IOSR Journal of Mechanical and Civil Engineering (IOSR-JMCE)
e-ISSN: 2278-1684,p-ISSN: 2320-334X, Volume 13, Issue 5 Ver. VIII (Sep. - Oct. 2016), PP 65-78
www.iosrjournals.org

Analysis of A Propped Cantilever Under Longitudinal Vibration By A Modification of the System's Stiffness Distribution

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Abstract : *The use of Lagrange equations allow dynamical systems to be modeled as an assemblage of discrete masses connected by mass-less elements. The solution obtained is exact for such systems, but when a continuous system is modeled as having discrete masses connected by mass-less elements the results become approximate. The lumping of masses of a continuous system introduces an error in the system's mass distribution. This error can be redressed by redistributing the masses but an alternative solution will be making a corresponding modification in the systems' stiffness matrix. This was achieved by formulating the force equilibrium equations of discrete elements of a propped cantilever beam such systems under free longitudinal vibration using the Hamilton's principle and the principle of virtual work and the inherent forces causing vibration obtained. This was then equated to the corresponding equation of motion of the system and the stiffness matrix of the system necessary for such equality obtained. This was used to generate a table of stiffness modification factors for segments of the propped cantilever beam under longitudinal vibration. By applying the Lagrange equations to the lumped massed beam using these modification factors, we were able to obtain the accurate fundamental frequency of the beam irrespective of the position or number of lumped mass introduced.*

I. Introduction

Every system that has mass and elasticity is capable of vibration. Since no system is mass-less we can conveniently say that all structures experience vibration to some degree (Thomson 1996, Rajasekaran 2009). When vibration is not due to an external force on the system, the system is said to be under free vibration and vibrates at one or more of its natural frequencies. These natural frequencies of a system depend on the distribution of its mass and stiffness and hence are a property of the dynamical system (Ezeokpube 2002, Blake 2010). The number of independent coordinates required to describe the motion of a system is known as the degrees of freedom of the system. A continuous structure will have an infinite number of degrees of freedom and hence an infinite number of coordinates to analyze. However certain idealizations are made and a continuous system may be treated as one having a finite number of degrees of freedom (Blake 2010). For systems with few degrees of freedom, it is possible to formulate the equations of motion by an application of the Newton's laws of motion (Benaroya and Nagurka 2010; Chandrasekaran 2015). The method however becomes complicated for systems with a high degree of freedom and the energy methods provide a convenient alternative. One of the notable products of the energy method is the Lagrange's equations. The Lagrange's equation enables the analysis of structural elements as discrete masses connected together by mass-less elements (Ahmad and Campbell 2013, Lisjak and Grasselli 2014). With a proper selection of representative masses the results can be very close to the exact response. When the discretization is increased by the use of more number of lump masses, the accuracy of the response improves.

In order to obtain the exact response of structural systems it is necessary to analyse them as elements with continuously distributed masses. A continuous structure has infinite degrees of freedom and normal modes but generally the first few modes are of most importance.

The advent of fast digital computers has made the analysis of large simultaneous equations easy (Saad and Henk 2000). This can be put to use in the Finite Element Method. Just like in the Rayleigh-Ritz method, there is need to select a shape function. The accuracy of finite element method can be improved upon by the careful selection of better shape functions (p-version) and also by the introduction of more joints/nodes and hence more elements (h-version) (Houmat 2009; Beaurepaire and Schueller 2011; Tornabene et al 2015). The latter has the implication of increasing the size of the resulting equations and hence the computational cost.

II. Mathematical theory

Lagrange formulated a scalar equation in terms of generalized coordinates and is presented as

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$i = 1,2, \dots n$

Where q_1, q_2, …, q_n are a set of independent generalized displacements, T is the kinetic energy of the structure and U is the strain energy of the structure and Q_i is the non-conservative or the non-potential force on the system. The Lagrange's equations can be used to develop the matrix equation for the analysis of a free undamped n-degree of freedom discrete mass structure.

$$[m]\{\ddot{q}\} + [k]\{q\} = 0 \quad (2)$$

By pre-multiplying the equation with the structure's flexibility matrix [f]

$$[D]\{\ddot{q}\} + \{q\} = 0 \quad (3)$$

Where the dynamical matrix $[\boldsymbol{D}] = [\boldsymbol{f}][\boldsymbol{m}]$. (4)

A solution of equation (3) is given as

$$\{q\} = \{\phi\}\sin(wt + \delta) \quad (5)$$

By substituting equation (5) into equation (3) and rearranging we obtain

$$([D] - \lambda[I])\{\phi\} = 0 \quad (6)$$

Where [I] is an identity matrix and $\lambda = 1/_{w^2}$. (7)

Equation (6) represents a system of n-homogenous, linear algebraic equation in the amplitudes $\{\phi\}$ and can be solved to get the frequencies w_1, w_2, …, w_n for an n-degree of freedom system. For each distinct frequency w_j (or eigenvalue), there will be a set of amplitudes $\{\phi\}_j$ (or eigenvector).

The eigenvectors or relative amplitudes $\{\phi\}_j$ obtained from a free vibration satisfy certain orthogonality conditions (Tauchert 1974).

While Lagrange's equations provide a way of analyzing multi-degree of freedom system, a similar approach for continuous structures is an energy theorem known as the Hamilton's principle. The principle states that the motion of an elastic structure during the time interval $t_1 < t < t_2$ is such that the time integral of the total dynamic potential $U – T + V_E$ is an extremum.

$$\delta\int_{t_1}^{t_2}(U - T + V_E)dt = 0 \quad (8)$$

where U represents the strain energy of the system, T the kinetic energy and V_E the work done by the external forces. The partial differential equation and boundary conditions governing the free longitudinal vibration of a bar is derived as

$$c^2 u_1^{''} = \ddot{u}_1 \quad (9)$$

where $c^2 = \frac{EA}{\mu}$ (10)

$$N_o = [EAu_1^{'}]_{x_1=0} \ or \ \delta u_1(0,t) = 0 \quad (11a)$$

$$N_L = [EAu_1^{'}]_{x_1=0} \ or \ \delta u_1(L,t) = 0 \quad (11b)$$

Where $A(x_1)$ is the cross sectional area of the bar, $\mu(x_1)$ is the mass per unit length of the bar and E is the modulus of elasticity of the material of the bar.

For a normal mode vibration (where each particle of the bar vibrates harmonically at a circular frequency w)

$$u_1(x_1,t) = \phi_1(x_1)\sin(wt + \delta) \quad (12)$$

which upon substitution into equation (10) will give

$$\phi^{''} + \frac{w^2}{c^2}\phi = 0 \quad (13)$$

The general solution of equation (13) is

$$\phi(x_1) = C_1\cos\frac{\omega x_1}{c} + C_2\sin\frac{\omega x_1}{c} \quad (14)$$

By introducing the boundary conditions equation (14) results in an eigenvalue problem, the solution of which yields the natural circular frequencies ω_j and mode shapes (eigenvectors) ϕ_j. The general solution by mode superposition is

$$u_1(x_1,t) = \sum_{j=1}^{\infty}\phi_j(x_1)\left(A_j\cos w_j t + B_j\sin w_j t\right) \quad (15)$$

(Thomson and Dahleh 1998)

Where the constants A_j and B_j can be determined form the initial conditions.

The eigenfunctions ϕ_j also satisfy certain orthogonality relationships.

III. Methodology

The two essential components that determine the vibration of structural systems are the structure's mass distribution and the structure's stiffness.

These properties are captured in the structure's inertia matrix and stiffness matrix respectively. The prominent role these elements play can easily be appreciated by taking a look at the equations of motion of a vibrating system or the structural dynamics' eigenvalue problem.

If the mode shape ϕ_j and circular frequency ω_j are kept constant, then any variation in mass distribution μ will have a corresponding change in the element rigidity EA.

Two equations were compared. One is the force equilibrium equation written as

$$\{F\} + [S]\{D\} = \{F^*\} \qquad (16)$$

(when the external force vector {F*} acts at the element's nodes)

Where {F} is the vector of fixed end forces generated when nodal displacements are restrained. [S] is the element stiffness matrix and {D} a vector of nodal displacements (Okonkwo 2012).

The second is the equation of motion of a vibrating system written simply as

$$[m]\{\ddot{x}\} + [k]\{x\} = \{P\} \qquad (17)$$

(when the external force vector {P} acts at the element's nodes)

Where [m] is the inertia matrix, [k] is the element stiffness matrix and {x} a vector of nodal displacements.

By comparing equation (16) with (17) we see some similarities. Even though equation (16) has been largely applied in statics, it can also be applied in dynamics if the equations for the vector of fixed end moments/forces {F} can be formulated. The real structure (continuous system) was analyzed using the hamilton's principle and the equations for the fixed end forces {F} and nodal displacements {D} formulated for any arbitrary segment of the longitudinally vibrating beam at time t = 0. This was then substituted into equation (16) to get the vector of nodal force { F*} that is causing the vibration.

[K] in equations (17) was taken as the stiffness matrix of the lump-massed beam. If a vibrating element of the real beam (beam with continuous mass) and that of a corresponding element of a lump-massed beam are to be equivalent then their deformation must be equal and the force acting on their nodes {P} will also be equal. Therefore

$$\{D\} = \{x\} \qquad (18)$$

$$[m]\{\ddot{x}\} + [k]\{x\} = \{F^*\} \qquad (19)$$

For (propped cantilever) a prismatic bar fixed at one end but free at the other, by considering its boundary conditions, we obtain from equation (14) that

$$C_1 = 0 \qquad (20)$$

And for a non-trivial solution $C_2 \neq 0$

$$w_j = \frac{i\pi c}{2L} = \frac{i\pi}{2}\sqrt{\frac{EA}{\mu L^2}} \qquad (21)$$

$i = 1,3,5,7,9, \dots, \infty \qquad j = 1,2,3,4,5, \dots, \infty$

By taking C_2 to be equal to unity, the mode shape for the j[th] mode of vibration is obtained as

$$\emptyset_j = \sin\frac{i\pi x}{2L} \qquad (22)$$

The second derivative of equation (15) with respect to time is

$$\ddot{u}(x,t) = \sum_{j=1}^{\infty} -\omega_j^2 \emptyset_j \left(A_j \cos\omega_j t + B_j \sin\omega_j t\right) \qquad (23)$$

By substituting equation (23) into (15) at time t = 0 will give

$$\ddot{u}(x,0) = \sum_{j=1}^{\infty} -\omega_j^2 A_j \sin\gamma_2 x \qquad (24)$$

where $\gamma_2 = \frac{j\pi}{2L} = \frac{\omega_j}{c} \qquad (25)$

By treating the longitudinally vibrating bar like a beam segment pinned at one end and free at the other (see Figure 1), it is possible to obtain the fixed end forces (axial) of an arbitrary segment of the bar. The forces at the ends of an isolated segment are F_1 and F_2.

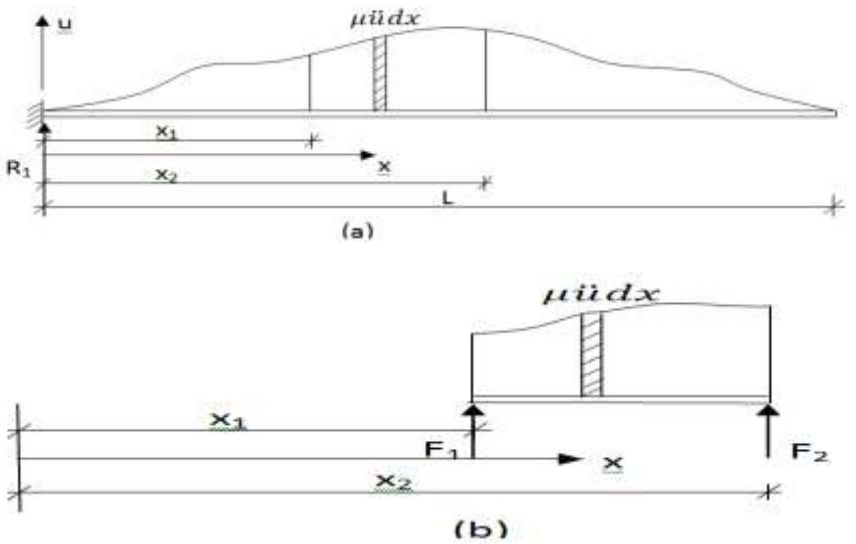

Figure 1

(a) A clamped-free bar under longitudinal vibration due to inertial forces $\mu\ddot{u}$

(b) A segment of the clamped-free bar under longitudinal vibration due to inertial forces $\mu\ddot{u}$

There are two possible cases for any arbitrary segment of the vibrating bar

a) Case 1

When $0 \le x_1 < L$ and $0 < x_2 < L$

Using the equations of external equilibrium

$\sum M_2 = 0; \quad F_1(x_2 - x_1) + \int_{x_1}^{x_2} \mu\ddot{u}(x_2 - x)dx = 0$

$$F_1 = \frac{1}{(x_2-x_1)}\sum_{j=1}^{\infty}\frac{\omega^2 A_j \mu}{\gamma_2^2}(\gamma_2 x_2 \cos\gamma_2 x_1 - \gamma_2 x_1 \cos\gamma_2 x_1 - \sin\gamma_2 x_2 + \sin\gamma_2 x_1) \quad . \quad . \quad (26)$$

$\sum F_y = 0; \quad F_1 + F_2 + \int_{x_1}^{x_2} \mu\ddot{u}dx = 0$

$$F_2 = \sum_{j=1}^{\infty}\frac{\omega_j^2 A_j \mu}{\gamma_2^2}\left[\frac{-\gamma_2 L\cos\gamma_2 x_2 + \gamma_2 L\cos\gamma_2 x_1}{L} - \frac{\gamma_2 x_2\cos\gamma_2 x_1 - \gamma_2 x_1\cos\gamma_2 x_1 - \sin\gamma_2 x_2 + \sin\gamma_2 x_1}{x_2 - x_1}\right] \quad . \quad (27)$$

The fixed end forces F_1 and F_2, can be expressed in terms of the axial rigidity EA rather than the circular frequency w_j

There is also need to normalize our distances so that the length of the bar L becomes equal to unity and the distances x_1 and x_2 expressed in dimensionless units.

By substituting equations (10) and (25) into equations (26) and (27) and normalizing the distances x_1 and x_2 we obtain

$$F_1 = \frac{EA}{L(\xi_2-\xi_1)}\sum_{j=1}^{\infty} A_j\left[\frac{i\pi\xi_2}{2}\cos\frac{i\pi\xi_1}{2} - \frac{i\pi\xi_1}{2}\cos\frac{i\pi\xi_1}{2} - \sin\frac{i\pi\xi_2}{2} + \sin\frac{i\pi\xi_1}{2}\right] \quad . \quad . \quad . \quad (28)$$

$$F_2 = \frac{EA}{L}\sum_{j=1}^{\infty} A_j\left[-\frac{i\pi}{2}\cos\frac{i\pi\xi_2}{2} + \frac{i\pi}{2}\cos\frac{i\pi\xi_1}{2} - \frac{\frac{i\pi\xi_2}{2}\cos\frac{i\pi\xi_1}{2} - \frac{i\pi\xi_1}{2}\cos\frac{i\pi\xi_1}{2} - \sin\frac{i\pi\xi_2}{2} + \sin\frac{i\pi\xi_1}{2}}{\xi_2-\xi_1}\right] \quad . \quad . \quad . \quad (29)$$

$j = 1,2,3,4,5, \ldots, \infty$

$i = 1,3,5,7,9, \ldots, \infty$

b) Case II

When $0 \le x_1 < L$ and $x_2 = L$

Since the far end of the bar is free (with respect to longitudinal vibration), $F_2 = 0$

For vertical force equilibrium

$$F_1 + \int_{x_1}^{L} \mu\ddot{u}dx = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

$F_1 = -\int_{x_1}^{L} \mu\ddot{u}dx$

$= -\sum_{j=1}^{\infty} -\omega_j^2 A_j \mu \int_{x1}^{L} \sin\gamma_2 x \, dx$

$= \sum_{j=1}^{\infty} \frac{\omega_j^2 A_j \mu}{\gamma_2} (-\cos\gamma_2 L + \cos\gamma_2 x)$ . . . . (31)

By expressing equation (31) in terms of EA

$F_1 = \frac{EA}{2L} \sum_{j=1}^{\infty} A_j i\pi \left(-\cos\frac{i\pi}{2} + \cos\frac{i\pi\xi_1}{2}\right)$ . . . . (32)

Where $j = 1,2,3,4,5, \ldots, \infty$

$i = 1,3,5,7,9, \ldots, \infty$

Recall that the constant A_j depends on the initial conditions of the vibrating bar.
The axial force due to self weight at any point x along the length of the bar is given by

$P_x = \mu g(L - x)$ . . . . . . (33)

μ is the mass per unit length of the bar and g is the acceleration due to gravity.
If the axial deformation on the infinitesimal element dx is du, then from Hooke's law

$P_x = EA\frac{du}{dx}$ . . . . (34)

By equating equation (30) to equation (31) and integrating

$u(x,0) = \frac{f}{L}\left(Lx - \frac{x^2}{2}\right)$ . . . . (35)

Where f is a dimensionless constant equal to $\frac{\mu g L}{EA}$

$A_j = \frac{\mu}{M_j}\int_0^L \frac{f}{L}\left(Lx - \frac{x^2}{2}\right)\sin\gamma_2 x \, dx = \frac{\mu f L^2}{M_j}\left(\frac{-\gamma_2^2 L^2 \cos\gamma_2 L - 2\cos\gamma_2 L + 2}{2\gamma_2^3 L^3}\right)$ . . . . (36)

The generalized mass can be expressed as

$M_j = \mu\int_0^L \emptyset_j^2 \, dx = \frac{\mu L}{2}$ . . . . (37)

Equation (36) above is an expression for the constant A_j for a bar under an initial displacement caused by its self weight. Equation (36) can be substituted into the equation (28) and (29) to obtain the values of the fixed end forces F_1 and F_2. With these equations the force equilibrium equations for segments of a vibrating beam can be written and the inherent forces in the system that is causing motion calculated at the nodes/junctions of the element. An arbitrary segment of a vibrating element is identified by means of the normalized distances ξ_1 and ξ_2 of its nodes from an origin. ξ_1 and ξ_2 are numbers between 0 and 1.

The force equilibrium equations for a segment of a longitudinally vibrating bar can be written as

$\{F\} + [k]\{u\} = \{P\}$ . . . . (38)

Where {F} is the fixed end forces, [k] is the stiffness of the segment under consideration and {u} is a vector of nodal displacements.

$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$ . . . . (39)

$[k] = \begin{bmatrix} \frac{EA}{l} & -\frac{EA}{l} \\ -\frac{EA}{l} & \frac{EA}{l} \end{bmatrix}$ . . . . . . (40)

$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$ . . . . (41)

U_1 is the total displacement at the position x_1 while u_2 is the total displacement at the position x_2. The total displacement is obtained by totaling the displacements due to all the modes of vibration.
{P} is the vector of nodal forces; they represent the forces acting on the nodes of the isolated segment.

$\{P\} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$ . . . . (42)

From equation (38) P_1 and P_2 can be expressed as

$P_1 = F_1 + \frac{EA}{\xi_2 - \xi_1}(u_1 - u_2)$ . . . . . (43)

$P_2 = F_2 + \frac{EA}{\xi_2-\xi_1}(-u_1 + u_2)$ . . . . . (44)

Just like we do in the decomposition of structures, a segment of a vibrating bar can be isolated and will be in equilibrium with the application of the force vector {P}. The force {P} represents the effect of the removed adjourning elements on the isolated segment.

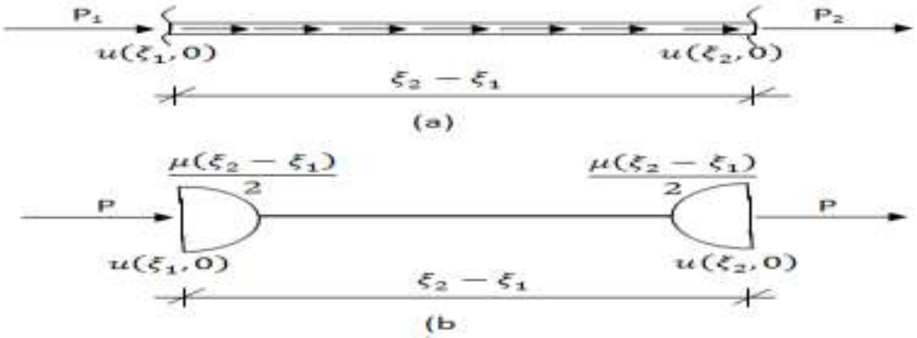

Figure 2

(a) An isolated segment of the longitudinally vibrating continuous bar showing the nodal forces P_1 and P_2

(b) An equivalent lumped massed segment showing the nodal forces

Figure 2a shows a segment of the vibrating continuous or real bar. The nodal forces on the bar P_1 and P_2 are calculated from the equilibrium equations (equation 43 and 44). When the continuous bar is represented by a lumped mass bar (a bar that has its distributed masses lumped at selected nodes), the equivalent segment of the bar is shown in Figure 2b. Just like the real segment the equivalent segment is supported by the same nodal forces P_1 and P_2 and has the same nodal displacements as the real bar. This implies that for the lumped massed beam to be equivalent to the real beam they must share the same inherent forces and displacements at the nodes. The equation of motion for the lumped massed bar is given as

$[m]\{\ddot{u}\} + [k_d]\{u\} = \{P\}$ . . . . . . (45)

Where [m] is the inertial matrix, {u} is a vector of nodal displacement and k_d is the stiffness of the lumped massed segment under consideration.

The proposed stiffness matrix for the lumped massed segment k_d is

$$[k_d] = \begin{bmatrix} \frac{EA}{l}\alpha_1 & -\frac{EA}{l}\alpha_2 \\ -\frac{EA}{l}\alpha_2 & \frac{EA}{l}\alpha_1 \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

where α_1 and α_2 are the stiffness modification factors for longitudinal vibration. They help redistribute the stiffness of the lumped massed bar in such a way as to annul the effect of the discretization of the bar due to the lumping of its distributed mass on selected nodes.

$$[m] = \begin{bmatrix} \frac{\mu(\xi_2-\xi_1)}{2} & 0 \\ 0 & \frac{\mu(\xi_2-\xi_1)}{2} \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad (47)$$

μ is the mass per unit length of the beam.

When treating the isolated segment of the vibrating beam alone the vector of nodal acceleration is written as

$$\{\ddot{u}\} = \begin{Bmatrix} \ddot{u}(\xi_1, 0) \\ \ddot{u}(\xi_2, 0) \end{Bmatrix} = \begin{Bmatrix} -\omega^2 u(\xi_1, 0) \\ -\omega^2 u(\xi_2, 0) \end{Bmatrix} = \begin{Bmatrix} -\omega^2 u_{11} \\ -\omega^2 u_{21} \end{Bmatrix} \quad . \quad . \quad . \quad . \quad . \quad (48)$$

ω is the fundamental frequency of the vibrating mass while u_{11} and u_{21} are the values of u_1 and u_2 for the first mode only.

$$\alpha_1 = \frac{-(\xi_2-\xi_1)u_{11}\left(P_1+\frac{(\xi_2-\xi_1)\pi^2 u_{11}}{8}\right)+(\xi_2-\xi_1)u_{21}\left(P_2+\frac{(\xi_2-\xi_1)\pi^2 u_{21}}{8}\right)}{u_{21}^2-u_{11}^2} \quad . \quad . \quad . \quad (49)$$

$$\alpha_2 = \frac{(\xi_2-\xi_1)u_{11}\left(P_2+\frac{(\xi_2-\xi_1)\pi^2 u_{21}}{8}\right)-(\xi_2-\xi_1)u_{21}\left(P_1+\frac{(\xi_2-\xi_1)\pi^2 u_{11}}{8}\right)}{u_{21}^2-u_{11}^2} \quad . \quad . \quad . \quad (50)$$

Equations (47) and (48) can be used to evaluate the stiffness modification factor for longitudinal vibration of a segment of a fixed-free or pinned-free bar located between ξ_1 and ξ_2 of the bar's total length. A numerical demonstration of their use is presented below. For ease of presentation the calculations will be presented in a tabular form.

Example 1: when $\xi_1 = 0, \xi_2 = 1.0$

Table 1: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0, \xi_2 = 1.0$ on a fixed-free bar under longitudinal vibration

$\xi_1 = 0, \ \xi_2 = 1.0$					
j	Aj	F_{1j}	F_{2j}	u_{1j}	u_{2j}
1	0.51602455093119	0.29454491820751	0.51602455093119	0	0.51602455093119
2	0.01911202040486	0.10917529475360	-0.0191120204049	0	-0.0191120204049
3	0.00412819640745	0.02829458235810	0.00412819640745	0	0.00412819640745
4	0.00150444475490	0.01804667881896	-0.0015044447549	0	-0.0015044447549
5	0.00070785260759	0.00929917787561	0.00070785260759	0	0.00070785260759
6	0.00038769688274	0.00708661811529	-0.0003876968827	0	-0.0003876968827
7	0.00023487690074	0.00456139214742	0.00023487690074	0	0.00023487690074
8	0.00015289616324	0.00375542713719	-0.0001528961632	0	-0.0001528961632
9	0.00010503247526	0.00269970617228	0.00010503247526	0	0.00010503247526
Total		**0.47746379558596**	**0.50004345111648**	**0**	**0.50004345111648**
$u_{11} = 0$ $u_{21} = 0.51602455093119$ From equations 4.6a and 4.6b $P_1 = -0.97750724670244$ $P_2 = 0$ From equations 4.27 and 4.28 the stiffness modification factors for longitudinal vibration of the element are $\alpha_1 = 1.23370055013617$ $\alpha_2 = 1.89430375926587$					

J is the mode number, j = 1 stands for the first mode, j = 2 for the second mode and so on. The values of the paramaters A_j, F_{1j}, F_{2j}, u_{1j} and u_{2j} are evaluated for modes 1 – 9 and summed to obtain end forces F_1 and F_2 and the end displacements u_1 and u_2.

Table 1 is an illustration on how the inherent nodal forces P_1 and P_2 and the stiffness modification factors α_1 and α_2 are calculated. The nodal forces P_1 and P_2 are the forces acting at the selected nodal point if the beam segment under consideration is decomposed. These nodal forces represent the effect of the removed adjacent beam segment on the beam segment under consideration. Using the methods presented in table 1 the values of stiffness modification factors at different values of ξ_1 and ξ_2 for the longitudinal vibration of a fixed-free bar are presented in Table A1 below. A sample matlab program for the calculation of the stiffness modification factors for a segment of a beam restrained at both end can be found the Appendix B.

Numerical Application and Discussion of Results

For the beam of Figure 3a the stiffness matrix and inertia matrix of the bar with respect to the coordinate of the lumped mass are

$$k = \frac{EA}{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

$$m = \frac{1}{2}\mu L \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

By substituting equations (51) and (52) into equation (6) and solving we obtain

$$\lambda = \frac{0.5\mu L^2}{EA} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53a)$$

$$\{\phi\} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53b)$$

$$\omega = 1.4142\sqrt{\frac{EA}{\mu L^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53c)$$

From table A1 the stiffness modification factors of the element of the bar are

$\xi_1 = 0, \xi_2 = 1, \alpha_1 = 1.233701, \alpha_2 = 1.894303$

By applying these stiffness modification factors, the modified stiffness matrix of the bar with respect to the coordinate of the lumped mass becomes

$$k = \frac{1.233701\, EA}{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

By using this modified stiffness on equation (6) the new values of λ, natural frequency and mode shape obtained are

$$\lambda = \frac{0.405284586784\ \mu L^2}{EA} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54a)$$

$$\{\phi\} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54b)$$

$$\omega = 1.5708\sqrt{\frac{EA}{\mu L^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54c)$$

For the beam of Figure 3b the stiffness matrix and inertia matrix of the bar with respect to the coordinate of the lumped mass are

$$k = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}\frac{EA}{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55a)$$

$$m = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}\mu L \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55b)$$

By substituting equations (55a) and (55b) into equation (6) and solving we obtain

$$\lambda_1 = \frac{0.42677669529664\ \mu L^2}{EA}, \lambda_2 = \frac{0.07322330470336\ \mu L^2}{EA} \quad . \quad . \quad . \quad . \quad . \quad . \quad (56a)$$

$$\{\phi_1\} = \begin{Bmatrix} 0.5773503 \\ 0.8164966 \end{Bmatrix}, \{\phi_2\} = \begin{Bmatrix} -0.5773503 \\ 0.8164966 \end{Bmatrix} \quad . \quad . \quad . \quad . \quad . \quad (56b)$$

$$\omega_1 = 1.5307\sqrt{\frac{EA}{\mu L^2}}\ ,\ \omega_2 = 3.6955\sqrt{\frac{EA}{\mu L^2}} \quad . \quad . \quad . \quad . \quad . \quad (56c)$$

The stiffness modification factors of the two segments/elements of the bar can be obtained from table A1 as
For element 1: $\xi_1 = 0$, $\xi_2 = 0.5$, $\alpha_1 = 0.995936$, $\alpha_2 = 1.339475$
For element2: $\xi_1 = 0.5$, $\xi_2 = 1$, $\alpha_1 = 0.995936$, $\alpha_2 = 0.972287$

By applying these stiffness modification factors, the modified stiffness matrix of the bar with respect to the coordinate of the lumped mass becomes

$$k = \begin{bmatrix} 3.983744 & -1.944574 \\ -1.944574 & 1.991872 \end{bmatrix}\frac{EA}{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

By using this modified stiffness on equation (6) the new values of λ, natural frequency and mode shape obtained are

$$\lambda_1 = \frac{0.40528456176396\ \mu L^2}{EA}, \lambda_2 = \frac{0.07425242373351\ \mu L^2}{EA} \quad . \quad . \quad . \quad . \quad . \quad (58a)$$

$$\{\phi_1\} = \begin{Bmatrix} 0.5773503 \\ 0.8164966 \end{Bmatrix}, \{\phi_2\} = \begin{Bmatrix} -0.5773503 \\ 0.8164966 \end{Bmatrix}. \quad . \quad . \quad . \quad . \quad . \quad (58b)$$

$$\omega_1 = 1.5708\sqrt{\frac{EA}{\mu L^2}}\ ,\ \omega_2 = 3.6698\sqrt{\frac{EA}{\mu L^2}} \quad . \quad . \quad . \quad . \quad . \quad (58c)$$

These were repeated for the bars of Figures 3(c), 3(d), 3(e) and 3(f) and a summary of the obtained natural frequencies presented in table 2 below

Table 2: Comparism of the obtained natural frequencies of different lump-massed fixed-free bar under longitudinal vibration with the exact results.

	Mode No	Hamilton (Exact)	Lagrange	Percentage Error (%)	Lagrange with modified stiffness	Percentage Error (%)
Figure 3 (a)	1	1.5708	1.4142	9.95	1.5708	0
Figure 3 (b)	1	1.5708	1.5307	2.55	1.5708	0
	2	4.7124	3.6955	21.43	3.6698	22.12
Figure 3 (c)	1	1.5708	1.4749	6.11	1.5708	0
	2	4.7124	4.9824	-5.73	4.6750	0.79
Figure 3 (d)	1	1.5708	1.5529	1.14	1.5708	0
	2	4.7124	4.2426	9.97	4.1054	12.88
	3	7.8540	5.7956	26.21	5.5171	29.75
Figure 3 (e)	1	1.5708	1.5755	-0.30	1.5708	0
	2	4.7124	3.7558	20.30	3.7100	21.27
	3	7.8540	6.1708	21.43	6.2305	20.67

From table 2, it would be observed that the natural frequencies obtained from the use of Lagrange equation on the continuous system had some measure of errors as seen in the percentage error column. But when the stiffness of the system was modified using the stiffness modification factors, the use of Lagrange equation was able to predict accurately the fundamental frequencies hence their percentage errors were zero. However the values of the higher frequencies remained approximate. Some of its predictions for higher frequencies were less accurate than that obtained without the application of the stiffness modification factors.

From this work we can infer that

1) In order to obtain an accurate dynamic response from a lumped massed beam under longitudinal vibration there must of necessity be a modification in the stiffness composition of the system (the finite element method actually does the opposite).

2) No linear modification of the stiffness distribution of lumped mass fixed-free beam under longitudinal vibration can cause them to be dynamically equivalent to the continuous beams. This is so because the values of α_1 and α_2 obtained for each segment as shown in Table A1 are not equal.

3) By modifying the stiffness distribution of a lumped mass propped cantilever it can be made to produce the same response at an equivalent continuous beam.

This work was limited to the longitudinal vibration of a propped cantilever. It can however be extended to any other beam of a different end constraint (boundary condition). It is possible to present the values from table A1 in the form of graphs. It was however presented as tables in order to allow researchers to pick the exact stiffness modification factors unlike the approximate values that would be obtained from graphs.

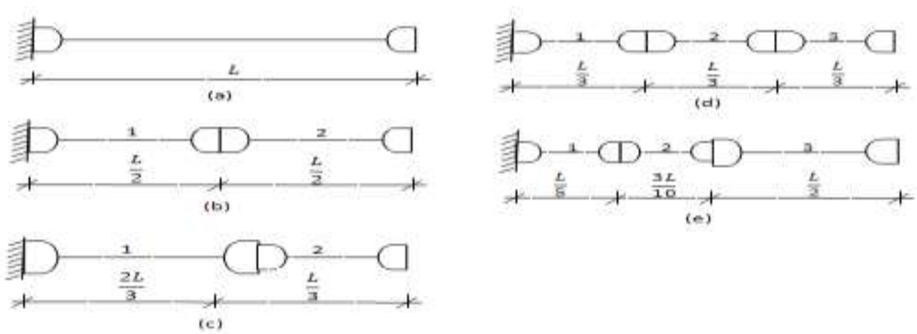

Figure 3: Some lumped massed beams fixed at one end and free at the other used for illustration of Lagrange equation

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APPENDIX A

Table A1: Stiffness modification factors for the longitudinal vibration of a fixed-free/pinned-free bar

			ξ_2						
			0	0.05	0.10	0.15	0.20	0.25	0.30
ξ_1	0	α_1	-	1.183968	1.132904	1.082305	1.049036	1.028611	1.010495
		α_2	-	1.207192	1.210925	1.217182	1.226019	1.237514	1.251769
	0.05	α_1	1.183968	-	1.148560	1.105646	1.073301	1.051217	1.031511
		α_2	1.207192	-	1.166776	1.161223	1.164937	1.173842	1.185349
	0.10	α_1	1.132904	1.148560	-	1.097758	1.070537	1.050429	1.031727
		α_2	1.210925	1.166776	-	1.109006	1.102963	1.105153	1.110747
	0.15	α_1	1.082305	1.105646	1.097758	-	1.052532	1.035185	1.018205
		α_2	1.217182	1.161223	1.109006	-	1.058351	1.052824	1.051551
	0.20	α_1	1.049036	1.073301	1.070537	1.052532	-	1.018377	1.002485
		α_2	1.226019	1.164937	1.102963	1.058351	-	1.021661	1.014191
	0.25	α_1	1.028611	1.051217	1.050429	1.035185	1.018377	-	0.988096
		α_2	1.237514	1.173842	1.105153	1.052824	1.021661	-	0.990709
	0.30	α_1	1.010495	1.031511	1.031727	1.018205	1.002485	0.988096	-
		α_2	1.251769	1.185349	1.110747	1.051551	1.014191	0.990709	-
	0.35	α_1	0.994012	1.013619	1.014289	1.001798	0.986647	0.972243	0.956345
		α_2	1.268915	1.199472	1.119325	1.053751	1.01535	0.981802	0.958528
	0.40	α_1	0.985971	1.003608	1.003820	0.991424	0.976089	0.961088	0.944345
		α_2	1.289113	1.217183	1.132300	1.061173	1.012784	0.979345	0.951433
	0.45	α_1	0.988424	1.003569	1.002547	0.989434	0.973234	0.957085	0.939016
		α_2	1.312556	1.238683	1.149784	1.073861	1.020974	0.983413	0.951386
	0.50	α_1	0.995936	1.008776	1.006373	0.992253	0.974911	0.957376	0.937743
		α_2	1.339475	1.263467	1.170592	1.090040	1.032848	0.991349	0.955378
	0.55	α_1	1.004505	1.015592	1.012025	0.996925	0.978416	0.959454	0.938178
		α_2	1.370146	1.291408	1.194182	1.108782	1.047147	1.001606	0.961602
	0.60	α_1	1.016998	1.026639	1.021980	1.005838	0.986075	0.965597	0.942562
		α_2	1.404893	1.323124	1.221360	1.131033	1.064944	1.015379	0.971368
	0.65	α_1	1.037287	1.045494	1.039679	1.022343	1.001172	0.979044	0.954112
		α_2	1.444100	1.359301	1.253032	1.157883	1.087505	1.034112	0.986292
	0.70	α_1	1.062890	1.070165	1.063363	1.044883	1.022319	0.998546	0.971697
		α_2	1.488219	1.400138	1.289150	1.189058	1.114365	1.057173	1.005593
	0.75	α_1	1.088804	1.095812	1.088531	1.069239	1.045528	1.020300	0.991651
		α_2	1.537785	1.445710	1.329355	1.223801	1.144407	1.083120	1.027518
	0.80	α_1	1.114402	1.121952	1.114768	1.095064	1.070505	1.044057	1.013765
		α_2	1.593431	1.496594	1.374144	1.262525	1.177976	1.112240	1.052300
	0.85	α_1	1.143503	1.152264	1.145657	1.125868	1.100720	1.073272	1.041498
		α_2	1.655911	1.553805	1.424742	1.306646	1.216679	1.146335	1.081934
	0.90	α_1	1.176530	1.187281	1.181830	1.162373	1.136980	1.108834	1.075822
		α_2	1.726125	1.618209	1.481971	1.356951	1.261292	1.186192	1.117231
	0.95	α_1	1.208020	1.221961	1.218586	1.200152	1.175071	1.146697	1.112807
		α_2	1.805153	1.690400	1.545967	1.413160	1.311167	1.230829	1.156885
	1.00	α_1	1.233701	1.252387	1.252289	1.235779	1.211728	1.183714	1.149381
		α_2	1.894304	1.771274	1.617235	1.475420	1.366137	1.279797	1.200175
			ξ_2						
			0.35	0.40	0.45	0.50	0.55	0.60	0.65
ξ_1	0	α_1	0.994012	0.985971	0.988424	0.995936	1.004505	1.016998	1.037209
		α_2	1.268915	1.289113	1.312556	1.339475	1.370146	1.404893	1.444100

	0.05	α_1	1.013619	1.003608	1.003569	1.008776	1..015592	1.026639	1.045494
		α_2	1.199472	1.217183	1.238683	1.263467	1.291408	1.323124	1.359301
	0.10	α_1	1.014289	1.003820	1.002547	1.006373	1.012025	1.021980	1.039680
		α_2	1.119325	1.132299	1.149784	1.170592	1.194182	1.221360	1.253032
	0.15	α_1	1.001798	0.991424	0.989434	0.992253	0.996924	1.005838	1.022344
		α_2	1.053752	1.061173	1.073861	1.090040	1.108782	1.131033	1.157883
	0.20	α_1	0.986647	0.976089	0.973234	0.974911	0.978416	0.986075	1.001172
		α_2	1.010535	1.012784	1.020974	1.032848	1.047147	1.064944	1.087505
	0.25	α_1	0.972243	0.961088	0.957085	0.957376	0.959454	0.965597	0.979044
		α_2	0.981802	0.979345	0.983413	0.991349	1.001606	1.015379	1.034112
	0.30	α_1	0.956345	0.944545	0.939016	0.937743	0.938178	0.942562	0.954113
		α_2	0.958528	0.951433	0.951386	0.955378	0.961602	0.971368	0.986292
	0.35	α_1	-	0.926411	0.919559	0.916524	0.915080	0.917445	0.926832
		α_2	-	0.927808	0.923634	0.923657	0.925826	0.931564	0.942654
	0.40	α_1	0.926411	-	0.903190	0.898132	0.894541	0.894626	0.901612
		α_2	0.927808	-	0.903857	0.900198	0.898599	0.900599	0.908154
	0.45	α_1	0.919559	0.903190	-	0.885113	0.879145	0.876737	0.881146
		α_2	0.923634	0.903857	-	0.885481	0.880507	0.879168	0.883610
	0.50	α_1	0.916524	0.898132	0.885113	-	0.866839	0.861753	0.863414
		α_2	0.923657	0.900198	0.885481	-	0.867144	0.862598	0.864070
	0.55	α_1	0.915080	0.894541	0.879145	0.866839	-	0.847616	0.846324
		α_2	0.925826	0.898599	0.880507	0.867144	-	0.847731	0.846137
	0.60	α_1	0.917445	0.894626	0.876737	0.861753	0.847616	-	0.832200
		α_2	0.931564	0.900599	0.879168	0.862598	0.847731	-	0.831992
	0.65	α_1	0.926832	0.901612	0.881146	0.863414	0.846324	0.832200	-
		α_2	0.942653	0.908154	0.883610	0.864070	0.846137	0.831992	-
	0.70	α_1	0.942205	0.914540	0.891480	0.870988	0.850911	0.833642	0.822958
		α_2	0.958177	0.920244	0.892680	0.870302	0.849437	0.832421	0.822552
	0.75	α_1	0.959998	0.929909	0.904267	0.880998	0.857861	0.837329	0.823390
		α_2	0.976081	0.934463	0.903746	0.878371	0.854400	0.834331	0.821773
	0.80	α_1	0.980031	0.947556	0.919357	0.893300	0.867020	0.843089	0.825747
		α_2	0.996541	0.950995	0.916887	0.888309	0.861001	0.837633	0.822161
	0.85	α_1	1.005784	0.971013	0.940362	0.911614	0.882246	0.854950	0.834285
		α_2	1.021751	0.972221	0.934731	0.902995	0.872386	0.845765	0.827498
	0.90	α_1	1.038313	1.001420	0.968523	0.937299	0.905026	0.874551	0.850842
		α_2	1.052548	0.999025	0.958237	0.923478	0.889710	0.860016	0.839257
	0.95	α_1	1.073758	1.034947	1.000009	0.966482	0.931388	0.897758	0.871082
		α_2	1.087311	1.029475	0.985165	0.947202	0.910070	0.877090	0.853728
	1.00	α_1	1.109068	1.068526	1.031696	0.995936	0.957926	0.920893	0.890893
		α_2	1.125036	1.062282	1.013941	0.972287	0.931215	0.894297	0.867711

			ξ_2						
			0.70	0.75	0.80	0.85	0.90	0.95	1.00
ξ_1	0	α_1	1.062890	1.088804	1.114402	1.143503	1.176530	1.208020	1.233701
		α_2	1.488219	1.537785	1.593431	1.655911	1.726125	1.805153	1.894303
	0.05	α_1	1.070165	1.095812	1.121952	1.152264	1.187281	1.221961	1.252387
		α_2	1.400138	1.445710	1.496594	1.553805	0.618209	1.690400	1.771274

	0.10	α_1	1.063363	1.088530	1.114768	1.145658	1.181830	1.218586	1.252289
		α_2	1.289150	1.329355	1.374144	1.424742	1.481971	1.545967	1.617235
	0.15	α_1	1.044883	1.069239	1.095064	1.125868	1.162373	1.200152	1.235779
		α_2	1.189058	1.223801	1.262525	1.306646	1.356951	1.413160	1.475420
	0.20	α_1	1.022319	1.045528	1.070505	1.100720	1.136980	1.175071	1.211728
		α_2	1.114365	1.144407	1.177976	1.216679	1.261292	1.311167	1.366137
	0.25	α_1	0.998563	1.020300	1.044057	1.073272	1.108834	1.146697	1.183714
		α_2	1.057173	1.083120	1.112240	1.146335	1.186192	1.230829	1.279797
	0.30	α_1	0.971697	0.991651	1.013765	1.041498	1.075822	1.112807	1.149382
		α_2	1.005593	1.027518	1.052300	1.081934	1.117231	1.156885	1.200175
	0.35	α_1	0.942205	0.959998	0.980031	1.005784	1.038313	1.075758	1.109068
		α_2	0.958177	0.976081	0.996541	1.021751	1.052548	1.087311	1.125036
	0.40	α_1	0.914540	0.929909	0.947556	0.971013	1.001420	1.034947	1.068526
		α_2	0.920224	0.934463	0.950995	0.972221	0.999025	1.029475	1.062282
	0.45	α_1	0.891480	0.904267	0919357	0.940362	0.968523	1.000009	1.031696
		α_2	0.892679	0.903746	0.916887	0.934731	0.958237	0.985165	1.031941
	0.50	α_1	0.870988	0.880998	0.893300	0.911614	0.937299	0.966481	0.995936
		α_2	0.870302	0.878371	0.888309	0.902994	0.923477	0.947202	0.972287
	0.55	α_1	0.850911	0.857861	0.867020	0.882246	0.905026	0.931388	0.957926
		α_2	0.849437	0.854400	0.861001	0.872386	0.889710	0.910070	0.931215
	0.60	α_1	0.833642	0.837329	0.843089	0.854950	0.874551	0.897758	0.920893
		α_2	0.832421	0.834331	0.837633	0.845765	0.860016	0.877090	0.894297
	0.65	α_1	0.822958	0.823390	0.825747	0.834285	0.850842	0.871082	0.890973
		α_2	0.822552	0.821773	0.822161	0.827498	0.839257	0.853728	0.867711
	0.70	α_1	-	0.815363	0.814326	0.819601	0.833297	0.850832	0.867716
		α_2	-	0.814953	0.812699	0.815590	0.825356	0.837841	0.849245
	0.75	α_1	0.815363	-	0.804410	0.806085	0.816614	0.831061	0.844219
		α_2	0.814953	-	0.803998	0.804186	0.811760	0.822021	0.830317
	0.80	α_1	0.814326	0.804410	-	0.793129	0.799969	0.810603	0.818564
		α_2	0.812699	0.803998	-	0.792585	0.797518	0.804936	0.808802
	0.85	α_1	0.819601	0.806085	0.793129	-	0.790081	0.797298	0.799783
		α_2	0.815590	0.804186	0.792585	-	0.789393	0.794567	0.793962
	0.90	α_1	0.833297	0.816614	0.799969	0.790081	-	0.797687	0.797349
		α_2	0.825356	0.811760	0.797518	0.789394	-	0.797002	0.794797
	0.95	α_1	0.850832	0.831061	0.810603	0.797298	0.797687	-	0.806968
		α_2	0.837841	0.822021	0.804936	0.794567	0.797002	-	0.806370
	1.00	α_1	0.867716	0.844219	0.818564	0.799783	0.797349	0.806968	-
		α_2	0.849245	0.830317	0.808802	0.793962	0.794797	0.806370	-

APPENDIX B

```
%Find the stiffness mod. factor for a segment of a fixed free bar under free  %vibration

e1 = 0
e2 = 1

for j=1:1:9
i = j*2-1;
Aj = 16/(i*i*i*pi*pi*pi);

F1j= Aj*(i*pi*e2/2*cos(i*pi*e1/2) - i*pi*e1/2*cos(i*pi*e1/2) ...
```

```
- sin(i*pi*e2/2)+sin(i*pi*e1/2))/(e2-e1);
F2j= Aj*(-i*pi/2*cos(i*pi*e2/2) + i*pi/2*cos(i*pi*e1/2)- ...
(i*pi*e2/2*cos(i*pi*e1/2) - i*pi*e1/2*cos(i*pi*e1/2) -...
sin(i*pi*e2/2)+sin(i*pi*e1/2))/(e2-e1));

u1j = Aj*sin(i*pi*e1/2);
u2j = Aj*sin(i*pi*e2/2);
if j==1
u11 = u1j;
u22 = u2j;
end

col1(j,1)=Aj;
col2(j,1)=F1j;
col3(j,1)=F2j;
col4(j,1)=u1j;
col5(j,1)=u2j;

end
Col1 = sum(col1);
Col2 = sum(col2);
Col3 = sum(col3);
Col4 = sum(col4);
Col5 = sum(col5);

if e1==0 && e2<1

format long;
Table=[col1 col2 col3 col4 col5;Col1 Col2 Col3 Col4 Col5];
F1 =- Col2;
F2 = -Col3;
u1 = Col4;
u2 = Col5 ;
P1 = F1+(u1-u2)/(e2-e1);
P2 = F2+(-u1+u2)/(e2-e1);
Q1 = ((e2-e1)*u22*(P2+pi*pi*(e2-e1)*u22/8) - ...
(e2-e1)*u11*(P1+pi*pi*(e2-e1)*u11/8))/(u22*u22-u11*u11)
Q2 = ((e2-e1)*u11*(P2+pi*pi*(e2-e1)*u22/8) - ...
(e2-e1)*u22*(P1+pi*pi*(e2-e1)*u11/8))/(u22*u22-u11*u11)
format short;

elseif e1>0 && e2<1
format long;
Table=[col1 col2 col3 col4 col5 ;Col1 Col2 Col3 Col4 Col5 ];
F1 = -Col2;
F2 = -Col3;
u1 = Col4;
u2 = Col5;
P1 = F1+(u1-u2)/(e2-e1);
P2 = F2+(-u1+u2)/(e2-e1);
Q1 = ((e2-e1)*u22*(P2+pi*pi*(e2-e1)*u22/8) - ...
(e2-e1)*u11*(P1+pi*pi*(e2-e1)*u11/8))/(u22*u22-u11*u11)
Q2 = ((e2-e1)*u11*(P2+pi*pi*(e2-e1)*u22/8) - ...
(e2-e1)*u22*(P1+pi*pi*(e2-e1)*u11/8))/(u22*u22-u11*u11)
format short;


else
format long;
```

```
Table=[col1 col2 col3 col4 col5;Col1 Col2 Col3 Col4 Col5];
F1 = -Col2;
F2 = -Col3;
u1 = Col4;
u2 = Col5 ;
P1 = F1+(u1-u2)/(e2-e1);
P2 = F2+(-u1+u2)/(e2-e1);
Q1 = ((e2-e1)*u22*(P2+pi*pi*(e2-e1)*u22/8) - ...
(e2-e1)*u11*(P1+pi*pi*(e2-e1)*u11/8))/(u22*u22-u11*u11)
Q2 = ((e2-e1)*u11*(P2+pi*pi*(e2-e1)*u22/8) - ...
 (e2-e1)*u22*(P1+pi*pi*(e2-e1)*u11/8))/(u22*u22-u11*u11)
format short;
end
```