

## A stable finite element method for compressible flows using enthalpic variables

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**Abstract:** *In this paper, we present a finite element method for the simulation of two-dimensional compressible flows. The Navier-Stokes equations are solved in terms of so-called enthalpic variables: static pressure, momentum per unit volume, and total specific enthalpy. The variational formulation is a variant of the Petrov-Galerkin method. The discretization of the variational formulation is done on a P1/P2 element and uses an implicit scheme. The algebraic system is solved using the GMRES algorithm with diagonal pre-conditioning. Several simulations have been carried out, in order to validate the methods proposed and the computation code developed. Subsequently, this computation code was used to model the flow in an intake system of a car engine.*

**Keywords -** *Navier-Stokes equations, enthalpic variables, Petrov-Galerkin, car engine intake system.*

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### I. Introduction

The compressible flows are governed by the conservation equations of mass, momentum and energy. These equations can be solved numerically in two distinct forms [1]. The so-called non-conservative form [1, 2] in which the conservation equations are simplified and transformed so as to reveal the so-called independent primitive variables: the static pressure  $p$ , the velocity  $\mathbf{u}$  and the temperature  $T$ . The so-called conservative form [3, 4, 5] where the conservation equations are solved as they derive from the conservation laws, without any transformation, and where the density  $\rho$ , the momentum per unit volume  $\mathbf{U}$  and the total energy per unit volume  $E$ , appearing in these equations, form the independent variables also called conservative variables. Often, to simulate the flows it is desired to apply Dirichlet boundary conditions on the static pressure  $p$ , the total pressure  $P_0$  or the total temperature  $T$ . However, in the conservative form, the boundary conditions concern a priori the conservative variables  $(\rho, \mathbf{U}, E)$ . Therefore, the desired values of pressure and temperature will not be easily imposed. On the other hand, this may be possible, while preserving the conservative form of the equations, by using, as independent variables, the so-called enthalpic variables: the static pressure  $p$ , the momentum per unit volume  $\mathbf{U}$  and the total specific enthalpy  $h$ .

The Navier-Stokes equations are of the convection-diffusion type. Galerkin finite element approximation of these equations can produce solutions with nonphysical oscillations. Additional diffusion then becomes necessary to stabilize the solution. This diffusion can be generated using the *Streamline Upwinding Petrov-Galerkin* method (SUPG). This method was introduced by Hughes et al. [1, 6]. It is a variant of the Petrov-Galerkin method based on the concept of optimal artificial diffusion. It was used initially to solve the Navier-Stokes and Euler equations in terms of so-called entropic variables [7, 8, 9]. This method is widely used in the field of computational fluid dynamics (CFD) [10, 11, 12, 13, 14, 15]. The method was then exploited and extended to conservative variables [16, 17, 18, 19, 20]. The latter approach has the advantage of being simpler to implement numerically and to easily apply the physical boundary conditions than that using the entropy variables. In this work, we show the possibility of extending the SUPG method to any type of independent variables, in particular the enthalpic variables.

The presentation of this article is structured as follows: after this introduction, we present the equations governing the compressible flows and we discuss the boundary conditions, often used for the simulation of the flows. In the third section, we present the variational formulation of the type SUPG which will be followed by the discretization and numerical resolution of the variational problem. The fourth section is dedicated to the different numerical tests for the simulation of two-dimensional compressible flows. Finally, we close this article with a general conclusion.

## II. Mathematical Formulation

The equations, in the a dimensional form, of conservation of the mass, the momentum and the energy governing the flows of Newtonian compressible fluids are respectively written as follows:

$$\left. \begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{U} &= 0 \\ \frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot (\mathbf{U} \otimes \mathbf{u}) + \nabla p &= \nabla \cdot \underline{\sigma} + \rho \mathbf{f} \\ \frac{\partial E}{\partial t} + \nabla \cdot [(E + p)\mathbf{u}] &= \nabla \cdot (\underline{\sigma} \cdot \mathbf{u}) - \nabla \cdot \mathbf{q} + \mathbf{f} \cdot \mathbf{U} \end{aligned} \right\} \quad (1)$$

$$\left. \begin{aligned} \mathbf{u} &= \frac{\mathbf{U}}{\rho} \\ T &= \frac{E}{\rho} - \frac{|\mathbf{U}|^2}{2\rho^2} \\ p &= (\gamma - 1) \left[ E - \frac{|\mathbf{U}|^2}{2\rho^2} \right] \\ \mathbf{q} &= -\frac{\gamma\mu}{Re Pr} \nabla \left[ \frac{E}{\rho} - \frac{|\mathbf{U}|^2}{2\rho^2} \right] \\ \underline{\sigma} &= \frac{\mu}{Re} \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^t - \frac{2}{3} (\nabla \cdot \mathbf{u}) \underline{I} \right] \end{aligned} \right\} \quad (2)$$

where  $\mathbf{U}$ ,  $\mathbf{u}$ ,  $\rho$ ,  $p$ ,  $T$ ,  $E$ ,  $\underline{\sigma}$ ,  $\mathbf{q}$  and  $\rho \mathbf{f}$  are respectively the vector of momentum per unit of volume, the velocity vector, the density, the static pressure, the temperature, the total energy per unit volume, the tensor of viscous stresses, the heat flux and the forces. Also,  $t$ ,  $\mu$ ,  $\gamma$ ,  $Re$ ,  $Pr$  and  $\underline{I}$  are respectively the time, the dynamic viscosity, the ratio of the specific heats to constant pressure ( $c_p$ ) and to constant volume ( $c_v$ ), the number of Reynolds, the number of Prandlt and tensor unit. The conservation equations (1) can be written in the following vector form:

$$\mathbf{V}_{,t} + \mathbf{F}_{i,i}^{conv}(\mathbf{V}) = \mathbf{F}_{i,i}^{diff}(\mathbf{V}) + \mathcal{F} \quad (3)$$

where  $\mathbf{V} = (\rho, \mathbf{U}, E)^t$  is the vector of so-called conservative variables. The vectors  $\mathbf{F}_i^{conv}$ ,  $\mathbf{F}_i^{diff}$  and  $\mathcal{F}$  are respectively the convective flow vectors, diffusion flux and the source vector. The repeated index, in equations (3), denotes a summation. A quasi-linear form of the system (3) is written:

$$\mathbf{V}_{,t} + \mathbf{A}_i \mathbf{V}_{,i} = (\mathbf{K}_{ij} \mathbf{V}_{,j})_{,i} + \mathcal{F} \quad (4)$$

where  $\mathbf{A}_i$  are the Jacobian matrices of transformation of the convective flow vector such that:

$$\mathbf{A}_i = \mathbf{F}_{i,V}^{conv} = \frac{\partial \mathbf{F}_i^{conv}}{\partial \mathbf{V}}$$

and  $\mathbf{K}_{ij}$  are the diffusion matrices defined as:

$$\mathbf{K}_{ij} \mathbf{V}_{,j} = \mathbf{F}_i^{diff}$$

In the flows studied, several types of boundary conditions are treated. At the entrance, often the flow is subsonic and parallel then, three conditions are necessary: total pressure or stagnation pressure  $P_0$ , total temperature or stagnation temperature  $T_0$ , zero transverse velocity  $u_2 = 0$ . At the outlet, for a subsonic flow a single Dirichlet condition is imposed. In general, this condition corresponds to the static pressure. In the case of a supersonic flow, no Dirichlet condition is imposed. The conditions with solid walls, the velocity of the fluid is zero:  $\mathbf{u} = 0$ . Moreover, if the wall is considered adiabatic:  $\mathbf{q} \cdot \mathbf{n} = 0$  where  $\mathbf{n}$  is the external normal unit vector, or isothermal:  $T_w = T_0$ .

However, in the conservative formulation (1), the boundary conditions concern a priori the conservative variables ( $\rho, \mathbf{U}, E$ ). As a result, the desired values of pressure and temperature will not be exactly enforced. On the other hand, this may be possible, while preserving the conservative form of the equations, by using, as independent variables, the so-called enthalpic variables: the static pressure  $p$ , the momentum per unit volume  $\mathbf{U}$  and the total specific enthalpy  $h$  defined as:

$$h = i + \frac{p}{\rho} + \frac{|\mathbf{U}|^2}{2\rho^2} = \frac{E}{\rho} + \frac{p}{\rho} \quad (5)$$

Indeed, by using the enthalpic variables, the static pressure becomes an independent variable. Also, to impose the total temperature  $T_0$  it amounts directly to imposing the corresponding total enthalpy:

$$h = \gamma T_0 \quad (6)$$

In addition, the desired total pressure  $P_0$  is ensured by imposing the corresponding static pressure such that:

$$p = P_0 - \frac{|\mathbf{U}|^2}{2\rho^2} \quad (7)$$

where the density  $\rho$  is expressed from the equation of a perfect gas. Thus, from the equation of a perfect gas, we obtain:

$$h = \frac{p}{(\gamma - 1)\rho} + \frac{p}{\rho} + \frac{|\mathbf{U}|^2}{2\rho^2} \quad (8)$$

The resolution of the equation of the second degree in  $\rho$  (8) gives:

$$\rho = \frac{\gamma p}{2(\gamma - 1)h} + \left( \left[ \frac{\gamma p}{2(\gamma - 1)h} \right]^2 + \frac{|\mathbf{U}|^2}{2h} \right)^{1/2} \quad (9)$$

By adopting the change of variables (5, 8 and 9), the conservation equations (1) are then written, in terms of enthalpic variables, as follows:

$$\left. \begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{U} &= 0 \\ \frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot (\mathbf{U} \otimes \mathbf{u}) + \nabla p &= \nabla \cdot \underline{\sigma} + \rho \mathbf{f} \\ \frac{\partial(\rho h)}{\partial t} - \frac{\partial p}{\partial t} + \nabla \cdot (h\mathbf{U}) &= \nabla \cdot (\underline{\sigma} \cdot \mathbf{u}) - \nabla \cdot \mathbf{q} + \mathbf{f} \cdot \mathbf{U} \end{aligned} \right\} \quad (10)$$

$$\left. \begin{aligned} \rho &= \frac{\gamma p}{2(\gamma - 1)h} + \left( \left[ \frac{\gamma p}{2(\gamma - 1)h} \right]^2 + \frac{|\mathbf{U}|^2}{2h} \right)^{1/2} \\ T &= h - \frac{|\mathbf{U}|^2}{2\rho^2} - \frac{p}{\rho} \\ \mathbf{q} &= -\frac{\gamma \mu}{Re Pr} \nabla \left[ h - \frac{p}{\rho} - \frac{|\mathbf{U}|^2}{2\rho^2} \right] \\ \underline{\sigma} &= \frac{\mu}{Re} \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^t - \frac{2}{3} (\nabla \cdot \mathbf{u}) \mathbf{I} \right] \end{aligned} \right\} \quad (11)$$

The vector and quasi-linear forms of the system (10) can be obtained respectively from the vectorial and quasi-linear (3) or (4) form, from the formulation into conservative variables, by adopting the following change of variables. :

$$\left. \begin{aligned} \mathbf{V}_{,t} &= \mathbf{A}_0 \mathbf{Y}_{,t} \\ \mathbf{V}_{,i} &= \mathbf{A}_0 \mathbf{Y}_{,i} \\ \mathbf{F}_i^{conv}(\mathbf{V}) &= \tilde{\mathbf{F}}_i^{conv}(\mathbf{Y}) \\ \mathbf{F}_i^{diff}(\mathbf{V}) &= \tilde{\mathbf{F}}_i^{diff}(\mathbf{Y}) \end{aligned} \right\} \quad (12)$$

where  $\mathbf{A}_0 = \mathbf{V}_{,\mathbf{Y}}$  is the Jacobian matrix of transformation of the vector of the conservative variables  $\mathbf{V} = (\rho, \mathbf{U}, E)^t$  to the vector of the enthalpic variables  $\mathbf{Y} = (p, \mathbf{U}, h)^t$ . The vector form and the quasi-linear form of the system (10) will then respectively be:

$$\mathbf{A}_0 \mathbf{Y}_{,t} + \tilde{\mathbf{F}}_{i,i}^{conv}(\mathbf{Y}) = \tilde{\mathbf{F}}_{i,i}^{diff}(\mathbf{Y}) + \tilde{\mathcal{F}} \quad (13)$$

$$\mathbf{A}_0 \mathbf{Y}_{,t} + \tilde{\mathbf{A}}_i \mathbf{Y}_{,i} = (\tilde{\mathbf{K}}_{ij} \mathbf{Y}_{,j})_{,i} + \tilde{\mathcal{F}} \quad (14)$$

with

$$\tilde{\mathbf{A}}_i = \tilde{\mathbf{F}}_{i,Y}^{conv} = \mathbf{A}_i \mathbf{A}_0 \quad \text{et} \quad \tilde{\mathbf{K}}_{ij} = \mathbf{K}_{ij} \mathbf{A}_0$$

### III. Finite Element Formulation

#### III.1. Variational formulation

The variational formulation of the problem (14), according to the method SUPG [5, 16, 17, 18, 19, 20], consists in finding  $\mathbf{Y}$  such that for any function of weighting  $\mathbf{W}$ , we have:

$$\begin{aligned} & \int_{\Omega} \{ \mathbf{W} \cdot [\mathbf{A}_0 \mathbf{Y}_{,t} + \tilde{\mathbf{A}}_i \mathbf{Y}_{,i} - \tilde{\mathcal{F}}] + \mathbf{W}_{,i} \cdot [\tilde{\mathbf{K}}_{ij} \mathbf{Y}_{,j}] \} d\Omega \\ & + \sum_e \int_{\Omega^e} \{ (\tilde{\mathbf{A}}_i^t \cdot \mathbf{W}_{,i}) \tilde{\tau} [\mathbf{A}_0 \mathbf{Y}_{,t} + \tilde{\mathbf{A}}_i \mathbf{Y}_{,i} - (\tilde{\mathbf{K}}_{ij} \mathbf{Y}_{,j}) \mathbf{Y}_{,j} - \tilde{\mathcal{F}}] \} d\Omega^e \\ & = \oint_{\Gamma} \{ \mathbf{W} \cdot [(\tilde{\mathbf{K}}_{ij} \mathbf{Y}_{,j}) \cdot \mathbf{n}_i] \} d\Gamma \end{aligned} \quad (15)$$

with  $\tilde{\tau}$  is the stabilization matrix given by

$$\tilde{\tau} = \mathbf{A}_0^{-1} \underline{\tau} \quad (16)$$

and  $\Omega^e$  is an element of the mesh of the domain  $\Omega$  of border  $\Gamma$ .

The variational formulation above (15) is distinguished by two important properties [5, 19, 20]. The first property is a weighted residual method in the sense that a regular exact solution of the original physical problem remains a solution to the variational problem. This ensures not only a good accuracy of the approximation, but also a spatio-temporal stability. The second property, stability is ensured thanks to the elliptic term:

$$\sum_e \int_{\Omega^e} (\tilde{\mathbf{A}}_i^t \cdot \mathbf{W}_{,i}) \tilde{\tau} (\tilde{\mathbf{A}}_i \mathbf{Y}_{,i}) d\Omega^e \quad (17)$$

Thus, this stability depends enormously on the structure of the matrix  $\tilde{\tau}$  and therefore on the structure of the matrix  $\underline{\tau}$ . The choice of this matrix is then fundamental for the good behavior of the SUPG method. The authors [5, 17, 19, 20] have proposed and validated numerically the definition of the matrix  $\underline{\tau}$  such that:

$$\underline{\tau} = \left[ \sum |c_{ij} \mathbf{A}_j| \right]^{-1} \check{\zeta}(Pe) \quad (18)$$

with

$$\check{\zeta}(Pe) = \min\left(1, \frac{Pe}{3}\right)$$

where  $c_{ij}$  are the coefficients of the inverse of the Jacobian matrix of geometrical transformation and  $Pe = \frac{ul}{2\nu}$  is the number of local Peclet where  $l$  is the size of the element and  $\nu$  is the kinematic viscosity of the fluid.

#### Note

The condition of Neuman  $(\underline{\sigma} \cdot \mathbf{u}) \cdot \mathbf{n} = 0$ , at the outlet of the flow, will be implicitly imposed. With this condition and the boundary conditions listed in Section 2, the contour terms in the variational form (15) vanish.

#### III.2. Discretization and numerical resolution

The numerical simulations carried out in this work aim to obtain stationary solutions (steady state). However, stationary conservation equations are hyperbolic-elliptic in nature in the case of Navier-Stokes equations. This change in nature makes the resolution of stationary conservation equations difficult. To avoid this difficulty, we solve the unsteady Navier-Stokes equations where the solution evolves in time until convergence corresponding to the steady state. Time discretization is performed using an implicit scheme. The

variables and their spatial derivatives are discretized using the finite element method on triangular elements with a quadratic approximation (six nodes) for the velocity and momentum components and a linear approximation (three nodes) for the other variables. The variational system (15) is transformed into an algebraic system that is solved using the GMRES method [21, 22, 23, 24] using a diagonal preconditioning.

#### IV. Numerical Simulations

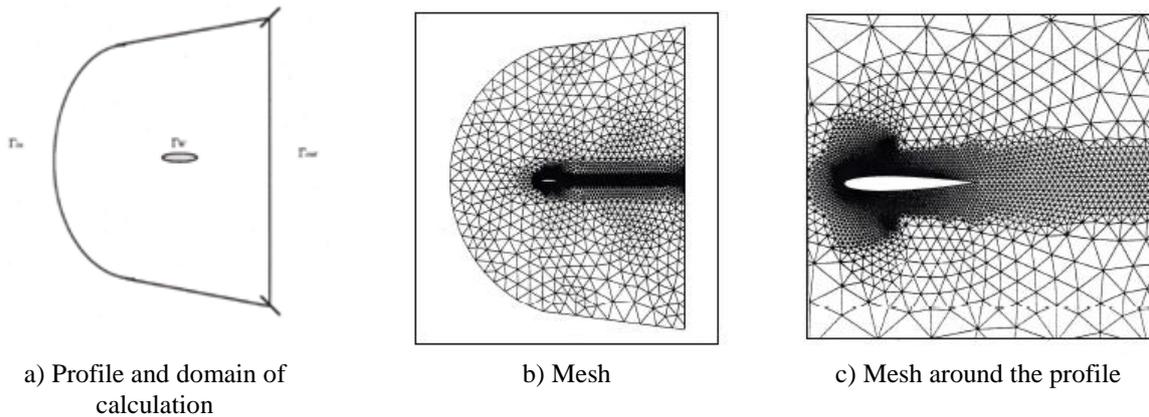
This section is devoted to the numerical validation of the conservative formulation using enthalpic variables and the SUPG stabilization technique. For this purpose, several types of two-dimensional compressible flows have been studied. The robustness of the calculation code has been validated by the study of the flow whose geometry and the physics of the flow are very complex. The results obtained are compared with the results of previous work.

##### IV.1. Flows around the profile NACA0012

Transonic flows around the NACA0012 profile with various difficulties have been solved using the SUPG stabilization method described above. The characteristics of these test cases vary between 500 and 2 000 for the Reynolds number with a Mach number equal to 0,85. The profile NACA0012 is symmetrical and the coordinates of the extrados are given by:

$$y(x) = 5t(0,2969x^{1/2} - 0,126x - 0,3516x^2 + 0,2848x^3 - 0,1015x^4)$$

where  $x$  is the distance along the chord from the leading edge ( $x = 0$ ),  $y$  is the ordinate on the intrados and  $t$  ( $= 0,12$ ) is the relative thickness of the profile. The mesh used has 8 150 triangular elements. The constraints imposed on this mesh are: dense mesh in the areas close to the profile, the wake, the leading edge and recirculation. Figure 1 shows: the geometry of the profile with the computational domain (figure 1a), the mesh (figure 1b) and an enlargement around the profile (figure 1c).



**Figure 1:** Profile NACA0012 - Domain of calculation and Mesh

At infinity upstream  $\Gamma_{in}$ , the boundary conditions are Dirichlet type:

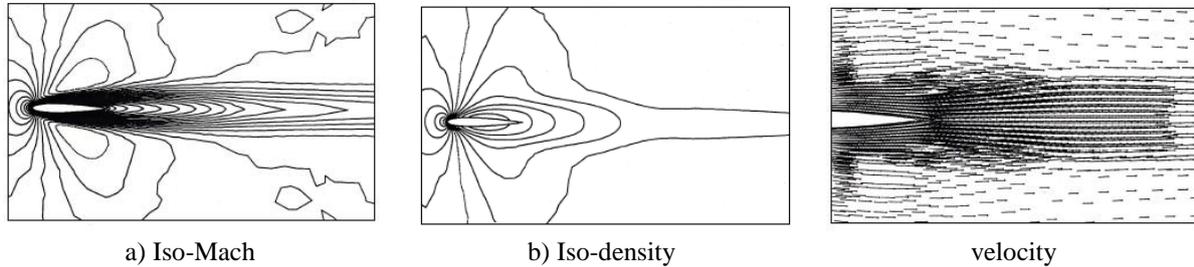
$$\begin{cases} M_\infty = 0,85 \\ p = p_\infty \\ U_1 = U_\infty \\ U_2 = 0,0 \\ E = E_\infty \end{cases}$$

At infinity downstream  $\Gamma_{out}$ , we impose a boundary condition of the Dirichlet type: let  $p = p_\infty$  and the condition  $(\underline{\sigma} \cdot \underline{n})_1 = 0$ . On the wall of the profile  $\Gamma_w$ , one imposes the condition of adhesion  $\mathbf{U} = 0$ . The initial solution used is a uniform field, with the exception of the wall of the profile where the condition  $\mathbf{U} = 0$  is imposed.

##### IV.1.1. Flow with $Re = 500$ and $M = 0,85$

Slightly transonic flow, no shock (Reynolds number is low and therefore no significant influence of viscous stresses). This simulation makes it possible to compare the results with those obtained by [2, 25, 26, 27], whose strategy of resolution is different. This flow was simulated with the two finite element methods, the

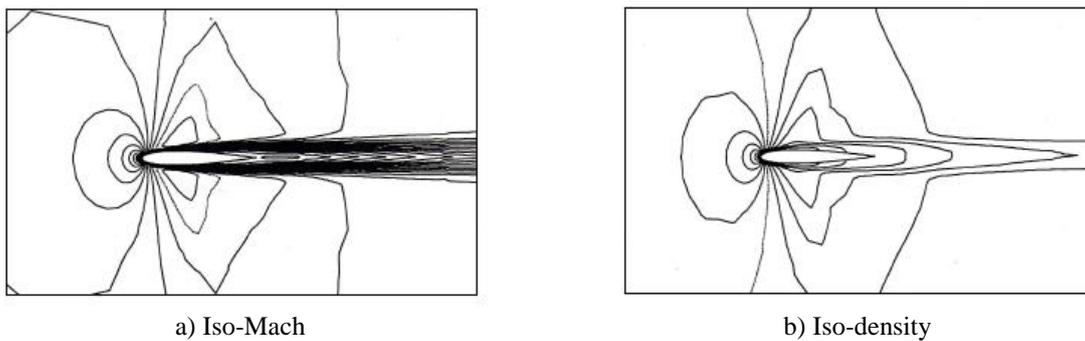
standard Galerkin method and the SUPG method. For both methods the results obtained are identical. This demonstrates the validity of the SUPG method that we developed because it is not diffusive for low Reynolds numbers. We present the iso-Mach, iso-density and velocity fields respectively in figures 2a, 2b and 2c. We can observe the presence of a thick boundary layer around the profile, as well as the presence of a stationary wake. These results correspond perfectly to the low value of the Reynolds number that we use for this test case.



**Figure 2:** Flow -  $Re = 500$  and  $M = 0,85$

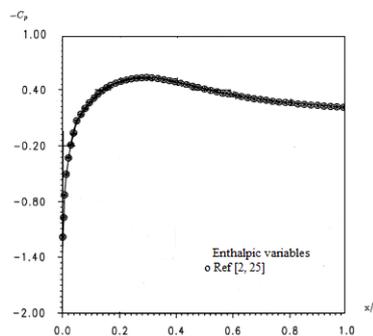
**IV.1.2. Flow with  $Re = 2\,000$  and  $M = 0,85$**

The resolution strategy consists of using the SUPG method and setting the Reynolds and Mach numbers to their maximum values, from the beginning of the resolution until convergence of the temporal schema. The iso-Mach and iso-density fields are shown respectively in figures 3a and 3b.



**Figure 3:** Flow -  $Re = 2\,000$  and  $M = 0,85$

In figure 4, the pressure coefficient on the profile was compared with those obtained by a finite element method using non-conservative variables [2, 25]. Note that for this test case, there are no differences between the two formulations.

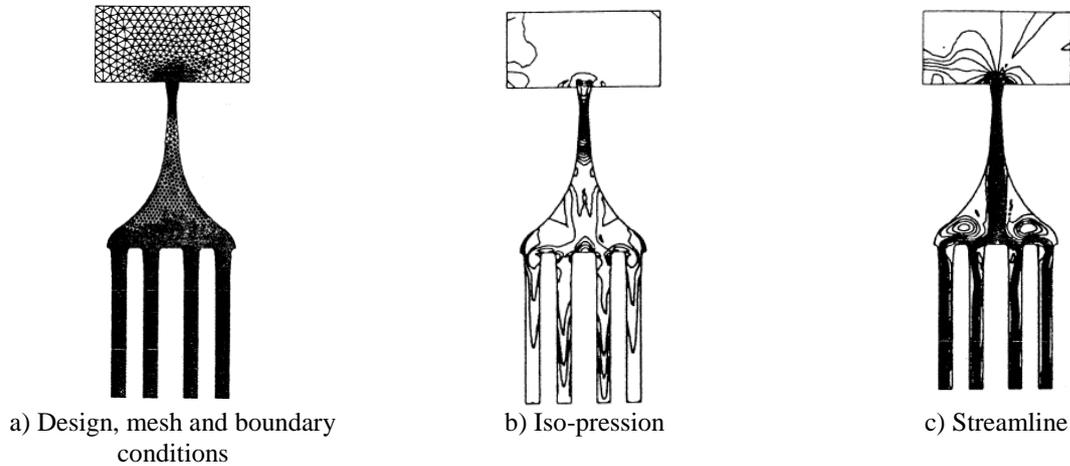


**Figure 4:** Pressure coefficient

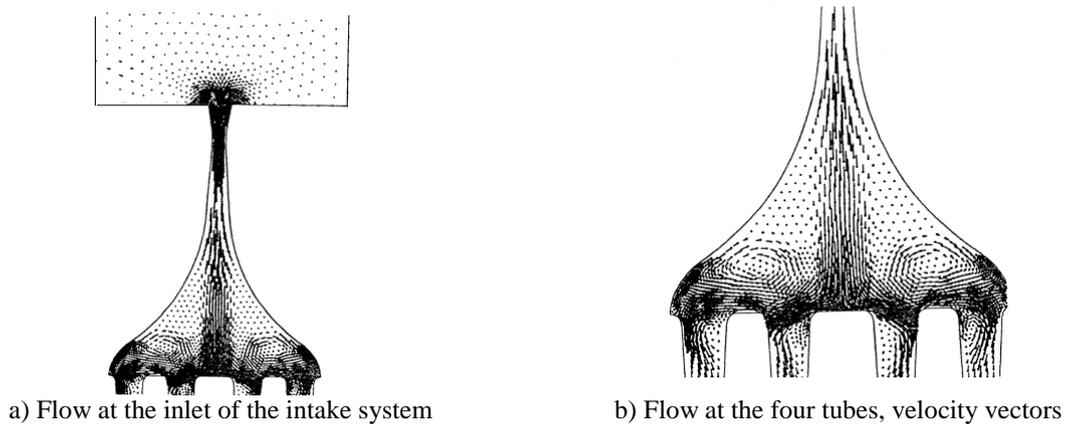
**IV.2. Flow in an intake system**

The primary motivation of this modeling is the North American competition of the SAE formula. The aim of the project is the design of a naturally aspirated engine to propel a small formula-type racing car. The intake system (see figure 5a) is designed to distribute air evenly throughout the four tubes while minimizing pressure losses. The inlet restriction of the intake system was designed according to the venturi principle to reduce pressure losses. The realization of these simulations confirms the validation of the developed computation code, and tests its robustness in the case of a relatively complex geometry and difficult boundary conditions. Thus, thanks to the simulations it was possible to predict the changes to be made to the initial design.

The mesh of this geometry and the boundary conditions are illustrated in figure 5a. The atmosphere is represented by the rectangle surrounding the inlet of the nozzle to which a constant pressure is fixed. At the exit of the four tubes is fixed a velocity profile. Still keeping the same mesh (9345 elements), the Reynolds number was gradually increased to 5000. The iso-pressure curves and the current lines are shown in figures. 5b and 5c. The flow in the intake system is shown schematically by the velocity vectors, with an enlargement at the inlet (figure 6a) and at the level of the four tubes (figure 6b).



**Figure 5:** Flow at the inlet of the intake system



**Figure 6:** Flow at the inlet of the intake system - velocity

## V. Conclusion

We have developed a finite element method for the simulation of two-dimensional compressible flows. We solve the Navier-Stokes equations, in conservative form, in terms of so-called enthalpic variables: the static pressure  $p$ , the momentum  $\mathbf{U}$  per unit volume and the total mass enthalpy  $h$ . With the use of these variables, the Dirichlet boundary conditions on the static or total pressure and the total temperature, often imposed for flow simulation, are applied directly and accurately, contrary to the standard formulation using conservative variables. : the density  $\rho$ , the momentum  $\mathbf{U}$  per unit volume and the total energy  $E$  per unit volume. The variational formulation, of the SUPG type, developed here, can use, in principle, any type of independent variables. The mathematical and numerical models, presented in this paper, have been validated on several types of geometries to simulate two-dimensional compressible flow. The robustness of the SUPG method has been confirmed, especially if the mesh is adapted. The results obtained are very similar to the results of previous work.

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