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# Gradient Solitons on GRW Spacetimes

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***Abstract***

*In this paper, we investigate several geometric soliton structures on generalized Robertson-Walker (GRW) spacetimes. First, we study GRW spacetimes admitting second order symmetric parallel tensors and prove that such tensors are necessarily constant multiples of the metric tensor. Further, we analyze gradient Ricci-Bourguignon soli-tons and almost gradient Ricci-Bourguignon solitons on GRW space-times and obtain relations connecting the potential function, scalar curvature and the warping function of the spacetime. We also examine gradient η-Ricci solitons and derive conditions satisfied by the potential function in terms of the curvature quantities of the GRW spacetime. In addition, gradient quasi-Yamabe solitons are studied and several characterizations are obtained. Finally, we introduce gradient η-Ricci-Bourguignon solitons in the setting of GRW spacetimes and establish corresponding structural results. The obtained results describe how various geometric soliton structures interact with the curvature proper-ties and warping function of generalized Robertson-Walker spacetimes.*

***Keywords:*** *Generalized Robertson-Walker spacetime, Ricci solitons, Ricci-Bourguignon solitons, η-Ricci solitons, quasi-Yamabe solitons, warped product spacetime.*

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## I. Introduction

We start with a Lorentzian manifold Mn whose Lorentzian metric $g$ is of signature $(+,+,\dots,+,-)$. In [1], Alias, Romero and Sánchez introduced the notion of generalized Robertson–Walker (GRW) spacetimes. A Lorentzian manifold $M$ is said to be a generalized Robertson–Walker spacetime if it can be expressed as a warped product of an open interval $\mathbb{I}$ of $\mathbb{R}$ and an $(n-1)$ -dimensional Riemannian manifold $M^*$, that is

$$M = -\mathbb{I} \times_{\phi} M^*, \tag{1.1}$$

where $\phi$ is a positive smooth function on $I$ called the warping function. The GRW spacetime reduces to the classical Robertson–Walker spacetime when $M^*$ is three–dimensional with constant sectional curvature. Thus GRW spacetime appears as a natural generalization of Robertson–Walker spacetime and plays an important role in relativistic cosmology. In particular, GRW spacetimes include several important cosmological models such as the static Einstein spacetime, the Einstein–de Sitter spacetime, the Friedman cosmological models and the de Sitter spacetime. Moreover, these spacetimes arise naturally in the study of inhomogeneous cosmological models describing isotropic radiation. For further geometrical and physical properties of GRW spacetimes we refer to ([11], [19]).

Ricci solitons arise naturally as self–similar solutions of the Ricci flow and play a fundamental role in the study of singularity formation and geometric evolution. The concept was introduced by Hamilton in his pioneering work on the Ricci flow [18]. A Riemannian manifold $(M,g)$ together with a vector field $V$ and a constant $\lambda$ is said to admit a Ricci soliton if

$$L_V g + 2Ric + 2\lambda g = 0, \tag{1.2}$$

where $L_V$ denotes the Lie derivative of the metric $g$ with respect to $V$. According to the sign of $\lambda$, Ricci solitons are classified as shrinking ($\lambda < 0$), steady ($\lambda = 0$) or expanding ($\lambda > 0$) [18]. If the vector field $V$ is the gradient of a smooth function $f$ , i.e. $V = \nabla f$ , then the soliton is called a gradient Ricci soliton and equation (1.2) reduces to

$$Hess\, f + Ric + \lambda g = 0, \tag{1.3}$$

where $Hess\ f$ denotes the Hessian of $f$. Ricci solitons have attracted considerable attention due to their applications in geometric analysis, mathe-matical relativity and theoretical physics.

Several generalizations of Ricci solitons have been introduced in the literature. Among them, the notion of $\eta$–Ricci solitons was proposed by Cho and Kimura [12] in the setting of contact geometry. A Riemannian manifold $(M, g)$ is said to admit a gradient $\eta$–Ricci soliton if

$$Hess\, f + Ric + \lambda g + \mu A \otimes A = 0, \tag{1.4}$$

where $f$ is a smooth function on $M$, $\lambda$ and μ are constants and $A$ is the 1–form associated with a unit vector field $\xi$.

Another related structure is the quasi–Yamabe soliton, which arises as a generalization of Yamabe solitons. A gradient quasi–Yamabe soliton on a Riemannian manifold (Mn, g) is defined by

$$Hess\, f = (r - \lambda)g + \alpha\, df \otimes df, \tag{1.5}$$

where f is a smooth function, $r$ denotes the scalar curvature, $\lambda$ is a constant and $\alpha$ is a real parameter. In the special case $\alpha = 0$, the above equation reduces to the classical gradient Yamabe soliton.

Another important geometric structure associated with geometric flows is the Ricci–Bourguignon flow, which can be viewed as a modification of the Ricci flow involving the scalar curvature term. Motivated by this flow, Ricci–Bourguignon solitons were introduced as natural generalizations of Ricci solitons. A Ricci–Bourguignon soliton is a triple $(g, \xi, \lambda)$ satisfying

$$L_\xi g + 2\, Ric - 2\rho r g + 2\lambda g = 0, \tag{1.6}$$

where $\rho$ is a real parameter, $r$ denotes the scalar curvature and $\lambda$ is the soliton constant. When the vector field $\xi$ is the gradient of a smooth function $f$ , the above equation reduces to the gradient Ricci–Bourguignon soliton equation

$$Hess\, f + Ric - \rho r g + \lambda g = 0. \tag{1.7}$$

Blaga [3] further introduced the concept of $\eta$–Ricci–Bourguignon solitons, which combines the structures of $\eta$–Ricci solitons and Ricci–Bourguignon solitons. It is defined by

$$Hess\, f + Ric + \rho r g + \lambda g + \mu A \otimes A = 0. \tag{1.8}$$

Later, motivated by the notion of almost Ricci solitons introduced by Pigola et al. [22], the concept of almost $\eta$–Ricci–Bourguignon solitons was considered by allowing the soliton constant to be a smooth function instead of a constant. In this case the defining equation takes the form

$$Hess\, f + Ric + \rho r g + \lambda(x) g + \mu A \otimes A = 0, \tag{1.9}$$

where $\lambda(x)$ is a smooth function on $M$. This structure provides a natural unification of Ricci–Bourguignon solitons and $\eta$–Ricci solitons and has recently attracted considerable interest in the study of geometric flows.

The Ricci–Bourguignon flow and its associated solitons have been stud-ied by several authors. Catino [7] established the short–time existence and uniqueness of solutions to the Ricci–Bourguignon flow. Later, Siddiqui and Siddiqui [23] constructed explicit examples of four–dimensional spacetimes admitting Ricci–Bourguignon solitons. Further investigations were carried out by Azami [2], who studied $\eta$–Ricci–Bourguignon solitons, while Mi [21] proved that under suitable conditions complete non–compact gradient Ricci–Bourguignon solitons possess non–negative scalar curvature. Blaga and Ta¸stan [5] obtained several rigidity results for Ricci–Bourguignon soli-tons, and Dhriti et al. [16] showed that gradient Ricci–Bourguignon soli-tons on paracontact metric manifolds reduce to Einstein manifolds under appropriate assumptions.

The investigation of geometric solitons in spacetime geometry originates from the study of Ricci flow and its self–similar solutions. In recent years, Ricci–type solitons have been extensively studied on different Lorentzian and contact manifolds. In particular, De and De [13] studied gradient soli-tons on para–Kenmotsu manifolds, while De, Chaubey and Shenawy [15] investigated Yamabe solitons in perfect fluid spacetimes. Further developments were obtained by Suh and De [25] on Ricci and Yamabe type solitons in contact geometry. Blaga [4] also examined several soliton structures in perfect fluid spacetimes.

Recently, De et al. [14] studied perfect fluid spacetimes admitting a con-circular vector field and obtained several interesting geometric properties. In particular, they showed that the velocity vector field annihilates the Weyl conformal curvature tensor and that, in dimension four, such spacetimes reduce to generalized Robertson–Walker spacetimes with Einstein fibre while admitting several Ricci–type soliton structures.

## II. Generalized Robertson-Walker Spacetime

The geometric properties of generalized Robertson-Walker spacetimes used in this section are well known and can be found in [1, 11, 19]. Let $(M, g)$ be an $n$-dimensional Lorentzian manifold. A spacetime $(M, g)$ is said to be a Generalized Robertson–Walker (GRW) spacetime if it can beexpressed as a warped product

$$M = \mathbb{I} \times_f M^*, \tag{2.1}$$

where $\mathbb{I} \subset \mathbb{R}$ is an open interval with coordinate t, (M ∗, g∗) is an (n − 1)-dimensional Riemannian manifold and $f: I \to (0, \infty)$ is a smooth positivefunction called the warping function. The Lorentzian metric $g$ on $M$ is given by

$$g = -dt^2 + f^2(t) g^*. \tag{2.2}$$

The vector field $\xi = \frac{\partial}{\partial t}$ is a unit timelike vector field satisfying

$$g(\xi, \xi) = -1. \tag{2.3}$$

For any vector field $X$ tangent to $M$ , the 1–form $A$ is given by

$$A(X) = g(X, \xi). \tag{2.4}$$

The Levi–Civita connection $\nabla$ of a GRW spacetime satisfies the following relations

$$\nabla_X \xi = \frac{f'}{f} X \tag{2.5}$$

for any vector field $X$ orthogonal to $\xi$, where $f' = \frac{df}{dt}$.

Using the definition of the curvature tensor

$$R(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi, \tag{2.6}$$

we obtain

$$R(X, Y)\xi = \frac{f''}{f} f\{A(X)Y - A(Y)X\}, \tag{2.7}$$

where $f'' = \frac{d^2 f}{dt^2}$.

Contracting above equation over $X$, we obtain the Ricci tensor

$$S(Y, \xi) = -(n-1)\frac{f''}{f} A(Y). \tag{2.8}$$

Thus, $\xi$ is an eigenvector of the Ricci tensor corresponding to the eigenvalue $-(n - 1)\,\frac{f''}{f}$.

Now consider the Weyl conformal curvature tensor defined by ([26], [27])

$$\begin{aligned} C(X, Y)Z &= R(X, Y)Z - \frac{1}{(n-2)}\{g(Y, Z)QX - g(X, Z)QY \\ &\quad + S(Y, Z)X - S(X, Z)Y\} - \frac{r}{(n-1)}(g(Y, Z)X - g(X, Z)Y), \end{aligned} \tag{2.9}$$

where $Q$ denotes the Ricci operator defined by $g(QX, Y) = S(X, Y)$. Using the above curvature relations of GRW spacetime, we obtain

$$C(X, Y)\xi = 0. \tag{2.10}$$

Hence we have the following result.

**Theorem 2.1.** *Let $(M, g)$ be a generalized Robertson–Walker spacetime. Then the unit timelike vector field $\xi$ annihilates the Weyl conformal curvature tensor.*

Furthermore, since

$$\nabla_X A(Y) = g(\nabla_X \xi, Y) \tag{2.11}$$

we obtain

$$\nabla_X A(Y) = \frac{f'}{f} g(X, Y), \tag{2.12}$$

which shows that the 1–form $A$ is closed.

## III. GRW Spacetime with Second Order Symmetric Parallel Tensor

In [14], the authors investigated perfect fluid spacetimes admitting a concircular vector field and studied the properties of second order symmetric parallel tensors. In particular, they showed that the existence of such a tensor imposes strong restrictions on the geometry of the spacetime and on the equation of state of the perfect fluid. Motivated by these results, in this section we consider the same type of second order symmetric parallel tensor on a generalized Robertson-Walker spacetime. Using the curvature properties of GRW spacetimes obtained in the previous section, we analyze the geometric consequences arising from the presence of such tensors and derive analogous structural results.

Now suppose that $P$ is a $(0, 2)$ symmetric tensor which is parallel with respect to the Levi–Civita connection, that is $\nabla P = 0$. Thus

$$(\nabla_X P)(Y, Z) = 0 \tag{3.1}$$

for all vector fields $X, Y, Z$ on $M$.

Hence we obtain

$$\nabla_X P(Y, Z) - P(\nabla_X Y, Z) - P(Y, \nabla_X Z) = 0. \tag{3.2}$$

Since $P$ is parallel, we have the well-known identity [14]

$$P(R(X, Y)Z, W) + P(Z, R(X, Y)W) = 0 \tag{3.3}$$

for any vector fields $X, Y, Z, W$ on $M$.

Let us take $Z = W = \xi$ in the above relation. Then we obtain

$$P(R(X, Y)\xi, \xi) = 0. \tag{3.4}$$

Using equation (2.7) in equation (3.4) we obtain.

$$P\left(\frac{f''}{f}\{A(X)Y - A(Y)X\}, \xi\right) = 0. \tag{3.5}$$

Hence

$$\frac{f''}{f}\{A(X)P(Y, \xi) - A(Y)P(X, \xi)\} = 0. \tag{3.6}$$

Replacing $X$ by $\xi$ and using equations (2.3) and (2.4), we obtain

$$\frac{f''}{f}\{-P(Y, \xi) - A(Y)P(\xi, \xi)\} = 0. \tag{3.7}$$

Thus we have

$$P(Y, \xi) = -A(Y)P(\xi, \xi). \tag{3.8}$$

Now using the parallelism of $P$, we obtain

$$(\nabla_X P)(Y, \xi) = 0. \tag{3.9}$$

Expanding this equation we get

$$\nabla_X P(Y, \xi) - P(\nabla_X Y, \xi) - P(Y, \nabla_X \xi) = 0. \tag{3.10}$$

Using the equation (3.8) in above equation, we obtain

$$-(\nabla_X A)(Y)P(\xi, \xi) - P(Y, \nabla_X \xi) = 0. \tag{3.11}$$

Using the equation (2.5) in above equation, we obtain

$$-(\nabla_X A)(Y)P(\xi,\xi) - \frac{f'}{f}P(Y,X) = 0. \tag{3.12}$$

From equation (2.4), we have

$$(\nabla_X A)(Y) = \frac{f'}{f}g(X,Y) \tag{3.13}$$

Substituting this into the equation (3.12), we obtain

$$-\frac{f'}{f}g(X,Y)P(\xi,\xi) - \frac{f'}{f}P(Y,X) = 0, \tag{3.14}$$

which provides,

$$P(X,Y) = -P(\xi,\xi)g(X,Y). \tag{3.15}$$

Since $P$ is parallel, it follows that $P(\xi,\xi)$ is constant. Therefore we obtain

$$P = cg, \tag{3.16}$$

where $c = -P(\xi,\xi)$ is a constant.

Hence, we have the following result.

***Theorem 3.1.*** *Let $(M,g)$ be a generalized Robertson–Walker spacetime admitting a second order symmetric parallel tensor $P$. Then the tensor $P$ is a constant multiple of the metric tensor $g$.*

## IV. Gradient Ricci-Bourguignon Solitons and Almost Gradient Ricci-Bourguignon Solitons on GRW Spacetime

If a Riemannian manifold $(M,g)$ is said to admit a gradient Ricci-Bourguignon soliton then with the help of equation (1.7), we have

$$\nabla_X Df = QX + (\rho r + \lambda)X. \tag{4.1}$$

Applying the curvature operator we have

$$R(X,Y)Df = \nabla_X\nabla_Y Df - \nabla_Y\nabla_X Df - \nabla_{[X,Y]}Df. \tag{4.2}$$

From equation (4.1), we obtain

$$\nabla_X\nabla_Y Df = \nabla_X(QY) + (\rho r + \lambda)\nabla_X Y + \rho(Xr)Y. \tag{4.3}$$

Similarly

$$\nabla_Y\nabla_X Df = \nabla_Y(QX) + (\rho r + \lambda)\nabla_Y X + \rho(Yr)X. \tag{4.4}$$

Substituting into equation (4.2), we obtain

$$R(X,Y)Df = \nabla_X(QY)Y - \nabla_Y(QX) + \rho\{(Xr)Y - (Yr)X\}. \tag{4.5}$$

Now contracting over $X$ we obtain

$$S(Y,Df) = (\frac{1}{2} + \rho(n-1))(Yr). \tag{4.6}$$

Putting $Y = \xi$, we obtain

$$S(\xi,Df) = (\frac{1}{2} + \rho(n-1))(\xi r). \tag{4.7}$$

Using equation (2.8) in (4.7),
we obtain

$$-(n-1)\frac{f''}{f}(\xi f) = (\frac{1}{2} + \rho(n-1))(\xi r). \tag{4.8}$$

Thus we obtain the following result.

***Theorem 4.1.*** *Let $(M,g)$ be a generalized Robertson–Walker spacetime admitting a gradient Ricci–Bourguignon soliton. Then the potential function f satisfies*

$$-(n-1)\frac{f''}{f}(\xi f) = (\frac{1}{2} + \rho(n-1))(\xi r).$$

If a manifold $(M,g)$ is said to admit an almost gradient Ricci-Bourguignon soliton if

$$Hess\, f - S - \rho r g - \lambda(x)g = 0, \tag{4.9}$$

where $\lambda$ is a smooth function.

Thus

$$\nabla_X Df = QX + (\rho r + \lambda(x))X. \tag{4.10}$$

From equation (4.2) and equation (4.10), we obtain

$$R(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X + \rho\{(Xr)Y - (Yr)X\} + (X\lambda)Y - (Y\lambda)X. \tag{4.11}$$

Contracting the above equation over $X$, we obtain

$$S(Y,Df) = \left(\frac{1}{2} + \rho(n-1)\right)(Yr) + (n-1)(Y\lambda). \tag{4.12}$$

Putting $Y = \xi$ in above equation, we obtain

$$S(\xi,Df) = \left(\frac{1}{2} + \rho(n-1)\right)(\xi r) + (n-1)(\xi\lambda). \tag{4.13}$$

Using equation (2.8), we obtain

$$-(n-1)\frac{f''}{f}(\xi f) = \left(\frac{1}{2} + \rho(n-1)\right)(\xi r) + (n-1)(\xi\lambda). \tag{4.14}$$

Hence we have the following result.

***Theorem 4.2.*** *Let $(M,g)$ be a generalized Robertson–Walker spacetime admitting an almost gradient Ricci–Bourguignon soliton. Then the potential function $f$ satisfies*

$$-(n-1)\frac{f''}{f}(\xi f) = \left(\frac{1}{2} + \rho(n-1)\right)(\xi r) + (n-1)(\xi\lambda).$$

# V. Gradient $\eta$-Ricci Solitons on GRW Spacetime

A Riemannian manifold $(M,g)$ is said to admit a gradient $\eta$-Ricci soliton if

$$Hess\, f + S + \lambda g + \mu A \otimes A = 0, \tag{5.1}$$

where $f$ is a smooth function on $M$ and $\lambda, \mu$ are constants. This equation can be written in the equivalent form

$$\nabla_X Df = -QX - \lambda X - \mu A(X)\xi \tag{5.2}$$

for any vector field $X$ on $M$.

From equation (4.2) and equation (5.2), we obtain

$$\nabla_X \nabla_Y Df = -\nabla X(QY) - \lambda\nabla_X Y - \mu X\big(A(Y)\big)\xi - \mu A(Y)\nabla_X\xi. \tag{5.3}$$

Similarly,

$$\nabla_Y \nabla_X Df = -\nabla_Y(QX) - \lambda\nabla_Y X - \mu Y(A(X))\xi - \mu A(X)\nabla_Y\xi. \tag{5.4}$$

Substituting these expressions into the curvature identity, we have

$$R(X,Y)Df = (\nabla_Y Q)X - (\nabla_X Q)Y + \mu\{X(A(Y)) - Y(A(X))\}\xi + \mu\{A(X)\nabla_Y\xi - A(Y)\nabla_X\xi\}. \tag{5.5}$$

Now using the equation (2.8), we obtain

$$R(X,Y)Df = (\nabla_Y Q)X - (\nabla_X Q)Y + \mu\{X(A(Y)) - Y(A(X))\}\xi + \mu f'f\{A(X)Y - A(Y)X\}. \tag{5.6}$$

Now consider an orthonormal frame field and perform contraction over $X$, then we obtain

$$S(Y,Df) = -\frac{1}{2}(Yr) + \mu(n-1)\frac{f'}{f}A(Y). \tag{5.7}$$

Putting $Y = \xi$ in above equation, we obtain

$$S(\xi,Df) = -\frac{1}{2}(\xi r) - \mu(n-1)\frac{f'}{f}A(\xi). \tag{5.8}$$

Using equations (2.2) and (2.3) in above equation, we obtain

$$S(\xi,Df) = -\frac{1}{2}(\xi r) - \mu(n-1)\frac{f'}{f}. \tag{5.9}$$

.Using equation (2.7), we have

$$S(\xi,Df) = -(n-1)\frac{f''}{f}(\xi f). \tag{5.10}$$

Equating the above two expressions we obtain

$$-(n-1)\frac{f''}{f}(\xi f) = -\frac{1}{2}(\xi r) - \mu(n-1)\frac{f'}{f}. \tag{5.11}$$

Thus we obtain the following result.

***Theorem 5.1.*** *Let $(M, g)$ be a generalized Robertson–Walker spacetimeadmitting a gradient $\eta$-Ricci soliton. Then the potential function $f$ satisfies*

$$-(n-1)\frac{f''}{f}(\xi f) = -\frac{1}{2}(\xi r) - \mu(n-1)\frac{f'}{f}.$$

***Corollary 5.2.*** *If the scalar curvature r of a generalized Robertson-Walker spacetime is constant, then*

$$(n-1)\frac{f'}{f}(\xi f) = \mu(n-1)\frac{f'}{f}.$$

*Hence the potential function is determined entirely by the warping function.*

***Corollary 5.3.*** *If the potential function f is invariant along the vector field $\xi$, that is*

$$\xi(f) = 0,$$

*then the gradient $\eta$-Ricci soliton reduces to a trivial soliton.*

***Corollary 5.4.*** *Let $(M, g)$ be a generalized Robertson–Walker spacetimeadmitting a gradient $\eta$-Ricci soliton. If the warping function satisfies $f'' = 0$, then either the scalar curvature is constant or the potential function is invariant along the unit timelike vector field $\xi$.*

# VI. Gradient Quasi-Yamabe Solitons on GRW Spacetime

A Riemannian manifold $(M, g)$ is said to admit a gradient quasi-Yamabe soliton if

$$Hess\, f = (r-\lambda)g + \alpha df \otimes df, \tag{6.1}$$

where $f$ is a smooth function on $M$, $r$ is the scalar curvature, $\lambda$ is a constantand $\alpha$ is a real constant. This equation can be written as

$$\nabla_X Df = (r-\lambda)X + \alpha(Xf)Df \tag{6.2}$$

for any vector field $X$ on $M$.

Now, taking covariant derivative of above equation along $X$, we have

$$\nabla_X \nabla_Y Df = (Xr)Y + (r-\lambda)\nabla_X Y + \alpha X(Yf)Df + \alpha(Yf)\nabla_X Df. \tag{6.3}$$

Similarly

$$\nabla_Y \nabla_X Df = (Yr)X + (r-\lambda)\nabla_Y X + \alpha Y(Xf)Df + \alpha(Xf)\nabla_Y Df. \tag{6.4}$$

Substituting into the curvature identity, we obtain

$$R(X,Y)Df = (Xr)Y - (Yr)X + \alpha\{(Yf)\nabla_X Df - (Xf)\nabla_Y Df\}. \tag{6.5}$$

Using equation (6.2) in above equation, we obtain

$$R(X,Y)Df = (Xr)Y - (Yr)X + \alpha(r-\lambda)\{(Yf)X - (Xf)Y\}. \tag{6.6}$$

Now contracting the above equation over $X$ with respect to an orthonormal frame field, we obtain

$$S(Y, Df) = -(n-1)(Yr) + \alpha(r-\lambda)\{(Yf) - (Df)A(Y)\}. \tag{6.7}$$

Putting $Y = \xi$ in the above equation, we obtain

$$S(\xi, Df) = -(n-1)(\xi r) + \alpha(r-\lambda)\{(\xi f) - (Df)A(\xi)\}, \tag{6.8}$$

which gives

$$S(\xi, Df) = -(n-1)(\xi r) + \alpha(r-\lambda)\{(\xi f) + (Df)\}. \tag{6.9}$$

Using equation (2.7) in the above equation, we obtain

$$-(n-1)\frac{f''}{f}(\xi f) = -(n-1)(\xi r) + \alpha(r-\lambda)\{(\xi f) + (Df)\}. \tag{6.10}$$

Thus we obtain the following result.

***Theorem 6.1.*** *Let $(M, g)$ be a generalized Robertson–Walker spacetimeadmitting a gradient quasi–Yamabe soliton. Then the potential function $f$ satisfies*

$$(n-1)\frac{f''}{f}(\xi f) = (n-1)(\xi r) - \alpha(r-\lambda)\{(\xi f) + (Df)\}.$$

***Corollary 6.2.*** *If the scalar curvature r of the GRW spacetime is constant, then the potential function satisfies*

$$(n-1)\frac{f''}{f}(\xi f) = -\alpha(r-\lambda)\{(\xi f) + (Df)\}.$$

***Corollary 6.3.*** *If the potential function f is invariant along $\xi$, that is*

$$\xi(f) = 0,$$

*then the gradient quasi–Yamabe soliton reduces to a trivial soliton.*

***Corollary 6.4.*** *If $\alpha = 0$, then the gradient quasi-Yamabe soliton reducesto a gradient Yamabe soliton on a generalized Robertson-Walker spacetime.*

## VII. Gradient $\eta$-Ricci-Bourguignon Solitons on GRW Spacetime

A Riemannian manifold $(M, g)$ is said to admit a gradient $\eta$-Ricci-Bourguignon soliton if

$$Hess\, f + S + \rho r g + \lambda g + \mu A \otimes A = 0, \tag{7.1}$$

where $\rho,\ \lambda,\ \mu$ are real constants and $r$ denotes the scalar curvature.

This equation can be written as

$$\nabla_X Df = -QX - (\rho r + \lambda)X - \mu A(X)\xi \tag{7.2}$$

for any vector field $X$.

Applying the curvature identity and using this equation, we obtain

$$\nabla_X \nabla_Y Df = -\nabla_X(QY) - (\rho r + \lambda)\nabla_X Y - \rho(Xr)Y - \mu X(A(Y))\xi - \mu A(Y)\nabla_X \xi. \tag{7.3}$$

Similarly

$$\nabla_Y \nabla_X Df = -\nabla_Y(QX) - (\rho r + \lambda)\nabla_Y X - \rho(Yr)X - \mu Y(A(X))\xi - \mu A(X)\nabla_Y \xi. \tag{7.4}$$

Substituting equations (7.3) and (7.4) into the curvature identity, we obtain

$$\begin{aligned} R(X,Y)Df &= (\nabla_Y Q)X - (\nabla_X Q)Y + \rho\{(Xr)Y - (Yr)X\} \\ &\quad + \mu\{X(A(Y)) - Y(A(X))\}\xi + \mu\{A(X)\nabla_Y \xi - A(Y)\nabla_X \xi\}. \end{aligned} \tag{7.5}$$

Using equation (2.4), we obtain

$$\begin{aligned} R(X,Y)Df &= (\nabla_Y Q)X - (\nabla_X Q)Y + \rho\{(Xr)Y - (Yr)X\} \\ &\quad + \mu\{X(A(Y)) - Y(A(X))\}\xi + \mu f' f\{A(X)Y - A(Y)X\}. \end{aligned} \tag{7.6}$$

Now contracting over $X$ with respect to an orthonormal frame field, we obtain

$$S(Y, Df) = \tfrac{1}{2}(Yr) + \rho(n-1)(Yr) + \mu(n-1)\tfrac{f'}{f}A(Y). \tag{7.7}$$

Putting $Y = \xi$, we obtain

$$S(\xi, Df) = \left(\tfrac{1}{2} + \rho(n-1)\right)(\xi r) - \mu(n-1)\tfrac{f'}{f}. \tag{7.8}$$

Using equation (2.7), we obtain

$$-(n-1)\tfrac{f''}{f}(\xi f) = \left(\tfrac{1}{2} + \rho(n-1)\right)(\xi r) - \mu(n-1)\tfrac{f'}{f}. \tag{7.9}$$

Thus we arrive the following result.

***Theorem 7.1.*** *Let $(M, g)$ be a generalized Robertson–Walker spacetimeadmitting a gradient η–Ricci–Bourguignon soliton. Then the potential function $f$ satisfies*

$$(n-1)\frac{f''}{f}(\xi f) = -\left(\frac{1}{2} + \rho(n-1)\right)(\xi r) + \mu(n-1)\frac{f'}{f}.$$

***Corollary 7.2.*** *If the scalar curvature r is constant, then*

$$(n-1)\frac{f''}{f}(\xi f) = \mu(n-1)\frac{f'}{f}.$$

***Corollary 7.3.*** *If the potential function is invariant along $\xi$, that is*

$$\xi(f) = 0,$$

*then the gradient η-Ricci-Bourguignon soliton becomes trivial.*

***Corollary 7.4.*** *If $\mu = 0$, then the gradient η-Ricci-Bourguignon soliton reduces to a gradient Ricci-Bourguignon soliton on a generalized Robertson-Walker spacetime.*

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