IOSR Journal of Mechanical and Civil Engineering (IOSR-JMCE)
e-ISSN: 2278-1684,p-ISSN: 2320-334X, Volume 12, Issue 6 Ver. II (Nov. - Dec. 2015), PP 08-14
www.iosrjournals.org

The Interaction Forces between Two non-Identical Cylinders Spinning around their Stationary and ParallelAxes in an IdealFluid

Abdullah Baz[1,] Salman M. Al-Kasimi[2]
Assistant Professor College of Computer and Information System
UQU, Saudi Arabia
Associate Professor The Research Institute UQU, Saudi Arabia

Abstract: *This paper derives the equations that describe the interaction forcesbetween two non-identical cylinders spinningaround their stationary and parallel axes in a fluid that isassumed to be in-viscous, steady, in-vortical, and in-compressible. The paper starts by deriving the velocity field from Laplace equation,governing this problem,and the system boundary conditions. It then determines the pressure field from the velocity field using Bernoulli equation. Finally, the paper integrates the pressure around both cylinder-surface to find the force acting on their axes.All equations and derivationsprovided in this paper are exact solutions. No numerical analysesor approximations are used.The paper finds that such cylinders repel or attract each other in inverse relation with separation between their axes, according to similar or opposite direction of rotation, respectively.*

Keywords: *Rotating non-identical cylinders, ideal fluid, Laplace equation,velocity field,Bernoulli equation, pressure, force, inverse law, repulsion, attraction.*

I. Introduction

Determining the force acting on an objectdue to its existence in a fluid is an important topic, and has several important applications. One of these applications is evaluating the lift force acting on an aeroplane wing due to the flow of the air. The solution to such a problem might be analytical or numerical, depending upon the complexity of the system and the required level of accuracy of the solution. Cylindrical Objects in fluid-flows constitute one category of such problems and have vast applications.

According to the literature reviewed, several such systems have already been studied both numerically and analytically,while other systems have attracted no attention. An example ofsuch studiedsystems[1]is the lift force acting on a cylinder rollingin aflow. Another example[2] is the interaction forces between two concentric cylinders with fluid internal and/or external to them. A third example[3]is the interaction forces between two cylinders rotating around two parallel floating axes.A fourth example [4]is the interaction forces among two identical cylinders rotating in an ideal fluid around their fixed and parallel axes.No study to the knowledge of both authors has been done on the interaction forceswhen the two cylinders are non-identical.

This paper is dedicated to investigate such a system. For simplicity, the cylinders are assumed infinitely long, so asto have a two-dimensional problem in xy-plane, where rotational axes are parallel to the ignored z-axis.

II. Problem Statement

Fig.1 shows an example of the system targeted by this paper. It depictstwo non-identical circles (for the two non-identical cylinders)of radiiR_A&R_B. The distance between the two centres (for the two axes) is:$2a$, where: $2a > R_A + R_B$.

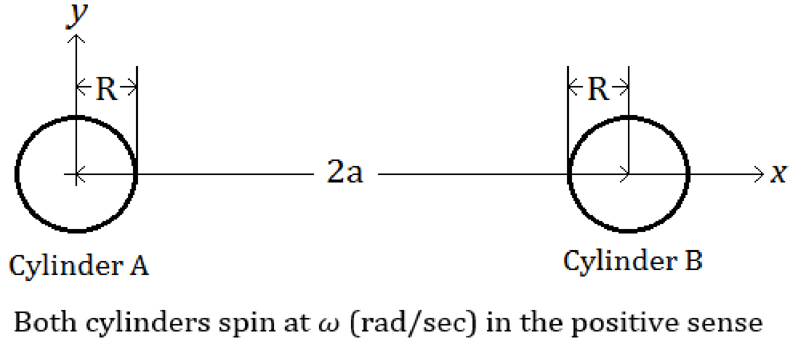

Fig.1: Top view of the system targeted by this paper

Bothcircles (cylinders)are rotating around their centres (axes) with fixed angular velocities,ω_A&ω_B (for Cylinder-A and Cylinder-B, respectively); in a fluid that is in-viscous, steady, in-vortical, and in-compressible. The aim of the paper is to derive the exact equations describing the forcesboth cylindersexert on their fixed axes after the entire system reached steady state.

The next section uses the Laplace equation[5]governing such problems, to find the velocity field of the fluid, satisfying its boundary conditions, which are:

1. the velocity of the fluid at circle-A resemblingthe surface of Cylinder-A is tangential to it, with a magnitude of: $\omega_A R_A$;
2. the velocity of the fluid at circle-B resemblingthe surface of Cylinder-Bis tangential to it, with a magnitude of: $\omega_B R_B$;
3. the velocity of the fluid at infinity is zero.

III. Fluid Velocity

As the governing Laplace equation is linear, super-position can be applied to simplify the solution. Considering Cylinder-A alone,the steady-state fluid-velocity vector:

$VA(x,y) = \langle VA_x(x,y), VA_y(x,y) \rangle$,(1)

isknown [6] to be as shown in Fig.2, where its two components are givenby:

$VA_x(x,y) = \frac{-\omega_A R_A{}^2 y}{x^2+y^2}$, and:(2)

$VA_y(x,y) = \frac{\omega_A R_A{}^2 x}{x^2+y^2}$, provided: $x^2 + y^2 > R_A{}^2$.(3)

These velocity equations satisfythe boundary conditions mentioned above. Furthermore, the velocity of the fluid due to the spinning of Cylinder-Ais seen to be directly proportional to ω_A, and inversely proportional tothe distance from the cylinder axis;i.e. the further from cylinder axis, the slower the fluid is.

Considering Cylinder-B alone, the steady-state fluid-velocity vector:

$VB(x,y) = \langle VB_x(x,y), VB_y(x,y) \rangle$,(4)

can be obtained from Eqs.2&3 (with a positive shift of: $2a$, along the x-axis) as:

$VB_x(x,y) = \frac{-\omega_B R_B{}^2 y}{(x-2a)^2+y^2}$,and:(5)

$VB_y(x,y) = \frac{\omega_B R_B{}^2 (x-2a)}{(x-2a)^2+y^2}$, provided: $(x-2a)^2 + y^2 > R_B{}^2$.(6)

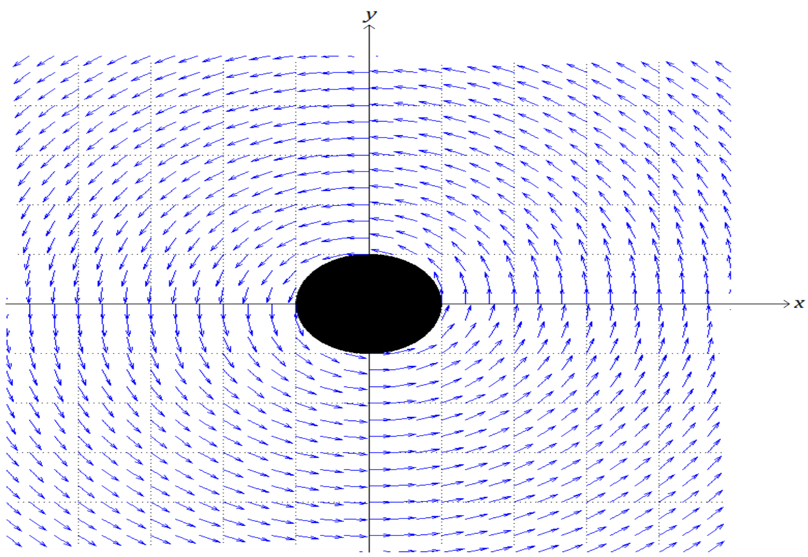

Fig.2: The velocity field of the ideal fluid due to the spinning of Cylinder-A

Hence, applying super-position;the fluid velocity for the system of both cylinders shown in Fig.1, can be found using Eqs.1-6 as:

$V(x,y) = VA(x,y) + VB(x,y) = \langle V_x(x,y), V_y(x,y)\rangle$,where:(7)

$V_x(x,y) = VA_x(x,y) + VB_x(x,y) = \frac{-\omega_A {R_A}^2 y}{x^2+y^2} - \frac{\omega_B {R_B}^2 y}{(x-2a)^2+y^2}$, and:(8)

$V_y(x,y) = VA_y(x,y) + VB_y(x,y) = \frac{\omega_A {R_A}^2 x}{x^2+y^2} + \frac{\omega_B {R_B}^2 (x-2a)}{(x-2a)^2+y^2}$,provided: (9)

$x^2 + y^2 > {R_A}^2$, and: $(x-2a)^2 + y^2 > {R_B}^2$; i.e. where fluid exists outside both cylinders.

A case of the above fluid-velocity is plotted as shown in Fig.3 below.The next section uses these fluid-velocity equations and Bernoulli equation to obtain the pressure field.

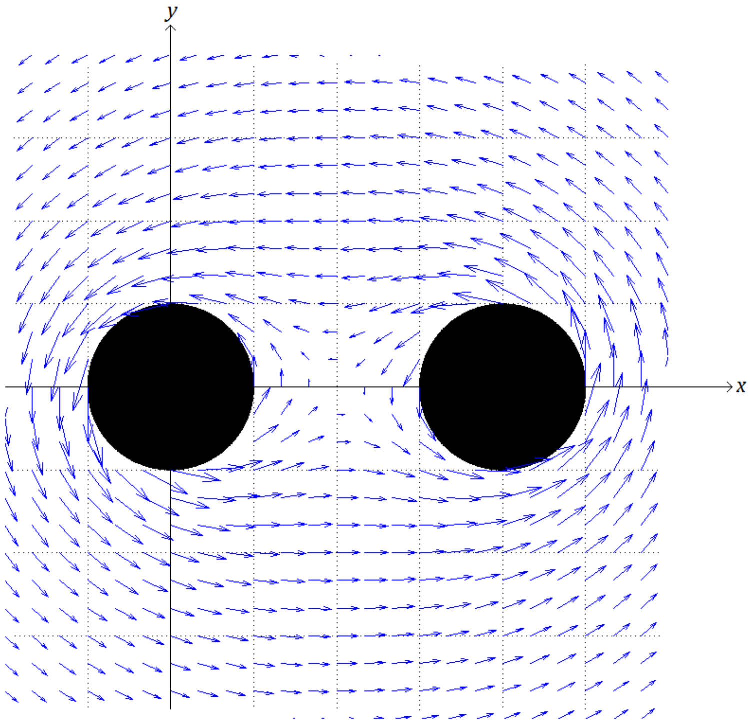

Fig.3: The velocity field of the ideal fluid due to the spinning of Cylinder-A and Cylinder-B in the same direction around their stationary and parallel axes

IV. Fluid Pressure

In this section, the pressure at the boundary of both cylinders is derived, in readiness to find the forces exerted on both axes. Ignoring the effect of the gravitational force in the fluid, Bernoulli equation relates the pressure magnitude, $P(x,y)$, to the velocity field, $V(x,y)$, as:

$P(x,y) + \frac{1}{2}\rho |V(x,y)|^2 = Constant$, where: ρ is the density of the fluid. (10)

The above equation can be read as:the summation of both static and dynamic pressures is constant everywhere in the fluid. In this respect, it is the square of the magnitude of the fluid velocity, $|V(x,y)|^2$, is what really matters for the fluid pressure.

Applying Eq.10 at Cylinder-A border & infinity (where velocity diminishes), then:

$PA(x,y) + \frac{\rho}{2} \left| V(x,y)|_{@\ Cylinder\ -A\ boundary} \right|^2 = P_\infty$, where:

$PA(x,y)$: is the pressure at Cylinder-A boundary, and:
P_∞: is the fluid pressure at ∞. Hence:

$PA(x,y) = P_\infty - \frac{\rho}{2}\left|V(x,y)\right|_{@\ Cylinder\ -A\ boundary}\Big|^2$.Using Eq.7, then:

$PA(x,y) = P_\infty - \frac{\rho}{2}\left[V_x^{\,2}(x,y) + V_y^{\,2}(x,y)\right]\Big|_{@\ Cylinder\ -A\ boundary}$.

This is substituted using Eqs.8&9 to:

$PA(x,y) = P_\infty - \frac{\rho}{2}\left[\left(\frac{\omega_A {R_A}^2 y}{x^2+y^2} + \frac{\omega_B {R_B}^2 y}{(x-2a)^2+y^2}\right)^2 + \left(\frac{\omega_A {R_A}^2 x}{x^2+y^2} + \frac{\omega_B {R_B}^2 (x-2a)}{(x-2a)^2+y^2}\right)^2\right]$.

This is reduced with: $x^2 + y^2 = {R_A}^2$ to:

$PA(x,y) = P_\infty - \frac{\rho}{2}\left[\left(\omega_A y + \frac{\omega_B {R_B}^2 y}{{R_A}^2+4a^2-4ax}\right)^2 + \left(\omega_A x + \frac{\omega_B {R_B}^2 (x-2a)}{{R_A}^2+4a^2-4ax}\right)^2\right]$, which is expanded to:

$PA(x,y) = P_\infty - \frac{\rho}{2}\left[{\omega_A}^2 {R_A}^2 + \frac{{\omega_B}^2 {R_B}^4 ({R_A}^2+4a^2-4ax)}{({R_A}^2+4a^2-4ax)^2} + \frac{2\omega_A\omega_B {R_A}^2 {R_B}^2 - 4a\omega_A\omega_B {R_B}^2 x}{{R_A}^2+4a^2-4ax}\right]$, giving:

$PA(x,y) = P_\infty - \frac{\rho}{2}\left[{\omega_A}^2 {R_A}^2 + \frac{{\omega_B}^2 {R_B}^4}{{R_A}^2+4a^2-4ax} + \frac{2\omega_A\omega_B {R_A}^2 {R_B}^2 - 4a\omega_A\omega_B {R_B}^2 x}{{R_A}^2+4a^2-4ax}\right]$. This is reduced to:

$PA(x,y) = P_\infty - \frac{\rho}{2}\left[{\omega_A}^2 {R_A}^2 + \frac{\omega_B {R_B}^2 (2\omega_A {R_A}^2 + \omega_B {R_B}^2 - 4a\omega_A x)}{{R_A}^2+4a^2-4ax}\right]$, then to:

$PA(x,y) = P_\infty - \frac{\rho {\omega_A}^2 {R_A}^2}{2}\left[1 + \frac{\omega_B {R_B}^2 (2\omega_A {R_A}^2 + \omega_B {R_B}^2 - 4a\omega_A x)}{{\omega_A}^2 {R_A}^2 ({R_A}^2+4a^2-4ax)}\right]$, where: $x \in$ Circle-A.

Converting to polar coordinates, with: $x = R_A \cos\theta$, then:

$PA(x,y) = PA(\theta) = P_\infty - \frac{\rho {\omega_A}^2 {R_A}^2}{2}\left[1 + \frac{\omega_B {R_B}^2 (2\omega_A {R_A}^2 + \omega_B {R_B}^2 - 4a\omega_A R_A cos\theta)}{{\omega_A}^2 {R_A}^2 ({R_A}^2+4a^2-4aR_A cos\theta)}\right]$, $\theta \in [0,2\pi]$. (11)

$PA(\theta)$is seen to be symmetrical about the$x$-axis.

V. Force Acting On The Rotational Axis Of Cylinder-A

The fluid pressure,$PA(\theta)$,expressed by Eq.11 is acting perpendicular to the surface of Cylinder-A as shown in Fig.4, and can be seen to cause infinitesimal force, $dFA(\theta)$, in the same direction, given by:

$dFA(\theta) = L \cdot R_A \cdot PA(\theta)d\theta$, where:

L: is the Length of either cylinder, which is assumed to be equal and infinitely long.

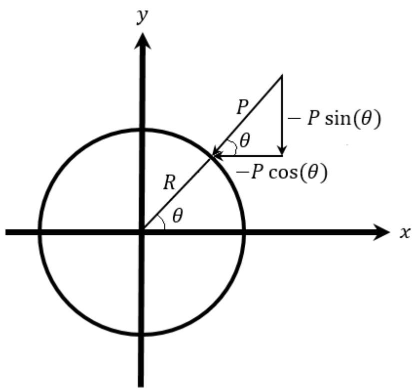

Fig.4: Top view of Cylinder-A showing fluid pressure

Decomposing: $dFA(\theta)$into two components, and ignoring its y-component due to the symmetry of $PA(\theta)$ about the x-axis; then:

$dFA_x(\theta) = -L \cdot R_A \cdot PA(\theta) \cos\theta \, d\theta$.(12)

Integrating $dFA_x(\theta)$around Circle-A (the surface of Cylinder-A) yields the force, F, exerted on the axis of rotation of Cylinder-A. Hence:

$F = \int_0^{2\pi} dFA_x(\theta)$. This is expressed using Eqs.11&12, as:

$$F = -L \cdot R_A \int_0^{2\pi} \left\{ P_\infty - \frac{\rho\omega_A{}^2 R_A{}^2}{2} \left[1 + \frac{\omega_B R_B{}^2 (2\omega_A R_A{}^2 + \omega_B R_B{}^2 - 4a\omega_A R_A cos\theta)}{\omega_A{}^2 R_A{}^2 (R_A{}^2 + 4a^2 - 4aR_A cos\theta)} \right] \right\} \cos\theta \, d\theta.$$

The first two of the above three integrandswill integrate to zero, leaving Fas:

$F = \rho\omega_B L \cdot R_A R_B{}^2 I/2$, where:(13)

$$I = \int_0^{2\pi} \frac{2\omega_A R_A{}^2 + \omega_B R_B{}^2 - 4a\omega_A R_A cos\theta}{R_A{}^2 + 4a^2 - 4aR_A cos\theta} \cdot \cos\theta \, d\theta.$$

VI. Solving Its Integral

The integral, I, can be solved [7] by usingthe complex transformation:

$z = e^{i\theta} = \cos\theta + i\sin\theta$, where: $i = \sqrt{-1}$, $i^2 = -1$, $d\theta = \frac{dz}{iz}$,and:

$\cos\theta = \frac{z + \frac{1}{z}}{2} = \frac{z^2+1}{2z}$;

whereby it converts to an integral over the closed contour, C, of the unit circle: $|z| = 1$, in the complex z-plane. Hence, removingθ, then:

$$I = \int_0^{2\pi} \frac{2\omega_A R_A{}^2 + \omega_B R_B{}^2 - 4a\omega_A R_A cos\theta}{R_A{}^2 + 4a^2 - 4aR_A cos\theta} \cdot \cos\theta \, d\theta = \oint_C \frac{2\omega_A R_A{}^2 + \omega_B R_B{}^2 - 4a\omega_A R_A \left(\frac{z^2+1}{2z}\right)}{R_A{}^2 + 4a^2 - 4aR_A \left(\frac{z^2+1}{2z}\right)} \cdot \left(\frac{z^2+1}{2z}\right) \cdot \frac{dz}{iz}.$$

This is simplified to:

$I = \frac{i}{2} \oint_C \frac{(2\omega_A R_A{}^2 + \omega_B R_B{}^2)z - 2a\omega_A R_A (z^2+1)}{z^2 [2aR_A z^2 - (R_A{}^2 + 4a^2)z + 2aR_A]} \cdot (z^2+1) dz$,and can be factored to:

$I = \frac{i}{4aR_A} \oint_C \frac{(2\omega_A R_A{}^2 + \omega_B R_B{}^2)z - 2a\omega_A R_A (z^2+1)}{z^2 (z-\alpha)(z-\beta)} \cdot (z^2+1) dz$,where: α and: β are the roots of:

$2aR_A z^2 - (R_A{}^2 + 4a^2)z + 2aR_A = 0$, given by:

$\alpha = \frac{R_A}{2a}$,and: (14)

$\beta = \frac{2a}{R_A} = \frac{1}{\alpha}$. (15)

Noting that:$2a > R_A$, then: $\alpha < 1$, and is inside C, while: $\beta > 1$, and is outside C.Hence, the integrand in I has three poles within C, two at the origin and one at α.

Using Cauchy Theorem in complex integrals[7], I is found as:

$$I = \frac{2\pi i \cdot i}{4aR_A} \left\{ \left[\frac{(2\omega_A R_A{}^2 + \omega_B R_B{}^2)z - 2a\omega_A R_A (z^2+1)}{(z-\alpha)(z-\beta)} \cdot (z^2+1) \right]' \Bigg|_{@z=0} + \left[\frac{(2\omega_A R_A{}^2 + \omega_B R_B{}^2)z - 2a\omega_A R_A (z^2+1)}{z^2 (z-\beta)} \cdot (z^2+1) \right] \Bigg|_{@z=\alpha} \right\}.$$

This gives:

$$I = \frac{-\pi}{2aR_A}\left[2\omega_A {R_A}^2 + \omega_B {R_B}^2 - 2a\omega_A R_A(\alpha+\beta) + \frac{(2\omega_A {R_A}^2+\omega_B {R_B}^2)\alpha - 2a\omega_A R_A(\alpha^2+1)}{\alpha^2(\alpha-\beta)}(\alpha^2+1)\right].$$

Removing β using Eq.15, this becomes:

$$I = \frac{-\pi}{2aR_A}\left[2\omega_A {R_A}^2 + \omega_B {R_B}^2 - 2a\omega_A R_A\left(\frac{\alpha^2+1}{\alpha}\right) + \frac{(2\omega_A {R_A}^2+\omega_B {R_B}^2)\alpha - 2a\omega_A R_A(\alpha^2+1)}{\alpha(\alpha^2-1)}(\alpha^2+1)\right].$$

Substituting for: α using Eq.14, and simplifying gives:

$$I = \frac{-\pi\omega_A R_A}{a}\left[1 - \frac{\omega_B {R_B}^2}{\omega_A(4a^2-{R_A}^2)}\right].$$

Hence, putting this, in Eq.13, and simplifying gives:

$$F = \frac{-\pi\rho L\omega_A\omega_B {R_A}^2{R_B}^2}{2a}\left[1 - \frac{\omega_B {R_B}^2}{\omega_A(4a^2-{R_A}^2)}\right],$$ remembering that: $2a > R_A$, hence: $4a^2 - {R_A}^2 > 0$. (16)

Defining the spin ratio, r, as:

$$r = \frac{\omega_A}{\omega_B};$$
(17)

then the force, F, will be zero (i.e. no attraction or repulsion) at a critical case, when:

$$r = r_c = \frac{{R_B}^2}{(4a^2-{R_A}^2)}.$$ (18)

This means that for any system of spacing ($2a$) and sizes (R_A & R_B); it is possible to cancel the force, F, at $r = r_c$. If $r > r_c$, F will be negative, i.e. the cylinders repel each other. On the other hand, if $r < r_c$, F will be positive, i.e. the cylinders attract each other.

For a given system of sizes and spin ratio, Eq.16 shows that at far enough spacing; the force goes asymptotically to:

$$F = \frac{-\pi\rho L\omega_A\omega_B {R_A}^2{R_B}^2}{2a}.$$

This is a repulsive force with an inverse relationship with the separation between axes of rotation.

VII. Effect Of Opposing Direction

Eq.16 shows that the effect of sense of rotation does not yield a different relation. ω_A can have the same sense as ω_B, or it can have different sense.

VIII. Conclusion

This paper derived the force acting on twonon-identical cylinders spinning atdifferent constantangular velocities around their stationary and parallel axes in anin-viscid, steady,in-vortical, and in-compressible fluid.The obtained equations showed that each cylinder axis,in such a system,experiences a repelling, attracting, or critically noforce.At far enough spacing, the magnitude of that force is inversely proportional to the separation between the two axes. It is also proportional to the density of the fluid, their radii, and the product of the angular velocities of the cylinders.

Nomenclature:

This section summarizes the symbols used in the paper in alphabetical order as follows:

ρ:	Density of the fluid
ω:	Angular speed of spinning of either cylinder
a:	Half the distance between axes of cylinders

$dFA(\theta)$: Infinitesimal force acting on Cylinder-A
$dFA_x(\theta)$: Component of: $dFA(\theta)$ along the x-axis
F: Interaction force acting on the axle of Cylinder-A
$F1$: Value of: F when both cylinders are about to touch each other
$F2$: Value of: F when both cylinders are spaced $2R$ apart
$F3$: Value of: F when both cylinders are spaced $4R$ apart
$F4$: Value of: F when both cylinders are spaced $6R$ apart
Fp: Peak value of: F
L: Length of either cylinder
$P(x,y)$: Pressure magnitude of the fluid
P_∞: Pressure magnitude of the fluidat ∞
$PA(x,y)$: Pressure magnitude of the fluidat Cylinder-A boundary in xy-coordinates
$PA(\theta)$: Pressure magnitude of the fluidat Cylinder-A boundary in $r\theta$-coordinates
R: Radius of either cylinder
$V(x,y)$: Velocity vector of the fluid due to the spinning of both cylinders
$V_x(x,y)$: Component of: $V(x,y)$ along the x-axis
$V_y(x,y)$: Component of: $V(x,y)$ along the y-axis
$VA(x,y)$: Velocity vector of the fluid due to the spinning of Cylinder-A
$VA_x(x,y)$: Component of: $VA(x,y)$ along the x-axis
$VA_y(x,y)$: Component of: $VA(x,y)$ along the y-axis
$VB(x,y)$: Velocity vector of the fluid due to the spinning of Cylinder-B
$VB_x(x,y)$: Component of: $VB(x,y)$ along the x-axis
$VB_y(x,y)$: Component of: $VB(x,y)$ along the y-axis

References

[1]. Fluid Mechanics: An Introduction, E. Rathakrishnan, 3rd edition. 2012.
[2]. Fluid Mechanics for Engineers: A Graduate Textbook, M. T. Schobeiri, 2010.
[3]. Hydrodynamics, Sir H. Lamb, 1945.
[4]. The Interaction Forces between Two Identical Cylinders Spinning around their Stationary and Parallel Axes in an Ideal Fluid, A. Baz, S. Al-Kasimi.
[5]. Solutions of Laplace's Equation, D. R. Bland, 1961.
[6]. Vortex Dynamics, P. G. Saffman, 1995.
[7]. Complex Integration and Cauchy's Theorem, G. N. Watson, 2012.