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LBM Simulation and Analytical Validation of Incompressible Couette and Poiseuille flows

D. Arumuga Perumal[1]
[1](*Department of Mechanical Engineering, National Institute of Technology Karnataka, Surathkal, Mangalore-575025, INDIA*)

Abstract: *The Lattice Boltzmann method (LBM) came in the Computational Fluid Dynamics (CFD) field as a tool for research, but its importance lies in its various areas of incompressible fluid flows. Owing to its excellent numerical method and kinetic theory boundary conditions it plays a vital role as a simulation technique for understanding incompressible fluid flow physics. In the present work, the study is extended to incompressible couette flow and poiseuille flow problems using the D2Q9 lattice model to gain some experience in the application of LBM. First, the couette flow is studied in some details with different lattice sizes and relaxation times. After having thus established the credibility of the code and the method including boundary conditions, LBM is then used to investigate the poiseuille flow. It is seen that, the present LBM results are excellent agreement with analytical results.*
Keywords: *Lattice Boltzmann method, Couette flow, Poiseuille flow, two-dimensional, Analytical solution.*

I. Introduction

Simulation of fluid dynamics has been a major topic of research for the past few decades [1].The fluid flow problems usually studied by three approaches in fluid dynamics such as pure experimental, pure theoretical and computational. The continuous growth of computer power has strongly motivated the scientific community and researchers to use computational fluid dynamics (CFD) for the design and testing of new technological solutions. Couette flow is used as one of the bench mark problem due to its practicality in many applications like chemical process engineering, aerospace engineering, automobile engineering etc. It is known that, couette flow is the flow between two infinite long parallel plates one of which is moving relative to the other. The flow field between the two plates is driven exclusively by the shear stress exerted on the fluid by the moving upper plate resulting in a velocity profile across the flow.

Liu *et al*. [2] described the applications of a finite particle method (FPM) to modeling incompressible flow problems such as poiseuille flow, couette flow, shear driven cavity and a dam collapsing problem. They compared their results with existing ones. Gibson *et al.* [3] computed a new equilibrium solution of plane couette flow at Reynolds number (*Re*) 400. Ravi *et al.* [4] presented an efficient parallel domain decomposition algorithm for non-equilibrium molecular dynamics (NEMD) simulations of large systems under planar couette flow. Very few numerical studies are available in the literature for incompressible couette flow using Lattice Boltzmann method (LBM). The main part of this section is to study theoretical background behind the incompressible planar couette flow and also validate our LBM code with existing results. Poiseuille flow is one of the bench mark problem used for fluid flow problems in the past few decades. Due to its simple and common geometry, boundary conditions can be easily incorporated. It is known that, poiseuille flow is created between two stationary walls when pressure gradient or body force is aligned with the walls. The fact is that the motion of the flow between the plates is caused by an imposed external pressure gradient. Pressure and viscosity forces for these kinds of flows are in equilibrium for a fluid element. Relationship for the flow velocity shows that the velocity profile between the plates represents a parabola. Serrin [5] presented results for couette and poiseuille flows by taking the coefficient of viscosity and cross viscosity as functions of second invariant of D. Loyalka and Hickey [6] reported explicit results for poiseuille flow between two parallel plates. They characterized bulk flow by the Burnett distribution. They have given results for a rigid gas in near continuum limit. Cercignani [7] analyzed plane poiseuille flow according to the method of elementary solutions. They discussed the limits of application of continuum and slip-flow theories.

II. Lattice Boltzmann Method Formulation

Over the years, finite difference method is frequently used in CFD [8]. The method consists in essentially setting up a grid in the problem domain, discretizing the governing equations with respect to the grid and solving them numerically. In the last one and a half decades Lattice Boltzmann Method has emerged as a new and effective approach of computational fluid dynamics and it has achieved considerable success in simulating fluid flows and heat transfer problems [9]. During one lattice time step, particles propagate to their adjacent lattice nodes and redistribute their momentum in the subsequent collisions. The macroscopic quantities such as density and velocity can be obtained by evaluating the hydrodynamic moments of the distribution function.

The general form of Boltzmann equation is written as [1]

$$\frac{\partial f}{\partial t} + \boldsymbol{c}.\frac{\partial f}{\partial r} + \boldsymbol{F}.\frac{\partial f}{\partial \mathbf{c}} = Q\left(f\right) \tag{1}$$

where $\boldsymbol{c}$ is the particle velocity and $\boldsymbol{F}$ is the body force. $Q\ (f)$ is the collision integral. The most widely known replacement is called BGK approximation:

$$Q_{BGK}\left(f\right) = -\frac{f - f^{eq}}{\tau} \tag{2}$$

The lattice Boltzmann equation (LBE) with BGK models can be written as [10]

$$f_i\left(\mathbf{x}+\mathbf{e_i}\Delta t, t+\Delta t\right) = f_i\left(\mathbf{x},t\right) - \frac{1}{\tau}\left(f_i\left(\mathbf{x},t\right) - f_i^{eq}\left(\mathbf{x},t\right)\right) \quad (i= 0,1,.....\mathrm{M}) \tag{3}$$

where $f_i\left(\mathbf{x},t\right)$ is the density distribution function, the particle discrete velocity $\boldsymbol{e_i}$ and time t; f_i^{eq} is its corresponding equilibrium state, which depends on the local macroscopic variables, ρ and $\boldsymbol{u}$. τ is the single relaxation parameter related to the hydrodynamic viscosity, Δt is the time step and M is the number of discrete particle velocity. For simulating two-dimensional flows, the two-dimensional nine-velocity model (*D2Q9*) with nine discrete velocities e_i (i = 0,1..,8) is commonly used. In a *D2Q9* square lattice each node has eight neighbours connected by eight links. The macroscopic density ρ and momentum density $\rho\boldsymbol{u}$ are defined as particle moments of the distribution function f_i :

$$\rho = \sum_{i=1}^{M} f_i ; \rho\mathbf{u} = \sum_{i=1}^{M} f_i e_i \tag{4}$$

Equation for the equilibrium distribution functions for two-dimensional, nine- velocity LB model (*D2Q9*):

$$f_i^{eq} = \rho w_i \left[1 + 3\frac{\mathbf{e_i}.\mathbf{u}}{\mathbf{c}^2} + 9\frac{\left(\mathbf{e_i}.\mathbf{u}\right)^2}{2\mathbf{c}^4} - \frac{3\mathbf{u}^2}{2\mathbf{c}^2}\right] \tag{5}$$

where the lattice weights are w_i = 4/9, i =0; w_i =1/9, i =1,2,3,4, ; w_i = 1/36, i = 5,6,7,8.

III. Couette Flow Description And Simulation Procedure

1.1. Problem Description

Consider the viscous flow between two plates separated by the vertical distance 'H', as shown in Figure 1. The upper plate is moving at the velocity U, and the lower plate is stationary; i.e. its velocity is u=0. The flow field between the two plates is generated exclusively by the shear stress exerted on the fluid by the moving upper plate, resulting in a linear velocity profile across the flow $u = u(y)$, as shown in Figure 1.

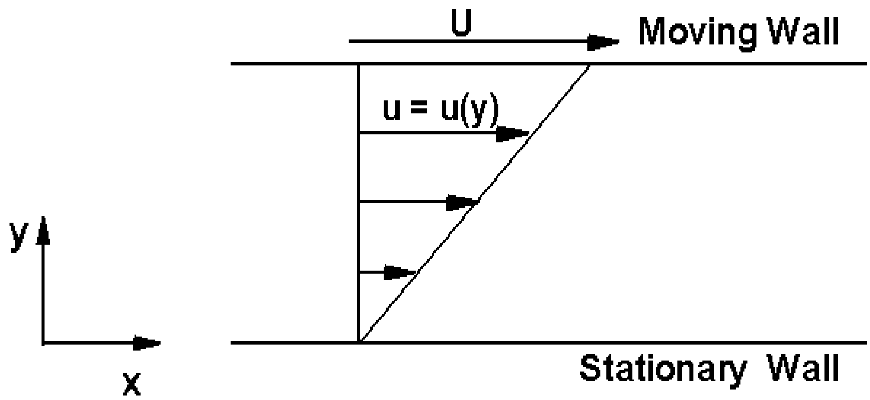

Figure 1: Schematic diagram of incompressible couette flow.

Assumptions

- Incompressible flow; ρ =constant.
- We assume that the plates are very long & wide, so that the flow is essentially axial, $u \neq 0$ but $v= w= 0$.
- Since there is no beginning or end of this flow, the flow field variables must be independent of x; i.e., $\partial / \partial x = 0$.
- There is no vertical component of velocity any where i.e. v=0.
- No body forces.
- There are no pressures gradients in either the x or y direction.

1.2. Analytical Solution

Governing equations: As incorporating the above assumptions into continuity and momentum equations, the reduced forms are given below

continuity
$$\frac{\partial v}{\partial y} = 0 \tag{6}$$

x-momentum
$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{7}$$

Boundary conditions: We assume a no–slip boundary condition for the fluid at each wall. Bottom wall: $u = 0$ for $y = 0$; Top wall: $u = U$ for $y = H$.

Therefore velocity profile obtained for incompressible steady couette flow is

$$u = U\,\frac{y}{H}. \tag{8}$$

Eq. (8) is the exact analytical solution for the incompressible steady couette flow.

1.3. Simulation Procedure

The lattice Boltzmann equation (LBE) with BGK model is used for simulation (described in section 2) in the present work. Relaxtation time formula used for computation is

$$\tau = (6\times \upsilon)+1)/2 \tag{9}$$

In this problem, I considered number of lattices along y-direction N_y = 81. Number of lattices along x-direction N_x = 81.

Initial Conditions

Initially the x direction velocity is assumed to be uniform throughout the channel except at the upper plate where the velocity is U = 0.1 and y-velocity is taken as 0. Density used during simulation was 1.0.

Boundary Conditions

At the bottom wall, bounce-back boundary condition is used [9]. As an example (*D2Q9* model), after streaming the unknown distributing functions are given as for the bottom wall

$$\begin{aligned} f_5(x,y,t) &= f_7(x,y,t) \\ f_2(x,y,t) &= f_4(x,y,t) \\ f_6(x,y,t) &= f_8(x,y,t) \end{aligned} \tag{10}$$

At moving (upper) wall all distribution functions are updated by equilibrium distribution function. Periodic boundary conditions are applied at the channel inlet and outlet as given by [9]. As an example the inlet boundary condition has written as

$$\begin{aligned} f_1(1,y,t) &= f_1(L,y,t) \\ f_5(1,y,t) &= f_5(L,y,t) \\ f_8(1,y,t) &= f_8(L,y,t) \end{aligned} \tag{11}$$

1.4. Code Validation

The geometry and boundary conditions of the incompressible couette flow is shown in Figure 1. First, the developed LBM code is used to compute the flow on a 81×81 lattice arrangement. Figure 2 shows the steady-state *u*-velocity profile along a vertical line passing through the centre of the channel at *Re* = 100 and the comparison of velocity profile with the analytical solution. It is seen that present LBM results agrees well with existing results.

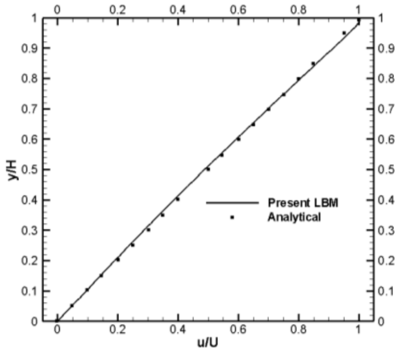

Figure 2: Velocity profile of incompressible steady couette flow by LBM.

3.5 Results And Discussion

A lattice node resolution study was carried out using three lattice sizes composed of coarse lattice size 65×65, fine lattice size 81×81 and very fine lattice size 101×101 nodes as shown in Figure 3. In the present work, Reynolds number *Re* = 100 is considered. The numerical results were equivalent for the 81×81 and 101×101 lattice nodes. Therefore 81×81 lattice size was considered in the present problem. In this section, we investigated the effect of different relaxation times by changing the Reynolds number of the flow as shown in Figure 4. It is clearly seen that as the relaxation time decreases the velocity profile changes its linearity position and the reason behind is that may be the increase in inertial effect.

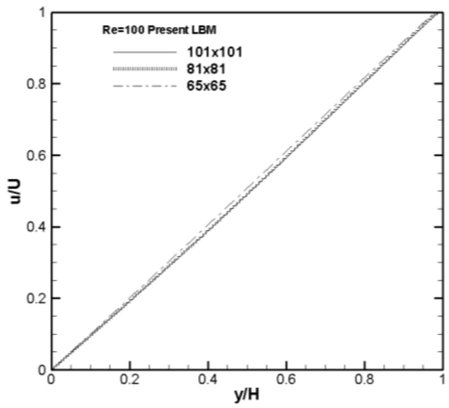

Figure 3: Velocity profile of incompressible steady couette flow by LBM with different lattice sizes.

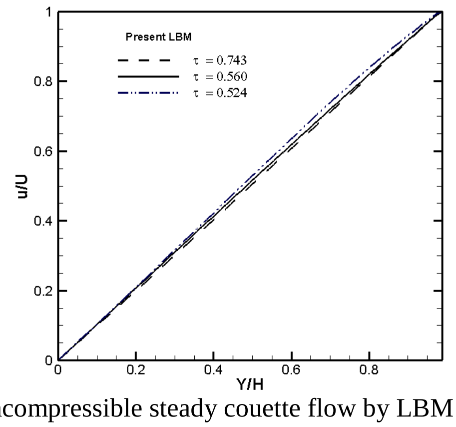

Figure 4: Velocity profile of incompressible steady couette flow by LBM with different relaxation times.

IV. Poiseuille Flow Description And Simulation Procedure

1.5. Problem Description

Consider the viscous flow between two stationary plates separated by the vertical distance '*H*', as shown in Figure 5. The flow field between the two plates is exclusively by the imposed pressure gradient from left to right, so that velocity profile is in parabolic shape as shown in Figure 5.

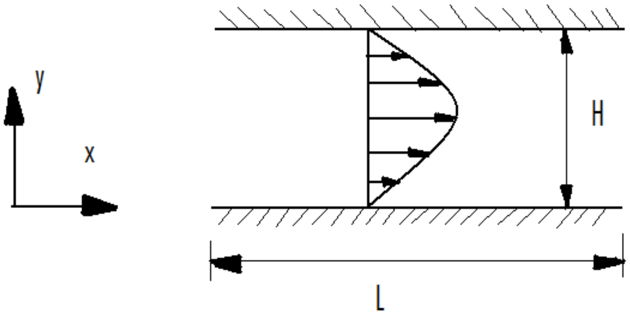

Figure 5 Schematic diagram of incompressible poiseuille flow.

Assumptions

1. Steady state $\frac{\partial}{\partial t}=0$
2. Incompressible flow; ρ =constant
3. No flow in *y* or *z* direction. i.e., *v=w=0*
4. If no pressure gradient in *z* direction and plates are long in *z* direction, then flow in middle (*z* direction) can be considered to have only a *z* component, i.e., ignore edge effects of plates in *z* direction.

1.6. Analytical Solution

Governing equations: After incorporating the above assumptions into continuity and momentum equations, the reduced forms are given below

continuity $$\frac{\partial v}{\partial y} = \mathbf{O} \qquad (12)$$

x-momentum $$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x} \qquad (13)$$

From Eq. (3.8) we can get

$$u = \left(\frac{1}{2\mu} \frac{dP}{dx} \right) y^2 + C_1 y + C_2 \qquad (14)$$

where C_1 and C_2 are integration constants.

Boundary conditions:

We assume a no–slip boundary condition for the fluid at each wall.

Bottom wall $u = 0$ for $y = 0$; Top wall $u = 0$ for $y = H$

We further take the pressure gradient $dP/_{dz}$ to be a given value. After substituting the boundary conditions into Eq. (14) we can get the following equation for incompressible poiseuille flow.

$$u = \frac{1}{2\mu} \frac{dP}{dx} \left(y^2 - Hy \right) \qquad (15)$$

It is known that, the velocity profile is a quadratic in *y* for poiseulle flow between two stationary parallel plates. Then the shear stress is given by substituting the velocity (*u*) into the constitutive equation

$$\tau_{yx} = \mu \frac{du}{dy} = \frac{dP}{dx} \left(y - \frac{H}{2} \right) \qquad (16)$$

The shear stress is a maximum at either wall, and zero at the centre. Interestingly, the shear stress does not depend on the viscosity coefficient (μ).

1.7. Lbm Simulation
Relaxation time used for LBM computation is

$$\tau = (6\times\upsilon)+1)/2 \qquad (17)$$

Here I considered number of lattices along *y*-direction N_y =81. Number of lattices along *x*-direction N_x =81.

Initial conditions
Initially the *x* direction velocity is assumed to be uniform throughout the channel.
Boundary conditions:
At the top wall, bounce-back boundary condition is imposed
Top wall Boundary Condition:

$$\begin{aligned} f_7(x,y,t) &= f_5(x,y,t) \\ f_4(x,y,t) &= f_2(x,y,t) \\ f_8(x,y,t) &= f_6(x,y,t) \end{aligned} \qquad (18)$$

Boundary conditions at inlet/exit as discussed by Succi et al. [1] are implemented in the present work. Inlet boundary condition can be written as

$$\begin{aligned} u &= 1-\left[\frac{f_0+f_2+f_4+2(f_3+f_6+f_7)}{rhoi}\right] \\ v &= 0 \\ f_1 &= f_3+2*rhoi*u/3 \\ f_5 &= f_7-\frac{f_2}{2}+\frac{f_4}{2}+\frac{rhoi*u}{6} \\ f_8 &= f_6+\frac{f_2}{2}-\frac{f_4}{2}+\frac{rhoi*u}{6} \end{aligned} \qquad (19)$$

1.8. Results
Figure 6 shows the velocity profile of incompressible poiseuille flows by LBM. In this section, different relaxation times are used to compare analytical solution. It is found that the results for 0.7≤ τ ≤ 3.0 agree well with the analytical solution. As the relaxation time (τ) increases we observed that the change in velocity profile curvature and the reason behind is that may be the effect of viscosity.

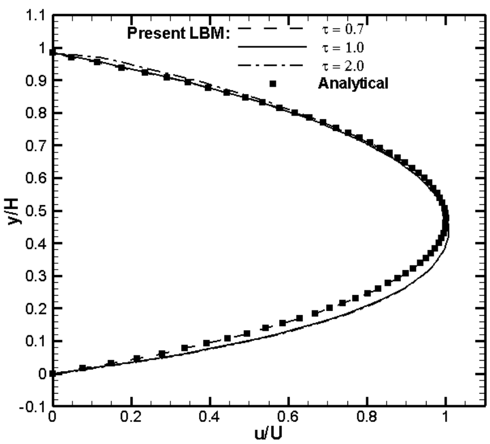

Figure 6 Velocity profile of incompressible poiseuille flow by LBM.

V. Conclusion

In the present work, mesoscopic LBM simulation of incompressible couette and poiseuille fluid flow problems are presented in detail. In the above two test cases, I compared present results with existing analytical results and also studied with different relaxation times. Whenever comparison is possible the presented results are found to be in good agreement with the analytical results reported by other researchers. The numerical results show that the present LBM is as accurate as the conventional numerical methods like Finite Difference (FD), Finite Volume (FV), Finite Element (FE), Spectral method. This verification gives confidence to apply the present method to solve other fluid flow problems. To sum up, the present study reveals many interesting features of couette l and poiseuille flows and demonstrates the capability of the LBM to capture this features.

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