IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-0661, p-ISSN: 2278-8727, Volume 21, Issue 5, Ser. 2 (Sept. – Oct. 2025), PP 64-66
www.iosrjournals.org

On Isomorphisms in the Theory of Fuzzy Fields

C.P. Santhosh
Associate Professor, Department of Mathematics
KMM Government Women's College, Kannur - 670004, Kerala, India

Abstract
Field homomorphisms are fundamental mappings between fields that preserve the algebraic structure of addition and multiplication. This research delves into the theory of fuzzy field homomorphisms, exploring their properties. The paper also investigates the concept of isomorphic fuzzy fields.
***Key Words**: Fuzzy field; homomorphism of fuzzy fields, isomorphism of fuzzy fields.*

Date of Submission: 12-10-2025 Date of Acceptance: 22-10-2025

I. Introduction

The theory of fuzzy sets, introduced by Lotfi A. Zadeh [8] in 1965, has significantly broadened the mathematical framework for dealing with uncertainty, vagueness and imprecision. By allowing elements to possess degrees of membership within a set, fuzzy set theory has enabled the development of generalized structures in logic, topology, algebra, etc. Within the algebraic domain, the integration of fuzzy set theory into classical structures has led to the development of fuzzy algebraic structures.

Fuzzy algebraic structures play a crucial role in computer science, particularly in knowledge representation, fuzzy automata and artificial intelligence. In engineering, they enhance control systems and fault diagnosis through robust handling of vague or noisy data. Additionally, fuzzy algebra supports decision-making in economics, social sciences, and multi-criteria analysis, where human reasoning and preferences often lack crisp boundaries. Their theoretical significance is also notable, providing a flexible framework for extending traditional algebraic concepts to better model real-world complexities.

This paper is to introduce the notion of homomorphism and isomorphism of fuzzy fields based on the concepts of fuzzy field proposed by Gu Wenxiang and Lu Tu [3]. Section 2 gives a brief summary of fuzzy fields and some preliminaries. Homomorphism and isomorphism of fuzzy fields are introduced in sections 3 and 4. Some fundamental properties of fuzzy field homomorphism and isomorphic fuzzy fields are discussed. It is investigated that the relation of isomorphism in the set of all fuzzy fields is an equivalence relation.

II. Preliminaries

This section gives a brief summary of fuzzy fields.

Definition 2.1 [3] Let X be a field and F a fuzzy set in X with membership function μ_F. Suppose the following conditions hold:
(i) $\mu_F\ (a + b) \geq \min\{\mu_F(a), \mu_F(b)\}, a, b \in X$
(ii) $\mu_F\ (a) = \mu_F(-a), a \in X$
(iii) $\mu_F\ (ab) \geq \min\{\mu_F\ (a), \mu_F(b)\}, a, b \in X$
(iv) $\mu_F\ (a) = \mu_F\ (a^{-1}), a\ (\neq 0) \in X$.

Then F is called a *fuzzy field* in X and it is denoted by (F, X). Also (F, X) is called a *fuzzy field* of X.
Note that $\mu_F\ (0) \geq \mu_F\ (a)$ for all $a \in X$ and $\mu_F\ (1) \geq \mu_F\ (a)$ for all $a\ (\neq 0) \in X$.

Proposition 2.1 [3] *Let X and Y be fields and f be a homomorphism of X into Y. Suppose that (F, X) is a fuzzy field of X and (G, Y) is a fuzzy field of Y. Then*
(i) *$(f(F), Y)$ is a fuzzy field of Y*
(ii) *$(f^{-1}(G), X)$ is a fuzzy field of X.*

III. Homomorphism Of Fuzzy Fields

In this section, the concept of homomorphisms between fuzzy fields is formally introduced, and some fundamental structural properties are examined.
If $f : X\rightarrow Y$ is a function, A is a fuzzy set in X and $a \in X$, then

$\mu_{f(\mathrm{A})}(f(a)) = \sup\{\mu_A(a') : f(a') = f(a),\ a' \in X\} \geq \mu_A(a')$ for all $a' \in X$ with $f(a') = f(a)$.

In particular, $\mu_{f(\mathrm{A})} f(a) \geq \mu_A(a)$. Also, if f is injective, then $\mu_{f(\mathrm{A})} f(a) = \mu_A(a)$. A field homomorphism is either injective or maps every element to 0. Hence in the case of field homomorphisms and fuzzy fields, the following result holds.

Proposition 3.1 Let X_1 and X_2 *be fields,* $f : X_1 \to X_2$ *be a homomorphism and* F *be a fuzzy field of* X_1. *Then*

$$\mu_{f(F)}(f(a)) = \begin{cases} \mu_F(a) & if f is\ injective \\ \mu_F(0) & if\ f(a) = 0\ for\ all\ a \in X_1 \end{cases}.$$

Proof. If f is injective, $\mu_{f(F)}(f(a)) = \sup\{\mu_F(a') : f(a') = f(a),\ a' \in X\} = \mu_F(a)$.

If $f(a) = 0$ for all $a \in X_1$, then $\mu_{f(F)} f(a) = \sup\{\mu_F(a') : f(a') = 0,\ a' \in X\} = \mu_F(0)$ since $f(0) = 0$ and $\mu_F(0) \geq \mu_F(a')$.

Theorem 3.1 *Every homomorphism of a field* X_1 *into another field* X_2 *together with a fuzzy field in* X_2 *induces a fuzzy field in* X_1.

Proof. Let $f : X_1 \to X_2$ be a homomorphism and (F, X_2) be a fuzzy field of X_2.

$\mu_F \circ f : X_1 \to [0, 1]$ is a fuzzy set in X_1 and it satisfies

(i) $(\mu_F \circ f)(a + b) = \mu_F(f(a + b)) = \mu_F(f(a) + f(b)) \geq \min\{\mu_F(f(a)), \mu_F(f(b))\}$
$= \min\{(\mu_F \circ f)(a), (\mu_F \circ f)(b)\},\ a, b \in X_1$

(ii) $(\mu_F \circ f)(-a) = \mu_F(f(-a)) = \mu_F(-f(a)) = \mu_F(f(a)) = (\mu_F \circ f)(a),\ a \in X_1$

(iii) $(\mu_F \circ f)(ab) = \mu_F(f(ab)) = \mu_F(f(a)\, f(b)) \geq \min\{\mu_F(f(a)), \mu_F(f(b))\} = \min\{(\mu_F \circ f)(a), (\mu_F \circ f)(b)\},\ a, b \in X_1$

(iv) $(\mu_F \circ f)(a^{-1}) = \mu_F(f(a^{-1})) = \mu_F((f(a))^{-1}) = \mu_F(f(a)) = (\mu_F \circ f)(a),\ a\ (\neq 0) \in X_1$.

Hence the fuzzy set in X_1 with membership function $\mu_F \circ f$ is a fuzzy field of X_1.

Definition 3.1 Two fuzzy fields (F_1, X_1) and (F_2, X_2) are said to be homomorphic if there exists a field homomorphism $f : X_1 \to X_2$ such that $\mu_{F_1} = \mu_{F_2} \circ f$. In this case, $(f, (F_1, X_1), (F_2, X_2))$ is called a fuzzy field homomorphism.

Proposition 3.2 *If two fuzzy fields* (F_1, X_1) *and* (F_2, X_2) *are homomorphic, then*

(i) $\mu_{F_1}(0) = \mu_{F_2}(0)$

(ii) $\mu_{F_1}(1) = \mu_{F_2}(1)$, where 0 and 1 on left sides are respectively the zero and unity in the field X_1 and those on right sides are respectively the zero and unity in the field X_2.

Proof. Since (F_1, X_1) and (F_2, X_2) are homomorphic, there exists a homomorphism $f : X_1 \to X_2$ such that $\mu_{F_1} = \mu_{F_2} \circ f$. So

(i) $\mu_{F_1}(0) = (\mu_{F_2} \circ f)(0) = \mu_{F_2}(f(0)) = \mu_{F_2}(0)$.

(ii) $\mu_{F_1}(1) = (\mu_{F_2} \circ f)(1) = \mu_{F_2}(f(1)) = \mu_{F_2}(1)$.

Proposition 3.3 *If* $(f, (F_1, X_1), (F_2, X_2))$ *is a fuzzy field homomorphism, then*

(i) $f(F_1) \subseteq F_2$

(ii) $f^{-1}(F_2) = F_1$.

Proof. For all $a \in X_1$, $\mu_{F_1}(a) = (\mu_{F_2} \circ f)(a) = \mu_{F_2}(f(a))$.

(i) Since $f : X_1 \to X_2$ is a field homomorphism, either f is injective or f takes every element of X_1 to 0 in X_2.

Case 1: f is injective.

Let $b \in X_2$.

If $f^{-1}(b) \neq \phi$, then there exists unique $a' \in X_1$ such that $f(a') = b$. Therefor $\mu_{f(F_1)}(b) = \sup\{\mu_{F_1}(a) : f(a) = b, a \in X_1\} = \mu_{F_1}(a') = \mu_{F_2}(f(a')) = \mu_{F_2}(b)$.

If $f^{-1}(b) = \phi$, then $\mu_{f(F_1)}(b) = 0 \leq \mu_{F_2}(b)$.

Case 2: $f(a) = 0$ for all $a \in X_1$.

Let $b \in X_2$.

If $b = 0$, then since $f(0) = 0$ and $\mu_{F_1}(0) \geq \mu_{F_1}(a)$,

$\mu_{f(F_1)}(b) = \sup\{\mu_{F_1}(a) : f(a) = 0, a \in X_1\} = \mu_{F_1}(0) = \mu_{F_2}(f(0)) = \mu_{F_2}(0) = \mu_{F_2}(b)$.
If $b \neq 0$, then $f^{-1}(b) = \phi$ and so $\mu_{f(F_1)}(b) = 0 \leq \mu_{F_2}(b)$.
In either case, $\mu_{f(F_1)}(b) \leq \mu_{F_2}(b)$ for all $b \in X_2$. Hence $f(F_1) \subseteq F_2$.
(ii) For all $a \in X_1$, $\mu_{f^{-1}(F_2)}(a) = \mu_{F_2}(f(a)) = \mu_{F_1}(a)$. Therefore $f^{-1}(F_2) = F_1$.

Remark 3.1 The proof of Proposition 3.3(i) leads to the following conclusion.
Let $(f, (F_1, X_1), (F_2, X_2))$ is a fuzzy field homomorphism. Then

$$\mu_{f(F_1)}(b) = \begin{cases} \mu_{F_2}(b) & \text{if } b \in \text{Range}(f) \\ 0 & otherwise. \end{cases}$$

Corollary 3.1 *If $(f, (F_1, X_1), (F_2, X_2))$ is a fuzzy field homomorphism, then for all $\alpha \in [0, 1]$,*
(i) $(f(F_1))_\alpha \subseteq (F_2)_\alpha$ *and* $(f(F_1))_{\alpha^+} \subseteq (F_2)_{\alpha^+}$
(ii) $(f^{-1}(F_2))_\alpha = (F_1)_\alpha$ *and* $(f^{-1}(F_2))_{\alpha^+} = (F_1)_{\alpha^+}$.
Proof follows from the definitions of α-cut and strong α-cut and from proposition 3.3.

IV. Isomorphism Of Fuzzy Fields

Definition 4.1 If $(f, (F_1, X_1), (F_2, X_2))$ is a fuzzy field homomorphism and if f is a bijection, then (F_1, X_1) and (F_2, X_2) are said to be isomorphic and $(f, (F_1, X_1), (F_2, X_2))$ is called a fuzzy field isomorphism. In this case we write $(F_1, X_1) \cong (F_2, X_2)$.

Proposition 4.1 *If $(f, (F_1, X_1), (F_2, X_2))$ is a fuzzy field isomorphism, then $f(F_1) = F_2$.*

Proof. If $b \in X_2$, then there exists a unique $a \in X_1$ such that $f(a) = b$ and consequently $\mu_{f(F_1)}(b) = \mu_{F_1}(a) = \mu_{F_2}(f(a)) = \mu_{F_2}(b)$.

Corollary 4.1 *If $(f, (F_1, X_1), (F_2, X_2))$ is a fuzzy field isomorphism, then $(f(F_1))_\alpha = (F_2)_\alpha$ and $(f(F_1))_{\alpha^+} = (F_2)_{\alpha^+}$ for all $\alpha \in [0, 1]$.*

Theorem 4.1 *The relation of isomorphism in the set of all fuzzy fields is an equivalence relation.*

Proof. *Reflexive.*

If (F, X) is any fuzzy field, then $(F, X) \cong (F, X)$ since the identity mapping $I : X \to X$ is an isomorphism and $\mu_F \circ I = \mu_F$.

Symmetric.

If $(F_1, X_1) \cong (F_2, X_2)$, then there exists an isomorphism $f : X_1 \to X_2$ such that $\mu_{F_1} = \mu_{F_2} \, o \, f$. So $f^{-1} : X_2 \to X_1$ is an isomorphism and for each $b \in X_2$, there exists unique $a \in X_1$ such that $f^{-1}(b) = a$. Hence
$(\mu_{F_1} o \, f^{-1})(b) = \mu_{F_1}(f^{-1}(b)) = \mu_{F_1}(a) = \mu_{F_2}(f(a)) = \mu_{F_2}(b)$.
This implies $\mu_{F_2} = \mu_{F_1} o \, f^{-1}$ and so $(F_2, X_2) \cong (F_1, X_1)$.

Transitive.

If $(F_1, X_1) \cong (F_2, X_2)$ and $(F_2, X_2) \cong (F_3, X_3)$, then there exist isomorphisms $f : X_1 \to X_2$ such that $\mu_{F_1} = \mu_{F_2} \, o \, f$ and $g : X_2 \to X_3$ such that $\mu_{F_2} = \mu_{F_3} \, o \, g$. These give the isomorphism $g \circ f : X_1 \to X_3$ with $\left(\mu_{F_3} \, o \, (gof)\right)(a) = \left((\mu_{F_3} \, o \, g) o f\right)(a) = (\mu_{F_3} \, o \, g)(f(a)) = \mu_{F_2}(f(a)) = \mu_{F_1}(a)$ for all $a \in X_1$ so that $\mu_{F_1} = \mu_{F_3} \, o \, (gof)$, which implies that $(F_1, X_1) \cong (F_3, X_3)$.

Corollary 4.2 *The relation of isomorphism in the set of all fuzzy fields of a field is an equivalence relation.*

References

[1] A. Rosenfeld, Fuzzy Groups, Journal of Mathematical Analysis and Applications 35(3) (1971), 512 517
[2] G.J. Klir and B. Yuan, Fuzzy Sets and Fuzzy Logic : Theory and Applications, Prentice-Hall Of India Private Limited, New Delhi, 2002.
[3] Gu Wenxiang and Lu Tu, Fuzzy Linear Spaces, Fuzzy Sets and Systems 49 (1992) 377- 380.
[4] H.J. Zimmermann, Fuzzy Set Theory-And Its Applications (2nd Revised Edition), Allied Publishers Limited, New Delhi, 1996.
[5] I.N. Herstein, Topics in Algebra (2nd Edition), Wiley, 2022.
[6] J. B. Fraleigh and N. Brand, A First Course in Abstract Algebra (8th Edition), Pearson, 2022.
[7] K.H. Lee, First Course On Fuzzy Theory and Applications, Springer Verlag, Heidelberg, 2005.
[8] L. A. Zadeh, Fuzzy Sets, Information and Control 8(3) (1965), 338– 353.