y}T_1^{(1)}(x,0^+)=0 && (0\le|x|<c)\\ &T_1^{(1)}(x,0^+)=T_2^{(1)}(x,0^-) && (c\le|x|<\infty)\\ &\frac{\partial}{\partial y}T_1^{(1)}(x,0^+)=\frac{\partial}{\partial y}T_2^{(1)}(x,0^-) && (0\le|x|<\infty)\end{aligned}\right\} \qquad (39)$$

Because the particular integrals $T_i^{(1)}(x,y)\ (i=1,2)$ do not satisfy the second and third boundary conditions of the equations (29), we have to analyze of the complementary functions $T_i^{(2)}(x,y)\ (i=1,2)$ by the same method used in the electric field analysis.

The governing equations and the boundary conditions for the complementary functions $T_i^{(2)}(x,y)\ (i=1,2)$ are

$$\frac{\partial^2}{\partial x^2}T_i^{(2)}(x,y)+\frac{\partial^2}{\partial y^2}T_i^{(2)}(x,y)=0 \quad (i=1,2) \qquad (40)$$

$$\left.\begin{aligned}&\frac{\partial}{\partial y}T_1^{(2)}(x,0^+)=0 && (0\le|x|<c)\\ &T_1^{(2)}(x,0^+)=T_2^{(2)}(x,0^-) && (c\le|x|<\infty)\end{aligned}\right\} \qquad (41)$$

$$\left.\begin{aligned}&\frac{\partial}{\partial y}T_1^{(2)}(x,0^+)=\frac{\partial}{\partial y}T_2^{(2)}(x,0^-)\\ &T_1^{(2)}(x,h_1)=-T_1^{(1)}(x,h_1)\\ &T_2^{(2)}(x,-h_2)=-T_2^{(1)}(x,-h_2)\end{aligned}\right\}(0\le|x|<\infty) \qquad (42)$$

The complementary functions $T_i^{(2)}(x,y)\ (i=1,2)$ of the equations (40) can be obtained as follows:

$$T_i^{(2)}(x,y)=\frac{2}{\pi}\int_0^\infty\{B_{i1}(s)\exp(-s|y|)+B_{i2}(s)\exp(s|y|)\}\cos(sx)ds \quad (i=1,2) \qquad (43)$$

where $B_{ij}(s)\ (i,j=1,2)$ are the unknown functions to be solved. We also define the following new unknown function $G_H(x)$:

$$G_H(x)=\begin{cases}\dfrac{\partial}{\partial x}\left[T_1^{(2)}(x,0^+)-T_2^{(2)}(x,0^-)\right] & (0\le|x|<c)\\ 0 & (c\le|x|<\infty)\end{cases} \qquad (44)$$

Using the boundary conditions (42), the relationships between $B_{ij}(s)\ (i,j=1,2)$ and $G_H(\xi)$ can be obtained as follows:

$$\left.\begin{aligned}
B_{11}(s) &= -\frac{1+\exp(-2sh_2)}{4\{1-\exp(-2sh)\}} I_H(s) + \frac{\exp(-sh_2)}{1-\exp(-2sh)}\left[\exp(-sh)D_1(s) - D_2(s)\right] \\
B_{12}(s) &= \frac{\exp(-2sh_1)\{1+\exp(-2sh_2)\}}{4\{1-\exp(-2sh)\}} I_H(s) - \frac{\exp(-sh_1)}{1-\exp(-2sh)}\left[D_1(s) - \exp(-sh)D_2(s)\right] \\
B_{21}(s) &= \frac{1+\exp(-2sh_1)}{4\{1-\exp(-2sh)\}} I_H(s) - \frac{\exp(-sh_1)}{1-\exp(-2sh)}\left[D_1(s) - \exp(-sh)D_2(s)\right] \\
B_{22}(s) &= -\frac{\exp(-2sh_2)\{1+\exp(-2sh_1)\}}{4\{1-\exp(-2sh)\}} I_H(s) + \frac{\exp(-sh_2)}{1-\exp(-2sh)}\left[D_1(s) - \exp(-sh)D_2(s)\right]
\end{aligned}\right\} \tag{45}$$

where

$$I_H(s) = \frac{1}{s}\int_{-c}^{c} G_H(\xi)\sin(s\xi)d\xi \tag{46}$$

$$\left.\begin{aligned}
D_1(s) &= -\frac{1}{2}\left[\int_0^\infty f_1(\eta,h_1)\cos(\eta p)d\eta\right]^2 \int_0^\infty \cos(sp)dp + \frac{\pi}{2}h_1 f_1(s,h_1) \\
D_2(s) &= -\frac{1}{2}\left[\int_0^\infty f_2(\eta,-h_2)\cos(\eta p)d\eta\right]^2 \int_0^\infty \cos(sp)dp + \frac{\pi}{2}h_2 f_2(s,-h_2)
\end{aligned}\right\} \tag{47}$$

$$f_1(s,h_1) = \frac{1}{\pi}\left[\frac{1-\exp(-2sh_2)}{1-\exp(-2sh)}\right]\exp(-sh_1)I_H(s), \quad f_2(s,-h_2) = \frac{1}{\pi}\left[\frac{1-\exp(-2sh_1)}{1-\exp(-2sh)}\right]\exp(-sh_2)I_H(s) \tag{48}$$

Making use of the first boundary condition (41) with the equations (45), we have the following singular integral equation for the determination of the unknown function $G_H(\xi)$:

$$\frac{1}{2\pi}\int_{-c}^{c}\left[\frac{1}{\xi - x} + M_H(\xi,x)\right]G_H(\xi)d\xi = \frac{2}{\pi}\int_0^\infty \left[N_1(s)D_1(s) - N_2(s)D_2(s)\right]\cos(sx)ds \quad (0 \le |x| < c) \tag{49}$$

where the integral kernel $M_H(\xi,x)$ is given by

$$M_H(\xi,x) = \int_0^\infty \frac{1}{1-\exp(-2sh)}\{2\exp(-2sh) + \exp(-2sh_1) + \exp(-2sh_2)\}\sin\{s(\xi - x)\}ds \tag{50}$$

The known functions $N_j(s)\ (j = 1,2)$ are

$$\left.\begin{aligned}
N_1(s) &= \frac{s}{1-\exp(-2sh)}\left[\exp(-sh_1) + \exp\{-s(h_2+h)\}\right] \\
N_2(s) &= \frac{s}{1-\exp(-2sh)}\left[\exp(-sh_2) + \exp\{-s(h_1+h)\}\right]
\end{aligned}\right\} \tag{51}$$

The singular integral equation (49) is to be solved with the following subsidiary condition obtained from the second boundary condition (41).

$$\int_{-c}^{c} G_H(\xi)d\xi = 0 \tag{52}$$

We can also solve the singular integral equation (49) and the additional equation (52), and we get the solution $G_H(\xi)$ and the function $I_H(s)$.

Substituting the equations (45) to the equations (43), the complementary functions $T_i^{(2)}(x,y)\ (i = 1,2)$ can be obtained as follows:

$$T_i^{(2)}(x,y) = \frac{1}{2\pi}\int_0^\infty \left[(-1)^i \exp(-s|y|) - K_H(s,y)\right]I_H(s)\cos(sx)ds$$

$$+\frac{2}{\pi}\int_0^\infty \left[K_{D1}(s,y)D_1(s) + K_{D2}(s,y)D_2(s)\right]\cos(sx)ds \quad (i = 1,2) \tag{53}$$

where

$$\left.\begin{aligned}K_H(s,y)&=\frac{1}{1-\exp(-2sh)}\left[\exp\{-s(2h+y)\}-\exp\{-s(2h-y)\}+\exp\{-s(2h_2+y)\}-\exp\{-s(2h_1-y)\}\right]\\K_{D1}(s,y)&=\frac{1}{1-\exp(-2sh)}\left[\exp\{-s(h+h_2+y)\}-\exp\{-s(h_1-y)\}\right]\\K_{D2}(s,y)&=\frac{1}{1-\exp(-2sh)}\left[\exp\{-s(h+h_1-y)\}-\exp\{-s(h_2+y)\}\right]\end{aligned}\right\}\quad(54)$$

Furthermore, the temperature gradients are

$$\frac{\partial}{\partial y}T_i^{(2)}(x,y)=\frac{1}{2\pi}\int_0^\infty\left[s\exp(-s|y|)-K_H'(s,y)\right]I_H(s)\cos(sx)ds$$

$$+\frac{2}{\pi}\int_0^\infty\left[K_{D1}'(s,y)D_1(s)+K_{D2}'(s,y)D_2(s)\right]\cos(sx)ds\quad(i=1,2)\quad(55)$$

where

$$\left.\begin{aligned}K_H'(s,y)&=\frac{-s}{1-\exp(-2sh)}\left[\exp\{-s(2h+y)\}+\exp\{-s(2h-y)\}+\exp\{-s(2h_2+y)\}+\exp\{-s(2h_1-y)\}\right]\\K_{D1}'(s,y)&=\frac{-s}{1-\exp(-2sh)}\left[\exp\{-s(h+h_2+y)\}+\exp\{-s(h_1-y)\}\right]\\K_{D2}'(s,y)&=\frac{s}{1-\exp(-2sh)}\left[\exp\{-s(h+h_1-y)\}+\exp\{-s(h_2+y)\}\right]\end{aligned}\right\}\quad(56)$$

The first terms of the equations (53) and (55) are

$$\left.\begin{aligned}&\frac{(-1)^i}{2\pi}\int_0^\infty\exp(-s|y|)I_H(s)\cos(sx)ds=\left\{\begin{aligned}&\frac{(-1)^i}{2\pi}\int_{-c}^{c}\arctan\left(\frac{\xi-x}{|y|}\right)G_H(\xi)d\xi\quad(|y|>0)\\&\frac{(-1)^i}{4}\int_{-c}^{c}\frac{|\xi-x|}{\xi-x}G_H(\xi)d\xi\qquad(|y|=0)\end{aligned}\right\}\ (i=1,2)\\&\frac{1}{2\pi}\int_0^\infty s\exp(-s|y|)I_H(s)\cos(sx)ds=\frac{1}{2\pi}\int_{-c}^{c}\frac{\xi-x}{y^2+(\xi-x)^2}G_H(\xi)d\xi\end{aligned}\right\}\quad(57)$$

For the case of $y\to0^\pm$,

$$T_1^{(2)}(x,0^+)=T_2^{(2)}(x,0^-)\quad(c\le|x|<\infty)\quad(58)$$

$$\frac{\partial}{\partial y}T_1^{(2)}(x,0^+)=\frac{\partial}{\partial y}T_2^{(2)}(x,0^-)$$

$$=\frac{1}{2\pi}\int_{-c}^{c}\left[\frac{1}{\xi-x}+M_H(\xi,x)\right]G_H(\xi)d\xi-\frac{2}{\pi}\int_0^\infty\left[N_1(s)D_1(s)-N_2(s)D_2(s)\right]\cos(sx)ds\quad(0\le|x|<\infty)\quad(59)$$

Therefore, taking the singular integral equation (49) into consideration,

$$\frac{\partial}{\partial y}T_1^{(2)}(x,0^+)=\frac{\partial}{\partial y}T_2^{(2)}(x,0^-)=0\quad(0\le|x|<c)\quad(60)$$

For the cases of $y=h_1$ and $y=-h_2$,

$$\left.\begin{aligned}T_1^{(2)}(x,h_1)&=-\frac{2}{\pi}\int_0^\infty D_1(s)\cos(sx)ds=-T_1^{(1)}(x,h_1)\\T_2^{(2)}(x,-h_2)&=-\frac{2}{\pi}\int_0^\infty D_2(s)\cos(sx)ds=-T_2^{(1)}(x,-h_2)\end{aligned}\right\}\quad(61)$$

From the equations (58)~(61), the complementary functions $T_i^{(2)}(x,y)\,(i=1,2)$ satisfy the boundary conditions (41) and (42).

V. Temperature Fields For The Case Of $h_1 = h_2 \to \infty$

First, we consider the particular integrals $T_i^{(1)}(x, y)\ (i = 1, 2)$ for $h_1 = h_2 \to \infty$. It is convenient to analyze that we introduce the following dimensionless quantities.

$$x = cx',\ y = cy',\ \xi = c\xi',\ s = \frac{s'}{c},\ h = ch', h_1 = ch_1' = \frac{ch'}{2}, h_2 = ch_2' = \frac{ch'}{2},\ G_E^{\infty}(\xi) = \frac{J_0}{\kappa} G_E^{\infty\prime}(\xi') \qquad (62)$$

where the superscript ∞ indicates the solution for $h_1 = h_2 \to \infty$. In this case, the singular integral equation (12) and the subsidiary condition (14) become

$$\left.\begin{array}{l} \dfrac{1}{2\pi}\displaystyle\int_{-1}^{1} \frac{1}{\xi' - x'} G_E^{\infty\prime}(\xi') d\xi' = -1 \quad (0 \le |x'| < 1) \\ \displaystyle\int_{-1}^{1} G_E^{\infty\prime}(\xi') d\xi' = 0 \end{array}\right\} \qquad (63)$$

It is easy to show that the function $G_E^{\infty}(\xi')$ takes the form

$$G_E^{\infty\prime}(\xi') = -\frac{2\xi'}{\left(1 - \xi'^2\right)^{1/2}} \qquad (64)$$

Substituting the equations (64) into the equations (32), we obtain $\Gamma_i^{\infty}(cx', cy') = \Gamma_i^{\infty}(x, y)\ (i = 1, 2)$ as follows:

$$\Gamma_i^{\infty}(cx', cy') = \frac{(-1)^i J_0 c}{2\pi} \int_0^{\infty} \frac{1}{s'} \exp(-s'|y'|) \sin\{s'(\xi' - x')\} ds' \int_{-1}^{1} G_E^{\infty\prime}(\xi') d\xi'$$

$$= -(-1)^i J_0 c \int_0^{\infty} \frac{1}{s'} \exp(-s'|y'|) J_1(s') \cos(s'x') ds' \quad (i = 1, 2) \qquad (65)$$

In the equations (65), $J_1(\,)$ is the Bessel function of the first kind of order 1. Similarly, $\partial\Gamma_i^{\infty}(cx', cy') / \partial y\ (i = 1, 2)$ become

$$\frac{\partial}{\partial y} \Gamma_1^{\infty}(cx', cy') = \frac{\partial}{\partial y} \Gamma_2^{\infty}(cx', cy')$$

$$= \frac{J_0}{2\pi} \int_0^{\infty} \exp(-s'|y'|) \sin\{s'(\xi' - x')\} ds' \int_{-1}^{1} G_E^{\infty\prime}(\xi') d\xi'$$

$$= -J_0 \int_0^{\infty} \exp(-s'|y'|) J_1(s') \cos(s'x') ds' \qquad (66)$$

Therefore,

$$T_1^{(1)\infty}(cx', cy') = T_2^{(1)\infty}(cx', cy')$$

$$= -\frac{(J_0 c)^2}{\lambda\kappa} \frac{1}{2} \left[\int_0^{\infty} \frac{1}{s'} \exp(-s'|y'|) J_1(s') \cos(s'x') ds' \right]^2$$

$$-\frac{(J_0 c)^2}{\lambda\kappa} |y'| \int_0^{\infty} \frac{1}{s'} \exp(-s'|y'|) J_1(s') \cos(s'x') ds' \qquad (67)$$

$$\frac{\partial}{\partial y} T_1^{(1)\infty}(cx', cy') = \frac{\partial}{\partial y} T_2^{(1)\infty}(cx', cy')$$

$$= -\frac{J_0^2 c}{\lambda\kappa} \left[\int_0^{\infty} \exp(-s'|y'|) J_1(s') \cos(s'x') ds' \right]$$

$$\times \left[\int_0^{\infty} \frac{1}{s'} \exp(-s'|y'|) J_1(s') \cos(s'x') ds' \right]$$

$$-\frac{J_0^2 c}{\lambda\kappa} \int_0^{\infty} \frac{1}{s'} \exp(-s'|y'|) J_1(s') \cos(s'x') ds' - \frac{J_0^2 c}{\lambda\kappa} |y'| \int_0^{\infty} \exp(-s'|y'|) J_1(s') \cos(s'x') ds' \qquad (68)$$

For the case of $y' \to 0^{\pm}$, the equations (67) and (68) become

$$T_1^{(1)\infty}(cx',0^+) = T_2^{(1)\infty}(cx',0^-)$$
$$= -\frac{(J_0 c)^2}{\lambda\kappa}\frac{1}{2}\left[\int_0^\infty \frac{1}{s'}J_1(s')\cos(s'x')ds'\right]^2 = \begin{cases} -\dfrac{(J_0 c)^2}{\lambda\kappa}\dfrac{1}{2}\left(1-x'^2\right) & (0 \le |x'| < 1) \\ 0 & (1 \le |x'| < \infty) \end{cases} \tag{69}$$

$$\frac{\partial}{\partial y}T_1^{(1)\infty}(cx',0^+) = \frac{\partial}{\partial y}T_2^{(1)\infty}(cx',0^-) = 0 \tag{70}$$

Moreover, for the case of $x' = 0$, the solutions (67) become

$$T_1^{(1)\infty}(0,cy') = T_2^{(1)\infty}(0,cy') = -\frac{(J_0 c)^2}{2\lambda\kappa} \quad (0 \le |y'| < \infty) \tag{71}$$

Next, we consider the complementary functions $T_i^{(2)}(x,y)\ (i=1,2)$ for the case of $h_1 = h_2 \to \infty$. Using the equation (64) and $G_H^\infty(c\xi') = 0$, the equations (53) become

$$T_1^{(2)\infty}(cx',cy') = T_2^{(2)\infty}(cx',cy') = \lim_{h\to\infty}\frac{(J_0 c)^2}{\lambda\kappa}\int_0^\infty \delta(s')f(s')ds' \tag{72}$$

where

$$\left.\begin{aligned} \delta(s') &= \frac{h'\exp(-s'h')}{\{1+\exp(-s'h')\}^2} \\ f(s',x',y') &= \left[\exp(-s'|y'|) + \exp(s'|y'|)\right]\frac{J_1(s'/c)}{s'}\cos(s'x') \end{aligned}\right\} \tag{73}$$

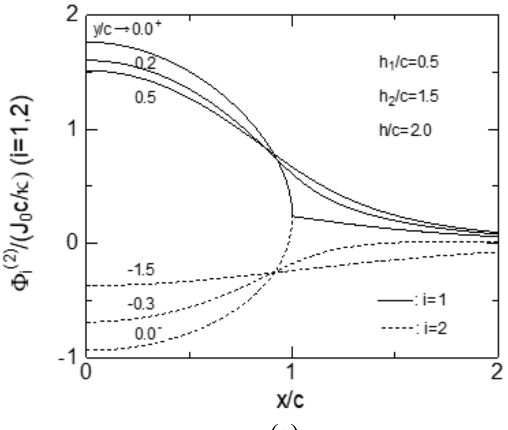
Because the function $\delta(s')$ is the Delta function sequence, the equations (72) become

$$T_1^{(2)\infty}(cx',cy') = T_2^{(2)\infty}(cx',cy') = \frac{(J_0 c)^2}{2\lambda\kappa} \tag{74}$$

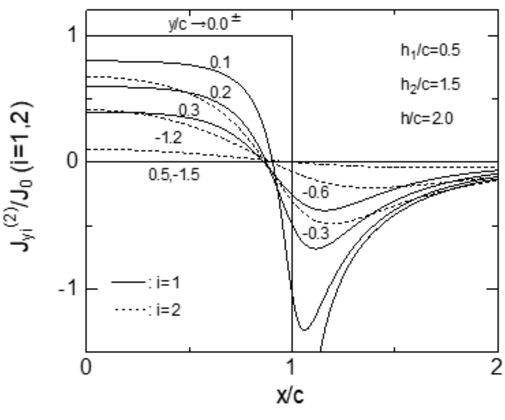
Therefore,

$$T_1^\infty(cx',0^+) = T_2^\infty(cx',0^-) = \begin{cases} \dfrac{(J_0 c)^2}{2\lambda\kappa}x'^2 & (0 \le |x'| < 1) \\ \dfrac{(J_0 c)^2}{2\lambda\kappa} & (1 \le |x'| < \infty) \end{cases} \tag{75}$$

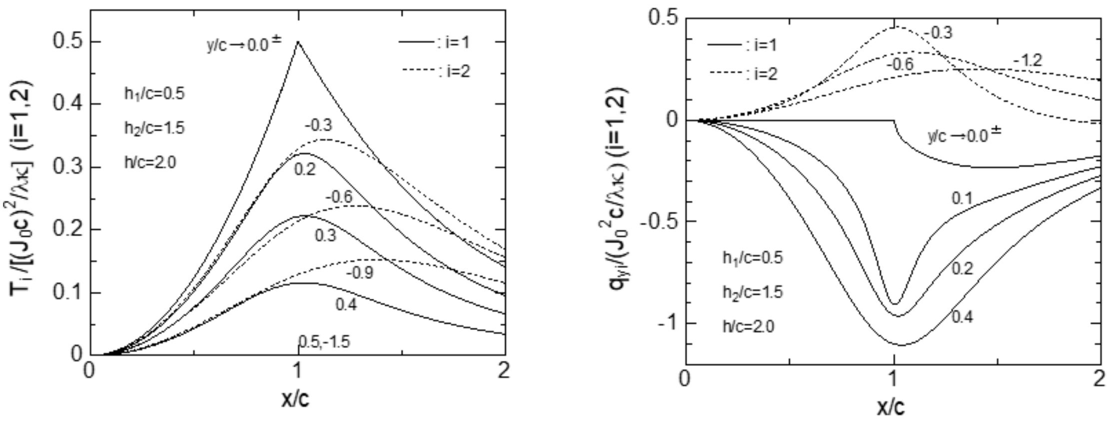
Numerical Results And Discussion



(a)



(b)



(c) (d)

Figure no2: The disturbed distributions of the electric potentials (a), the current densities in the *y*-direction (b), the temperatures (c) and the temperature gradients in the *y*-direction (d) for $h_1 / c = 0.5$ and $h_2 / c = 1.5$ $(h / c = 2.0)$.

First, we consider the entire trend of the disturbed electric and temperature fields and we will also check the validity of the solutions. Figures no2(a)~(d) show the normalized electric potentials $\Phi_i^{(2)}(x, y) / (J_0 c / \kappa)$, the current densities $J_{yi}^{(2)}(x, y) / J_0$, the temperatures $T_i(x, y) / \left[(J_0 c)^2 / \lambda\kappa \right]$ and the temperature gradients $q_{yi}(x, y)\ /(J_0^2 c / \lambda\kappa)$ $(i = 1,2)$ for $h_1 / c = 0.5$ and $h_2 / c = 1.5$ $(h / c = 2.0)$. In the figures, y / c shows the normalized position parameter and the solid and dotted lines indicate the fields of $0 \le y \le h_1$ $(i = 1)$ and $-h_2 \le y \le 0$ $(i = 2)$. Figures no2(a)~(d) show that the disturbed electric and temperature fields satisfy the electric boundary conditions (5), (6) and the temperature boundary conditions (28), (29), respectively. Thus, it is noted that the obtained solutions should be reasonable.

All the values are symmetric with respect to the *x*-axis and the values of $J_{yi}^{(2)}(x, y)$ $(i = 1,2)$ have singularity at the crack tip $(|x| \to c^+,\ y \to 0.0^{\pm})$. While the temperatures $T_1(x, 0^+)$ and $T_2(x, 0^-)$ on the crack surfaces $(0 \le |x| < c)$ are evaluated by the numerical calculations, the values of them are almost equal to $T_1^{\infty}(cx', 0^+) = T_2^{\infty}(cx', 0^-)$ given by the equations (75). Therefore, it may be that the top and bottom surfaces do not affect the temperatures, and the maximum values are $T_1(c, 0^+) = T_2(c, 0^-) = (J_0 c)^2 / 2\lambda\kappa$. The temperatures $T_1(x, 0^+)$ and $T_2(x, 0^-)$ $(c \le |x| < \infty)$ decrease and approach zero with increasing $|x|$. The distribution of the temperatures $T_i(x, y)\ (i = 1,2)$ tend to decrease with increasing $|y|$.

Next, we study the influence of the crack position on the temperature distribution. Figure no3 indicates the normalized temperature along the $y \to 0.0^{\pm}$ planes for the case of $h / c = 2.0$. On account of symmetry, we will consider only $T_1(x, 0^+)$ for the case of $0.0 < h_1 / c \le 1.0$. Because the temperature at the upper surface of the plate is thermally restricted, the temperature $T_1(x, 0^+)\ (|x| \ge c)$ decreases with decreasing h_1 / c. However, the crack face temperature $T_1(x, 0^+)\ (0 \le |x| < c)$ is independent of h_1 / c.

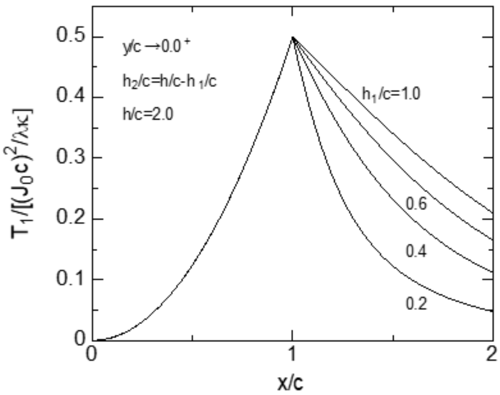


Figure no3: The influence of the crack location on the disturbed temperature distribution on the $y/c = 0^{+}$ plane for the case of $h/c = 2.0$.

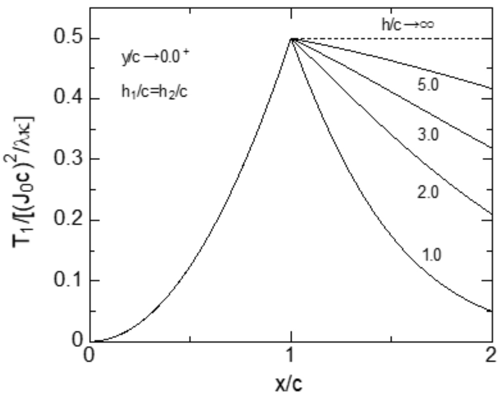


Figure no4: The influence of the thickness of the plate on the disturbed temperature distribution on the $y/c \to 0.0^{+}$ plane for the case of $h_1/c = h_2/c$.

Finally, we study the influence of the thickness of the plate on the disturbed temperature distribution. Figure no4 shows the effect of h/c on the normalized temperature distribution $T_1(x,0^{+})/\left[(J_0c)^2/\lambda\kappa\right]$ on the $y/c \to 0.0^{+}$ plane for the case of $h_1/c = h_2/c$. Because of $T_1(x,0^{+}) = T_2(x,0^{-})$, it is also sufficient to consider only the temperature $T_1(x,0^{+})$. The dotted line indicates the close-form solution $T_1^{\infty}(x,0^{+})$ given by the equation (75). Because the effect of the upper surface increases, the temperature $T_1(x,0^{+})$ $(c \le |x| < \infty)$ decreases with decreasing h/c.

VI. Conclusion

The temperature distribution in a plate with a through crack during resistance spot welding under a uniform current density is considered in this paper. Using the integral transform method, the electric and temperature fields disturbed by the crack are obtained. The following facts can be found from the analysis and the numerical results.

1. The temperatures along the crack surfaces should not depend on the geometric parameters.
2. The maximum values of the crack surface temperatures would be $T_1(c,0^{+}) = T_2(c,0^{-}) = (J_0c)^2/2\lambda\kappa$
3. As the crack approaches to the upper or lower plane, the temperature distributions tend to decrease except for the crack surface temperatures.
4. If the thickness of the plate is infinite, the temperatures along the crack surfaces can be obtained as closed form solution as follows:

$$T_1^{\infty}(cx',0^+) = T_2^{\infty}(cx',0^-) = \left\{ \begin{array}{ll} \dfrac{(J_0 c)^2}{2\lambda\kappa} x'^2 & (0 \le |x'| < 1) \\ \dfrac{(J_0 c)^2}{2\lambda\kappa} & (1 \le |x'| < \infty) \end{array} \right\}$$

Similarly, the temperatures along the y-axis are

$$T_1^{(1)\infty}(0,cy') = T_2^{(1)\infty}(0,cy') = 0 \quad (0 \le |y'| < \infty)$$

References

[1]. Liu, T. J-C., Electro-Thermo-Structural Coupled-Field Simulation Of Electric Connector With Edge Crack. Proceedings Of The 6th Asian Conference On Mechanics Of Functional Materials And Structures (ACMFMS2018), 2018, Pp.7-10.

[2]. Sneddon, I. N. And Lowengrub, M., Crack Problems In The Classical Theory Of Elasticity, 1969, John Wiley & Sons, Inc., New York

[3]. Erdogan, F., Gupta, G.D. And Cook, T.S., Methods Of Analysis And Solution Of Crack Problems 1972, G. C. Sih (Ed), Noordhoff, Leyden.