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Perfect Partitions

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Abstract
A partition, P, of the natural number, n, is perfect if (a) every natural number less than n has a partition of terms of P, and (b) these partitions are unique. Thus, 2 + 2 + 1 is a perfect partition of 5, while 3 + 2 + 1 is not a perfect partition of 6.

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I. Partitions

A *partition* of the natural number, *n*, is an expression of *n* as a sum of natural numbers. [1]

Example: 5 = 2 + 2 + 1. The *length* of a partition is the number of terms. Of course, 5 has several partitions: 5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1, and the *trivial* partition, 1 + 1 + 1 + 1 + 1.

An *odd* partition consists solely of odd numbers, while a *distinct* partition consists of distinct numbers. Euler showed that the natural number, *n*, has the same number of odd partitions and distinct partitions. [2]

Perfect Partitions

Behold the unordered partitions of 7.

7
6 + 1
5 + 2
5 + 1 + 1
4 + 3
4 + 2 + 1
4 + 1 + 1 + 1
3 + 3 + 1
3 + 2 + 2
3 + 2 + 1 + 1
3 + 1 + 1 + 1 + 1
2 + 2 + 2 + 1
2 + 2 + 1 + 1 + 1
2 + 1 + 1 + 1 + 1 + 1
1 + 1 + 1 + 1 + 1 + 1 + 1

The four partitions in bold font are *perfect partitions* of 7. In such a partition, each number from 1 to 7 can be expressed *uniquely* as a sum of terms in that partition.

II. Several Results

Theorem 1: Let $n \geq 5$, where $n = 2 \pmod 3$. Then *n* has the perfect partition, 3 + … + 3 + 1 + 1.

Proof: $n = 3k + 2 = 3k + 1 + 1 = 3 + 3 + \ldots + 3 + 1 + 1$, a partition with *k* ‘3’s. The reader should prove that it has the uniqueness property.

Theorem 2: Let $n = k - 1 \pmod k$ and let $n > k$. Then $n = mk + (k - 1)$. This produces the perfect partition, $k + \ldots + k + 1 + \ldots + 1$, containing *m* ‘*k*’s and $k - 1$ ‘1’s.

Note: The $k - 1$ ‘1’s in Theorem 2 can be replaced by any perfect partition of $k - 1$. Thus, since $15 = 3 \pmod 4$, we have 15 = 4 + 4 + 4 + 1 + 1 + 1 or 4 + 4 + 4 + 2 + 1, where 1 + 1 + 1 is replaced by 2 + 1.

Theorem 3: No term in a perfect partition of 2*k* is greater than or equal to $k + 1$.

Proof: If a partition of 2*k* starts with $(k + 1)$, the remainder of the terms must add up to $k - 1$. Then no terms of the partition has sum *k*. Done!

Theorem 4: No term in a perfect partition of 2*k* equals *k*.

Proof: The balance of the partition would add up to *k*, violating the uniqueness property, as *k* would have two different expressions.

Remark: $2^n - 1$ has the perfect partition, $2^{n-1} + 2^{n-2} + \ldots + 2 + 1$, a geometric sequence.

Theorem 5: Let M be the largest term of a perfect partition, $a_1 + a_2 + \dots + a_r$, of n. Then $M + n$ has the perfect partition, $M + a_1 + a_2 + \dots + a_r$.
Theorem 6: Let n have the perfect partition, $a_1 + a_2 + \dots + a_r$. Then $2n + 1$ has the perfect partition, $(n + 1) + a_1 + a_2 + \dots + a_r$. (Proof left to the reader.)
Theorem 7: Let a and b be terms in a perfect partition such that $a > b > 1$. Then $a - b \geq 2$.
Proof: Since 1 is a term of every perfect partition, if $a - b = 1$, we would have $a = b + 1$, violating the uniqueness requirement.

Ordered Factorizations and Perfect Partitions
Definition: Given the natural number, $n \geq 2$, an *ordered factorization* of n is a product of factors (excluding 1) that equals n, where order matters.
Example: 12 has the eight ordered factorizations,
12 2×6 6×2 3×4 4×3 $2\times2\times3$ $2\times3\times2$ $3\times2\times2$
Theorem 8: The number of perfect partitions of n equals the number of ordered factorizations of $n + 1$. [3]
Theorem 9: **(1)** Let p be prime. Then $p - 1$ has only the trivial perfect partition. **(2)** Let p and q be distinct primes. Then $pq - 1$ has three perfect partitions.
Proof: p is the only ordered factorization of p. Furthermore, pq, $p\times q$, and $q\times p$, are the three ordered factorization of pq. Then use Theorem 8.

References

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[2]. M.Lewinter, J.Meyer, Elementary Number Theory With Programming, Wiley, 2015.
[3]. Goulden And Jackson, Combinatorial Enumeration, Wiley, 1983.