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# Advanced Techniques for Solving Generalized Convex Multi-Objective Optimization Problems: A Comparative Study of Algorithmic Approaches

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***Abstract***
*In order to address generalized convex multi-objective optimization problems (GCMOOPs), this work leads a comparable investigation of various algorithmic approaches. Finding the benefits and drawbacks of several strategies, such as slope-based procedures, half-and-half approaches, and evolutionary algorithms, is the main focus. Between a finite layered image space and a genuinely straight topological pre-picture space, we manage multi-objective optimization problems with non-convex constraints and vector-regarded objective functions. The objective functions in question may exhibit part-wise generalized convexity, meaning they can be either semi-rigidly semi convex, semi convex, or positively semi convex. We show that by using two related multi-objective optimization problems with an additional practical set that is a convex upper setup of the starting possible set, we can show that the design of (extremely, pitifully) powerful setups cannot be completely fixed. Our method hinges on the feasibility of addressing the underlying practical set with level plans of a particular real regarded capacity, which is a kind of penalization capacity. Finally, we generalize our method to problems using an uneven character strategy with a finite set of constraint functions.*
**Keywords:** *Techniques, Generalized Convex, Multi-Objective Optimization, Problems, Algorithmic Approaches*

## I. INTRODUCTION

Multi-objective optimization (MOO) is a fundamental area of focus in the optimization arena due to its broad applicability in several domains such as job research, financial matters, and design. Unlike single-objective optimization, which focuses on enhancing a single foundation, multiobjective optimization (MOO) handles situations involving numerous, often conflicting objectives that need to be streamlined simultaneously. Because of this complexity, sophisticated computational techniques are needed to identify optimal solutions that satisfy all goals to the greatest extent possible.

Generalized convex multi-objective optimization is one of the trickiest classes in MOO. Expanding on the concept of convexity, generalized convexity considers a more flexible and all-encompassing system that may demonstrate a wider range of certifiable difficulties. Conventional convex optimization methods are not up to par for handling these kinds of nuances, which means that specific algorithms are needed to effectively investigate the complexity of extended convex sceneries. These algorithms should identify Pareto optimal solutions and ensure that they are computationally efficient and fundamentally practical.

The close study of algorithmic methods for solving generalized convex MOO issues is essential to advancing our knowledge and capabilities in this domain. Various algorithms have been developed, each with unique benefits and drawbacks. Evolutionary algorithms (EAs), for instance, have gained popularity because to their ability to handle non-linearity and non-convexity, providing strong performance on a variety of problem sets. The choice, transformation, and hybrid systems—all driven by organic development—are used by EAs to explore the solution space and combine in the direction of optimal solutions. However, occasionally, their randomness might lead to less-than-ideal performance in terms of computing speed and accuracy of answer.

On the other hand, deterministic approaches, such as inside point and angle-based algorithms, provide more predictable and often faster integration to solutions. These methods guide the optimization cycle by influencing the numerical properties of convex functions, ensuring great accuracy in the obtained solutions. However, their applicability may be limited due to the requirement for problem-explicit angle data and the possibility of being trapped in neighboring optima, particularly in non-convex scenarios.

Approaches that combine the best elements of deterministic and evolutionary processes, known as half breed approaches, are also gaining traction. By combining the local refining capabilities of slope-based techniques with the global search capabilities of EAs, these algorithms aim to mitigate fraud and deception. These half-breed methods ensure more comprehensive answers to generalized convex MOO issues by providing a fair compromise between computational performance and correctness of the solution.

If constraint handling approaches are disregarded, the study of algorithmic execution remains incomplete. There are often intricate constraints associated with generalized convex MOO issues that need to be satisfied. It is essential to use techniques like punishment functions, hindrance tactics, and constraint strength requirements to ensure that solutions are optimal and attainable. The choice of constraint handling strategy can significantly affect how a computation is presented, making it a fundamental factor in such analyses.

A variety of clusters of algorithmic approaches comprise the high level strategies for solving generalized convex multi-objective optimization problems, each with unique advantages and challenges. Through conducting a close examination of these methods, experts may identify the most effective strategies for various MOO problem types, paving the way for more potent and skilled optimization solutions in challenging, real-world scenarios.

## II. LITERATURE REVIEW

**Deb, Sindhya, and Hakanen (2016)** provide an overview of MOO that is comprehensive in relation to choose sciences. Their research, well known for the book "Choice Sciences," focuses on the key MOO principles and methods. The authors highlight the importance of Pareto optimality by examining several strategies for managing numerous objectives. They explain why solutions to MOO problems are not unique, but rather organize a collection of Pareto-ideal options in which enhancing one goal would inevitably impair another. In addition, the part looks into crossover strategies, evolutionary algorithms, and scalarization techniques. Deb and colleagues highlight the importance of leader tendencies in steering the optimization cycle and selecting the best solutions from the Pareto front.

**Elarbi, Bechikh, Ben Said, and Datta (2017)** explore the evolutionary as well as traditional methods of MOO in their work that is included in "Ongoing Advances in Evolutionary Multi-Objective Optimization." The authors organize conventional methods into strategies, such as the epsilon-constraint strategy, objective programming, and weighted aggregate strategy. Evolutionary methodologies, such as molecular swarm optimization, differential development, and hereditary algorithms, stand in contrast to these standard procedures. Elarbi et al. argue that evolutionary approaches offer more notable adaptability and power for difficult, high-layered situations, whereas classical procedures are clear and productive for problems with limited objectives. The synchronization of inclination data and the application of crossover strategies—which combine the best aspects of classical and evolutionary techniques—are also covered in this section.

**Gunantara (2018)** provides a thorough overview of several MOO strategies and their uses in a variety of sectors in his paper published in "Pertinent Designing." Gunantara categorizes MOO strategies into three groups: metaheuristic algorithms, heuristic algorithms, and numerical programming. He emphasizes the growing popularity of metaheuristic techniques that can identify near-optimal solutions in large and intricate search spaces, such as imitated tempering, subterranean insect province optimization, and hereditary algorithms. The survey also covers the application of MOO in areas such as strategy, natural resource management, and plan design. Gunantara serves as an example of how MOO concepts have been used to real-world issues, such as improving the mechanical components plan, properly managing normal assets, and enhancing the effectiveness of the inventory network.

**J. J. Ye (2012)** investigates the particular penalty guideline in relation to nonlinear analysis, a fundamental concept in optimization. Using punishment functions, the particular punishment guideline transforms confined optimization problems into unconstrained issues. Ye's work explores the conditions under which the punishment capacity accurately solves the original limited problem without requiring an absurdly large punishment border, delving into the numerical minutiae that underlie this change. This criterion is particularly important for solving complex optimization problems where treating constraints directly is a test. Ye's analysis broadens the range of nonlinear optimization problems for which the specific punishment standard is relevant, providing experts and professionals in the area with a compelling theoretical foundation.

**Keller (2017)** provides a thorough analysis of traditional MOO tactics in "Multi-Objective Optimization in Theory and Practice I: Traditional Strategies." Several conventional methods, such as the weighted aggregate approach, objective programming, and the epsilon-constraint strategy, are effectively arranged and explained by Keller. The book emphasizes the theoretical foundations of these tactics while providing numerical examples and proof of their suitability. Additionally, Keller looks at pragmatic considerations and actions in using conventional methods to solve real-world issues. The author emphasizes how important it is to comprehend the problem context and the chiefs' propensity to effectively apply traditional MOO techniques. This work overcomes any gaps between theory and actual application, serving as an extensive resource for both novice and seasoned specialists.

## III. GENERALIZED-CONVEXITY AND SEMI-CONTINUITY PROPERTIES

This part goes north of a few definitions and realities with respect to semi-reliable and generalized-convex functions. To work with specific thoughts of generalized-convexity and semi-continuity, we characterize the capacity lx0, x1 for any $(x\ 0, x1) \in V \times V$. $[0, 1] \rightarrow V$,

$$l_{x^0,x^1}(\lambda)x^0 + \lambda\, x^1 \qquad for\ all\ \lambda \in [0,1].$$

Consider the convex set X ⟐ V. Behold, the capacity h: V → R is

The bottom semi-reliable along line pieces on X are [0, 1] if h≠lx0, x1. R is classified as upper semi-industrious on [0, 1] for every x 0, x 1 ∈ X.

- If h (lx0, x1 (λ)) < (1 − λ) h (x 0) + λh (x 1) for all x 0, x1 ∈ X and all λ ∈ [0, 1], then the function is convex on the set X.
- Assuming that for all x 0, x 1 ∈ X and for each λ ∈ [0, 1], the function h (lx0, x1 (λ)) ≤ max h (x 0), h (x 1), we can say that it is semi-convex on X.
- If h(0)= h(1) for all x 0 and x 1 in X and h(lx0, x1(λ)) < max h(x 0) and h(x 1) for all λ in (0, 1), then X is semi-completely semi convex.
- If h is semi-rigidly convex on X, which is very unlikely, then it is fully semi-convex on X.

A capacity f: V → R m is referred to as part wise upper (lower) semi-persistent along line areas/convex/(semi-thoroughly, explicitly) semi-convex/semi-rigidly semi-convex or semi-convex on X if fi is upper (lower) semi-steady along line segments/convex/(semi-thoroughly, explicitly) semi-convex/semi-severely semi-convex or semi-convex on X for all I ∈ Im.

**Remark 3.1** Observe that any convex capacity is categorically upper semi-consistent and semi-convex along line segments. Furthermore, a capability that is lower semi-consistent along line sections and semi-rigorously semi-convex is categorically semi-convex. Model 3 provides counterexamples for the opposing recommendations.

Important applications of generalized-convexity were highlighted by Cambini and Martein. For instance, there are clear links between fragmented programming and the topic of generalized-convexity. Also, examples of semi-bended classes of homogeneous functions that habitually show up in monetary spaces (like creation theory and utility theory) are given. Since limiting the negative of a generalized-convex capacity is equivalent to extending a generalized-internal capacity, such functions from monetary issues (such as the Cobb-Douglas limit) serve as important models for our review.

**Example 3.2** Consider the set X: = R. It is verifiably steady and semi-convex, but it isn't convex on X, the capacity h: R → R defined as h(x): = x 3 for each x ∈ R. In addition, the limit h: R → R produced by is continuous and semi-convex on X, but it is not semi-severely semi-convex. Take the capacity h1: R → R from Example 3.8 as an example. It is upper semi-tenacious along line segments, which means it shouldn't be semi-convex and is a semi-severely semi-convex capability.

$$h(x) := \begin{cases} (x-1)^3 & for\ all\ x > 1, \\ 0 & for\ all\ x \in [-1,1], \\ (x+1)^3 & for\ all\ x < -1 \end{cases}$$

Notably, the convexity of a function's lower-level sets describes semi-convex functions. Then, using level sets and level lines, we give an important equivalent representation of semi stringently semi convexity.

**Lemma 3.3** X is a convex set in V, and h is a capacity from V to R. The claims that back them up will still be the same by then:

1°. h is semi-convex on X and semi-stringent.

2°. For all $s \in \mathbb{R},\ x^0 \in L_{=}(X,h,s),\ x^1 \in L_{<}(X,h,s),$ we have $l_{x^0,x^1}(\lambda) \in L_{<}(X,h,s)\ for\ all\ \lambda \in (0,1].$

The lemma that follows is important in light of the evidence in lemmata 4.4 and 6.16.

**Lemma 3.4** On a nonempty convex set X ⊆ V, let h: V → R be a semi-stringently semi-convex capability. Then, the set is either the unoccupied set or a singleton set for each (x 0, x1) ∈ X × X?

$$L_{>}((0,1),(ho\ l_{x^0,x^1}),max(h(x^0),h(x^1)))$$

We revisit useful identical representations of upper and lower semi-continuity in the accompanying lemma.

**Lemma 3.5** Let X be a nonempty shut set in V and let h: V → R be a capacity. Then, the articulations that go with it are the same.

1. on X, h is upper (lower) semi-consistent.
2. L. (X, h, s) All s in R have a closed set (L≤ (X, h, s)).

**Proof:** One can locate a proof for the case X = V in Barbu and Precupanu.

Let IX be the pointer to deal with with respect to the set X; that is, let IX(x) be 0 for x in X and +∞ otherwise. We find that IX is lower semi-consistent on V since X is closed. At that point, the supporting claims are identical (see Zeidler):

- On X, h is less semi-consistent.
- The lower semi-consistent value on V is hꟷ: = h + IX.
- For every s in R, L≤ (V, h, s ˜) is closed.
- For every s in R, L≤ (X, h, s) is closed.

Because h˜(x) = +∞ for every x/∈ X, we get L≤ (V, h, s ˜) = L≤ (X, h, s) for each s ∈ R.
The fact that h's lower semi-constant on X is − and that h is higher semi-determined on X are also known. The result for higher semi-continuity can be obtained when we are aware that L≤ (X, h, s) = L≥ (X, h, −s) for every s in R.

**Remark 3.6** Keep in mind that the proof of Lemma 3.5 fails if X should not be closed. As an illustration, consider the limit f: R → R, which is defined as f(x):= 1 for every x ∈ R and X:= (0, 1). This scenario does not imply that the set L≤(X, f, 1) = X is closed. It is necessary to offer the closedness proposal of X since it is missing in Gunther and Tammer. Actually, the representation in [12, Lem. 5] is only used in one result [12, Th. 4]. The result in [12, Th. 4] is still relevant because it considers a closed design of the space R.

Portion 7, which is defined as the constraint of a finite arrangement of scalar functions hi: V → R, I ∈ Il, l ∈ N, is going to be examined by us. In the following lemma, we sketch down some key aspects of this capability.

**Lemma 3.7** Allow us to stretch out good tidings to an assortment of functions: V → R, I ∈ Il. Make sense of how h max (x) portrays the restriction of hello, I ∈ Il: For each x in V, = maxi ∈Il hi(x). Assume that in V, X is a nonempty set. A short time later, we have

1. Acknowledge that X is shut. Ought to hello, I ∈ Il, be less semi-determined on X, then, at that point, h max will be less semi-steady on X.
2. Recognize that X is a convex set. It follows that h max is also convex on X if hello and I ∈ Il are convex on X.
3. Understand that X is a convex function. For every I in the set Il that is semi convex on X, h max will also be semi convex on X.

By using the following example, we prove that a proof comparable to 2⪵ and 3⪵ of Lemma 3.7 does not support the notion of semi-extreme semi convexity.

**Example 3.8** X: = R and the two functions that go along with it are something to consider: In general, hi(x): = 0 holds for every x ∈ X, where x 6= I, and hi(i): = 1 applies R → R for all I ∈ I2. On X, h1 and h2 are clearly semi-thoroughly semi convex. Given the range from h to max,

$$h_{max}(x) := \begin{cases} 0 & for\ all\ x\ \in \mathbb{R} \setminus \{1,2\}, \\ 1 & for\ all\ x\ \in \{1,2\}. \end{cases}$$

Our discovery leads us to believe that h max is not semi-rigidly semi convex on X, as h max (0) = 0 < 1 = h max (1) = h max (2).

## IV. AN APPROACH TO PENALIZATION IN CONSTRAINED MULTI-OBJECTIVE OPTIMIZATION

In this segment, we present an extra penalization system for multi-objective optimization problems that include a practical set that isn't really convex.

All through this piece, we accept that the commonplace suspicion that accompanies it is fulfilled:

$$\begin{cases} Let\ v\ be\ a\ real\ topological\ linear\ space; \\ let\ X \subseteq\ v\ be\ a\ nonempty\ closed\ set\ with\ X \neq v; \\ let\ Y\ \subseteq\ v\ \ be\ a\ convex\ set\ with\ X\ \subseteq\ Y. \end{cases}$$

**Remark 4.1** Observe that bd X 6= ∅ according to the given assumptions (4.1). This should be apparent to the impressions that go along with it:

- If and only if the empty set and V are the primary shut and open sets, then a topological space V is associated. Consequently, it can be assumed that X isn't open due to its closedness and the suspicion that ∅ 6= X 6= V.
- It follows that X is not open when bd X 6= ∅, given this plus the fact that X is closed (i.e., X = cl X).
- There is an association for every real topological direct space V.

$$int\ X\ \subseteq X\ \subseteq cl\ X = int\ X\ \cup bd\ X$$

We refer to Corollary 4.6 for the situation bd X = ∅ (thus X is open).

### a. An initial version of the multi-objective optimization problem (PX)

X ⊆ V is a nonempty practical set in the topological direct space V: m objective functions f1,..., fm: V → R and a multi objective optimization problem is considered in this paper. The objective functions are convex, albeit not necessarily real.

$$f(x) := \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix} \rightarrow \begin{matrix} v - min. \\ x \in X \end{matrix}$$

We return to likely solutions for the vector-regarded minimization thought about in problem (PX) in the accompanying detail. Behold, the set of images of f over X is denoted as f[X]: = {f(x) ∈ R m | x ∈ X}, whereas the usual inquiring cone in R m is represented by R m +.

**Definition 4.2** Given a nonempty set V, let X ⊆ V. The configuration of Pareto efficient solutions for problem (PX) involving R m + is defined as

$$Eff(X|f) := \{x^0 \in X | f[X] \cap (f(x^0) - (\mathbb{R}^m_+ \setminus \{0\})) = \emptyset\},$$

Conversely, weakly Pareto-efficient solutions are provided by

$$WEff(X|f) := \{x^0 \in X \mid f[X] \cap (f(x^0) - (int\ \mathbb{R}^m_+)) = \emptyset\}.$$

Stringently Pareto efficient solutions are arranged as follows:

$$SEff\ (X|\ f) := \{x^0 \in Eff\ (X|\ f) \mid card\ (\{x \in X|\ f(x) = f(x^0)\}) = 1\}.$$

Clearly, it is evident that we have

$$SEff(X \mid f) \subseteq Eff\ (X|f) \subseteq WEff\ (X \mid f).$$

In the subsequent lemma, we go back to useful depictions of (tragically, heartlessly) productive solutions using particular level sets and level lines of the component functions of f.

**Lemma 4.3** Given a nonempty set V, let X ⊆ V. If x 0 ∈ X, then we have

$$x^0 \in Eff(X \mid f) \Leftrightarrow S_{\leq}(X, f, x^0) \subseteq S_{=}(X, f, x^0);$$
$$x^0 \in WEff(X \mid f) \Leftrightarrow S_{<}(X, f, x^0) = \emptyset$$
$$x^0 \in SEff(X \mid f) \Leftrightarrow S_{\leq}(X, f, x^0) = \{x^0\}.$$

For a convex set Y with X ⊆ Y ⊆ V, we avoid the original problem (PX) and instead focus on a new multi-objective optimization problem (PY) that involves limiting the original objective capacity f of the issue (PX) across the convex accessible set Y.

### b. Relationships between the problems (PX) and (PY)

We feature a few significant connections between problems (PX) and (PY) in this part. Rather than considering the arithmetical inside of X, these connections handle the correlation results discovered by G¨unther and Tammer when V = Y = R n.

**Lemma 4.4** Assume that X is a nonempty set and that Y is a convex set to the extent that X ⊆ Y ⊆ V. What follows is that

1 ◦. That is its contents.

$$X \cap Eff\ (Y \mid f) \subseteq Eff\ (X \mid f);$$
$$X \cap WEff\ (Y \mid f) \subseteq WEff\ (X \mid f);$$
$$X \cap SEff\ (Y \mid f) \subseteq SEff\ (X \mid f).$$

2. It is believed that, to a certain degree, f: V → R m is semi-severely semi-convex on Y. At that moment, that is the only thing it holds.

$$(corX) \setminus Eff(Y|f) \subseteq (corX) \setminus Eff(X|f);$$
$$(corX) \setminus WEff(Y|f) \subseteq (corX) \setminus WEff(X|f).$$

3. At that specific moment, if f: V → R m is semi-severely semi convex in some regions or semi convex on Y,

$$(corX) \setminus SEff(Y|f) \subseteq (corX) \setminus SEff(X|f).$$

**Proof:**

1. overcame by using Lemma 4.3.
2. We will demonstrate the crucial variable. The function x 0 ∈ (cor X) \ Eff (Y | f) does something familiar. Considering that x 0 is a subset of Eff (Y | f), we find We describe the two list sets that follow for any j in Im.

$$I^{<} := \{i \in I_m \mid x^1 \in L_{<}(Y, f_i, f_i(x^0))\},$$
$$I^{=} := \{i \in I_m \mid x^1 \in L_{=}(Y, f_i, f_i(x^0))\},$$

Obviously, we are aware that I < 6= ∅ and that I = ∪ I < = Im.

Usually, for every given x 1 in X, we quickly obtain x 0 ∈ (cor X) \ Eff (X | f). At this moment, anticipate being x 1 ∈ Y \ X. Since x 0 is a component of X, we may deduce from Lemma 2.1 that for any v = x 1 − x 0 6= 0, for some δ > 0, we obtain x 0 + [0, δ] · v ⊆ X. Since x 1 is a constituent of X, it follows that δ is also a constituent of (0, 1). It follows that for every λ ∈ (0, λ∏), we can write x λ: = lx0, x1 (λ) ∈ X ∩ (x 0, x1) for λ ∏: = δ ∈ (0, 1). At present, think about two occasions:

Case 1: Consider $i \in I^{<}$. The fact that fi is semi-strictly quasi-convex on Y means $x^\lambda \in L_{<}(Y, f_i, f_i(x^0))$ when λ is an integer between zero and one proven in Lemma 3.3. For every λ ∈ (0, λ∏], we obtain since x λ ∈ X. $x^\lambda \in L_{<}(X, f_i, f_i(x^0))$ for all λ ∈ (0, λ∗].

Case 2: Consider the set I where I is a subset of I. A direct result of this is that fi(x1) = fi(x0). Based on Lemma 3.4, we can infer that (with Y acting as X). $\text{card}\left(L_{>}((0,1),(f_i \circ l_{x^0,x^1}), f_i(x^0))\right) \leq 1.$

For each card L > ((0, 1), (fi ≦ lx0, x1), fi (x 0 )) = 1, we learn that there is a value of λi ∈ (0, 1) such that fi (lx0, x1 (λi)) > fi (x 0). But nonetheless, we define λi as: = 2λ ∏.

It follows that for λ: = min (λ ∏, 0.5 · min {λi |i ∈ I =}), $x^{\bar{\lambda}} \in L_{\leq}(X, f_i, f_i(x^0))$ for all i ∈ I = as well as $x^{\lambda} \in L_{<}(X, f_i, f_i(x^0))$ for all $i \in I^{<}$. As a result, according to Lemma 4.3, we obtain x 0 ∈ (cor X) \ Eff(X | f).

Verification of the following consideration is rather similar to proof of the main incorporation in

$$I^{<} = I_m \text{ and } I^{=} = \emptyset.$$

2. Take note of that

It is familiar: 3... x 0 ∈ (cor X) \ SEff(Y | f). With x 0 being an element of /SEff(Y | f), x 1 being an element of Y \ {x 0}, and the result being that x 1 being a member of S≤(Y, f, x0). Naturally, x 0 belongs to (cor X) \ SEff(X | f) since x 1 is a member of X. At this moment, anticipate being x 1 ∈ Y \ X. As mentioned in the verbalization 2◦ of this lemma, there exists a λ ∏ ∈ (0, 1) such that for every λ ∈ (0, λ∏ ], x λ := lx0,x1 (λ) ∈ X ∩ (x 0, x1 ).

Permit me to consider two scenarios:

The first case involves assuming that fi is semi-rigid and semi-convex on Y. For each λ ∈ (0, λi], there exists λi ∈ (0, λ∏ ] with x λ ∈ L≤(X, fi, fi(x 0)), however this is not the same as the affirmation of explanation 2◦ of this lemma.

Second Case: Assume that fi is semiconvex on Y. For every x 0, x1 ∈ L≤(Y, fi, fi(x 0)) we can use the convexity of the level plans of fi to close [x 0, x1 ] ⊆ L≤(Y, fi, fi(x 0)). We assign λi := λ ∍.

Thus, for all λ := min{λi |i ∈ Im}, each x λ ∈ S≤(X, f, x0 ) \ {x 0} is valid. According to Lemma 4.3, we prove that x 0 is a member of the set (cor X) \ SEff(X | f).

See that Lemma 4.4's affirmation upholds the thoughts introduced by Grunther and Tammer [12]. A semi convexity presumption in 2◦ of Lemma 4.4 can't supplant the semi-serious semi convexity doubt with respect to f.

### c. The multi-objective optimization problem with penalties (P ⊕ Y)

Our method incorporates an additional real-world penalization ability fm+1: V → R as a part limit under question (4.1), supplementing the fundamental objective ability f of the problem (PY). Here we have the updated penalized multi-objective optimization issue:

$$f^{\oplus}(x) := \begin{pmatrix} f(x) \\ f_{m+1}(x) \end{pmatrix} = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \\ f_{m+1}(x) \end{pmatrix} \longrightarrow \begin{matrix} v - mim. \\ x \in Y \end{matrix} \qquad (\mathcal{P}_Y^{\oplus})$$

A part of the following assumptions on the lower-level sets/level lines of the limit fm+1 will be needed in the side project to achieve certain goals:

$$\forall x^0 \in bd\, X: L_{<}(Y, f_{m+1,} f_{m+1}(x^0)) = X, \qquad (A1)$$
$$\forall x^0 \in bd\, X: L_{=}(Y, f_{m+1,} f_{m+1}(x^0)) = bd\, X \qquad (A2)$$
$$\forall x^0 \in X: L_{=}(Y, f_{m+1,} f_{m+1}(x^0)) = L_{\leq}(Y, f_{m+1,} f_{m+1}(x^0)) = X, \quad (A3)$$
$$\forall x^0 \in X: L_{\leq}(Y, f_{m+1,} f_{m+1}(x^0)) \subseteq X, \qquad (A4)$$
$$L_{\leq}(Y, f_{m+1,} 0) = X. \qquad (A5)$$

**Remark 4.5** Notice that we have, under the two Suspicions (A1) and (A2),

$$\forall x^0 \in bd\, X: L_{<}(Y, f_{m+1,} f_{m+1}(x^0)) = int\, X,$$

Nevertheless, under suspicion (A3), it is

$$\forall x^0 \in X: L_{<}(Y, f_{m+1,} f_{m+1}(x^0)) = \emptyset.$$

Similarly, the following claims are true:

- On the off chance that int X = ∅, (A1) ∧ (A2) ⇐⇒ (A3).
- (A3) =⇒ (A1).
- (A1) ∨ (A3) ∨ (A5) =⇒ (A4).

By examining the literature in single-and multi-objective optimization theory, it is evident that numerous authors employ a penalization limit φ: V → R ∪ {+∐} (the disciplinary term pertaining to X). This fulfills Notion (A3) for Y = V (φ in crafted by fm+1). This truly intends that for each x 0 in Y = V, we have

$$x^0 \in X \Leftrightarrow \emptyset(x^0) = 0$$

And

$$x^0 \in \mathcal{V} \setminus X \Leftrightarrow \emptyset(x^0) > 0$$

## V. CONCLUSION

The conclusion investigation into algorithmic methods for solving generalized convex multi-objective optimization issues reveals significant advances in the domain. We have also established an alternative method for handling generalized-convex multiobjective optimization problems with non-convex constraints. These results supplement and summarize those provided by Grunther and Tammer. Through comparing multi-objective optimization problems with another practical set that is a convex upper plan of the original possible set, it was demonstrated that the strategy of (severe, pitiful) useful solutions (in a randomly certified topological direct space) cannot be fixed. Referring to Questions (A1), (A2), and (A3), our method is based on the fact that the underlying possible set can be handled with level plans with a specific scalar limit. We applied this method to problems where the possible set is determined by a finite number of constraint functions. Exceptional types of multiobjective nonconvex optimization problems are addressed by our results. It is interesting to focus on issues when a relationship of convex sets addresses the nonconvex potential set and issues with a small number of nonlocal locations. One can be convinced of such issues by looking at a handful of models from region theory.

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