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Generalized Siamese Primes And The Goldbach Conjecture

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Abstract

The odd primes, p and q, are called Siamese primes, [1], if there exists a natural number, n, such that $p = n^2 - 2$ *and* $q = n^2 + 2$. *The odd primes, p and q, are called Generalized Siamese primes, [1], if* ***(a)*** *there exists natural numbers, n and* $a > 2$, *such that* $p = n^2 - a$ *and* $q = n^2 + a$, *and/or* ***(b)*** *there exists natural numbers, n, a, an k such that* $p = n^k - a$ *and* $q = n^k + a$. *Various theorems are presented.*

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I. Siamese Primes

The odd primes, p and q, are called *Siamese primes*, [1], if there exists a natural number, n, such that $p = n^2 - 2$ and $q = n^2 + 2$. For $n = 3$, for example, we have $7 < 3^2 < 11$. Since $n^2 = p + 2$, n must be odd. Furthermore, $n^2 = 1 \pmod 4$. As $q - p = 4 = 0 \pmod 4$, we have $q = p \pmod 4$. Since $p = n^2 - 2$ and $q = n^2 + 2$, we see that $p = 3 \pmod 4$ and $q = 3 \pmod 4$, so $p = q = 3 \pmod 4$. The following theorem is of interest.

Theorem 1: $3 \mid n$.

Proof: Observe that $n^2 = 0$ or $1 \pmod 3$. If $n^2 = 1 \pmod 3$, then $q = n^2 + 2 = 0 \pmod 3$, which is impossible. Then $n^2 = 0 \pmod 3$ implying that $n = 0 \pmod 3$. ■

The reader is reminded that Goldbach conjectured that every even number greater 2 is the sum of two not necessarily distinct primes. Some even integers have several such expressions. For example, $22 = 3 + 19 = 5 + 17 = 11 + 11$. We present the following theorem.

Theorem 2: Let $n \geq 3$. If Goldbach's Conjecture is true, there exists at least one pair of primes, p and q, such that $p = n^2 - a$ and $q = n^2 + a$.

Proof: We have $p + q = 2n^2$. This expresses the even number, $2n^2$, as the sum of two primes. ■

II. Generalized Siamese Primes Involving Distance

We consider the system, $p = n^2 - 3$ and $q = n^2 + 3$. It follows that n is even. For $n = 4$, for example, we have $13 < 4^2 < 19$. We bypass the system $p = n^2 - 4$ and $q = n^2 + 4$, since $n^2 - 4 = (n - 2)(n + 2)$ which is composite for $n > 3$. By the same reasoning, if $a = t^2$ and $n - t > 1$, then we have no solutions to $p = n^2 - t^2$ and $q = n^2 + t^2$. For the system, $p = n^2 - 5$ and $q = n^2 + 5$, we have $6 \mid n$. [1]

We have the following rather general theorem.

Theorem 3: Let $p = n^2 - a$ and let $q = n^2 + a$. **(a)** If a is even, then $p = q \pmod 4$. **(b)** If a is odd, either $p = 1 \pmod 4$ and $q = 3 \pmod 4$, or $p = 3 \pmod 4$ and $q = 1 \pmod 4$.

Proof of (a): $q - p = 2n^2 = 0 \pmod 4$, implying that $p = q \pmod 4$. **(b)** Since $p = n^2 - a$, n^2 is even, in which case $n^2 = 0 \pmod 4$. Then $p + q = 2n^2 = 0 \pmod 4$. Since p and q are odd, the result follows. ■

III. Generalized Siamese Primes Involving *k*-powers of *n*

We consider the system

$$p = n^k - a \qquad q = n^k + a$$

where p and q are odd primes, and $k \geq 2$.

Theorem 4: GCD(a, n) = 1 is a necessary condition for solving the above system.

Proof: Let GCD(a, n) = $d > 1$. Then $d \mid a$ and $d \mid n$, implying that $d \mid n^k$. Since $p = n^k - a$, we have $d \mid p$, contradicting the primality of p. ■

Corollary: a and n have opposite parity. ■

We generalize Theorem 2 with the next theorem whose proof is identical to that of Theorem 2.

Theorem 5: Let $n \geq 3$. If Goldbach's Conjecture is true, there exists at least one pair of primes, p and q, such that $p = n^k - a$ and $q = n^k + a$. ■

Theorem 6: Let k be even. If $a \neq 0 \pmod 3$, then $3 \mid n$.

Proof: By contradiction. We assume that $n = \pm 1 \pmod 3$, in which case $n^k = 1 \pmod 3$ Then we have $p = n^k - a = 1 - (\pm 1) \pmod 3$ and $q = n^k + a = 1 + (\pm 1)$, one of which is 0 (mod 3), a contradiction. ■

Lemma: Let $a = 1 \pmod 5$, and let k be even. Then $n = 0 \pmod 5$.

Proof: Since k is even, $n^k = 0$, 1, or 4 (mod 5). If $n^k = 1 \pmod 5$, then $p = n^k - a = 0 \pmod 5$ which is impossible. If $n^k = 4 \pmod 5$, then $q = n^k + a = 0 \pmod 5$ which is also impossible. Then $n = 0 \pmod 5$. ■

It follows from Theorem 6 and the Lemma, that if $a = 1 \pmod 5$ and $n \neq 0 \pmod 3$, then $30 \mid n$.

References

[1] M.Lewinter and S.Reyman, *Generalized Siamese Primes*, www.ijmsi.org Vol. 13, Issue 5, Sep.–Oct. 2025, pp. 20-21.
[2] M.Lewinter, J.Meyer, *Elementary Number Theory with Programming*, Wiley & Sons. 2015.