IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-0661, p-ISSN: 2278-8727, Volume 21, Issue 5, Ser. 2 (Sept. – Oct. 2025), PP 34-46
www.iosrjournals.org

Application of the Cauchy-Goursat Theorem

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Abstract:
This research focuses on an essential aspect of complex analysis, specifically complex integration. The focus is on the Cauchy-Goursat theorem, which states that the line integral of a holomorphic function along a closed contour equals zero when the function is holomorphic within and on that contour, in a domain with a single connection. Various applications derived from this study are analyzed and discussed, aimed at solving integrals of functions that may present singularities. In this regard, more advanced research would enable the contribution of tools in the theory of control and potential flow.
Key Words: *Complex integral, residue theory, complex analysis, Cauchy-Goursat theorem, potential flow.*

Date of Submission: 01-10-2025 Date of Acceptance: 11-10-2025

I. Introduction

In the study of definite integrals, there are some very complicated integrals that cannot be solved by known methods. The Cauchy- Goursat theorem is an answer to a question in complex analysis: When is it true that the integral of an analytic function f on a closed curve is zero? The references show that this occurs if f has a defined antiderivative along the entire curve, and sometimes it does not occur.

Cauchy's Theorem can also be applied to closed curves that are not simple, but can be split into simple closed curves. The orientation of a curve $\mathcal{C}$ induces an orientation on each of the split curves, and it can be seen from the definition of the complex integral that the integral of $\mathcal{C}$ is the sum of the integrals of the pieces. Thus, if the function f is analytical inside each of the simple curves into which the curve is split $\mathcal{C}$, then the integral of f o the curve $\mathcal{C}$ must be equal to 0, [1].

Not every complex function $f(z)$ that is continuous in a region $\mathcal{R}$ is the complex derivative of a function. $F(z)$ in $\mathcal{R}$, by the fundamental theorem, for this to happen it is necessary that the integral of $f(z)$ be 0 on every closed curve in $\mathcal{R}$. Cauchy's Theorem tells us that this happens if $f(z)$ is differentiable as a complex function and its derivative is continuous. Requiring that f it has a complex derivative and that it be continuous seems a very strong hypothesis compared to the hypothesis in the real case (that f it is continuous). Goursat was able to prove Cauchy's Theorem without requiring that the derivative of f be continuous, and this small change has very important consequences as it translates into the following result: "If f is an analytic function in a simply connected region $\mathcal{R}$ then for every closed curve $\mathcal{C}$ in $\mathcal{R}$, $\int_{\mathcal{C}} f(z)dz = 0$.

II. Material and Methods

It is an applied approach, as it will solve certain real-world integrals using the remainder theorem. The design is descriptive and comparative, using the Cauchy integral as a tool to achieve the proposed objectives.

The documented data collection technique is appropriate for each of the variables considered in the research, allowing us to obtain information to enrich the theoretical framework and analyze the properties of the Cauchy integral and some of its consequences. All the information will help develop the methodological strategies to obtain the results of this research.

III. Result

3.1 Fundamental theorems of calculus

Theorem 1. Let $\boldsymbol{f}: \Omega \subset \mathbb{C} \longrightarrow \mathbb{C}$, $\Re = Ran(f)$, be such that f is injective in Ω and such that its inverse function $z = f^{-1}(\boldsymbol{w}): \Omega \subset \mathbb{C} \longrightarrow \mathbb{C}$ is continuous. Then, if $f(z)$ is differentiable at $z_0 \in \Omega$ and if $f'(z_0) \neq 0$, then $f^{-1}(w)$ is differentiable at $w_0 = f(z_0)$ and $(f^{-1})'(w_0) = \frac{1}{f'(z_0)}$.

Definition 1. An arc $\Gamma \subset \mathbb{C}$ is the range of a $f: [a, b] \longrightarrow \mathbb{C}$ continuous function. We consider every arc to have an orientation that must correspond to the direction of growth of t, [2], [3], [4].

If $f(t) = u(t) + iv(t)$, $t \in [a, b]$, f we will call the parameterization of Γ and t the parameter. I f $\alpha: [c, d] \to [a, b]$ is an increasing bijective function, then Γ is also the range of $g = f \circ \alpha: [c, d] \to \mathbb{C}$. Every Jordan curve Γ separates the plane into two domains, one bounded (called the interior of Γ) and one unbounded (the exterior of Γ); furthermore, if any other curve joins the interior with the exterior, it must intersect Γ.

Definition 2. A curve $\Gamma \subset \mathbb{C}$ that has a differentiable parameterization $f: [a, b] \longrightarrow \mathbb{C}$ at $\langle a, b \rangle$ and whose derivative $f'(t) \neq 0, \forall t \in [a, b]$. Every regular curve is rectifiable and its length is given by $\ell = \int_a^b \sqrt{((u'(t))^2 + ((v'(t))^2} dt$.

Theorem 2 (Second Fundamental Theorem). Let be $f: \Omega \longrightarrow \mathbb{C}$ a continuous and integrable function on $\Omega \subset \mathbb{C}$ a domain and let be $F(z) = \int f(z)dz, \forall z \in \Omega$. If $\Gamma \subset \mathbb{C}$ is a rectifiable curve joining two points z_1 y z_2 in Ω, then $\int f(z)dz = F(z_2) - F(z_1)$.

Corollary. If f is integrable over a domain Ω and $\Gamma \subset \Omega$ is a closed rectified curve, then $\int_\Gamma f(z)dz = 0$.

Theorem 3 (First Fundamental Theorem). Let be $f: \mathbb{C} \longrightarrow \mathbb{C}$continuous in $\Omega \subset \mathbb{C}$ domain, such that the integral of f is independent of the path in Ω. Let be $z_0 \in \Omega$ fixed, we define $F(z) = \int_{z_0}^z f(z)dz, \forall z \in \Omega$, and then $F'(z)$exists, $\forall z \in \Omega$ and $f(z) = F'(z)$, $\forall z \in \Omega$, that is, f it is integrable in Ω, [5], [6].

Theorem 4. Let be $f: \mathbb{C} \longrightarrow \mathbb{C}$ continuous on $\Omega \subset \mathbb{C}$ a domain. Then the following three statements are equivalent: a) f is integrable in $\Omega \subset \mathbb{C}$; b) The integral of f is path-independent in $\Omega \subset \mathbb{C}$; c) The integral of f around every closed piecewise regular curve $\Gamma \subset \Omega$ is zero.

3.2 On Cauchy´s theorem and Goursat 's lemma

Cauchy's theorem in $\mathbb{C}$, is closely related to Green's theorem in the plane. To see this relationship, we prove the slightly weaker version of Cauchy's theorem, [4], [5], [6].

Theorem 5. Let $f: \mathbb{C} \longrightarrow \mathbb{C}$, $f \in C^1(\Omega)$, $\Omega \subset \mathbb{C}$ a simply connected domain, then, for every $\Gamma \subset \Omega$ piecewise regular Jordan curve

$$\int_\Gamma f(z)dz = 0 \tag{1}$$

Proof: Let $\Re = Int(\Gamma)$. Clearly $\Re$ is simply connected. If $f = u + iv$, then by Green's Theorem ($u$and $v$are $C^1(\mathbb{R})$)

$$\int_\Gamma udx - vdy = \iint_\Re \left[\frac{\partial(-v)}{\partial x} - \frac{\partial(u)}{\partial y}\right] dxdy \tag{2}$$

$$\int_\Gamma vdx + udy = \iint_\Re \left[\frac{\partial(u)}{\partial x} - \frac{\partial(v)}{\partial y}\right] dxdy,$$

but since f it is analytic, it satisfies the Cauchy-Riemann conditions $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Therefore, (1) and (2) cancel out; and $\int_\Gamma f(z)dz = \int_\Gamma (udx - vdy) + i\int_\Gamma (vdx + udy) = 0$.

The objective is to prove Cauchy's theorem, for this the following Lemma will be useful.

Goursat 's Lemma. Let, be continuous on a $f:\mathbb{C}\longrightarrow\mathbb{C}$ simply connected $\Gamma\subset\Omega$ domain, and let , be Ω a piecewise regular Jordan curve, then $\forall\varepsilon>0$, there exists a polygon $P_\varepsilon\subset\Omega$ such that $\left\|\int_\Gamma f(z)dz-\int_{P_\varepsilon} f(z)dz\right\|<\varepsilon$. Closed polygon: $\exists z_0,\ldots,z_n$, $z_0=z_n$ in Ω such that $P_\varepsilon=\overline{z_0z_1}\cup\overline{z_1z_2}\cup\ldots\cup\overline{z_{n-1}z_n}$.

Proof: Since Ω is open and $\Gamma\subset\Omega$ is compact (closed and bounded), then there exist $\rho>0$ and $E\subset\Omega$ compact such that $\Gamma\subset E$ and E contains all neighborhoods $B(z,\rho)$, with $z\in\Gamma$. Let $\varepsilon>0$, as E is compact and f, is continuous on Ω, then f is uniformly continuous on E, that is, $\exists\eta>0$ such that for all $z^1,z^2\in E$with $|z^1-z^2|<\eta\Rightarrow|f(z^1)-f(z^2)|<\frac{\varepsilon}{2L}$, where $L=Long(\Gamma)$. For $\varepsilon>0$, by line integral $\exists\delta>0$ such that for all $P_\varepsilon=\overline{z_0z_1}\cup\overline{z_1z_2}\cup\ldots\cup\overline{z_{n-1}z_n}$, $z_n=z_0$, closed polygon with vertex in Γ with $\|z_k-z_{k-1}\|<\delta$; $k=1,2,\ldots,n$, we have $\left\|\int_\Gamma f(z)dz-\sum_{k=1}^n f(\xi_k)(z_k-z_{k-1})\right\|<\frac{\varepsilon}{2}$, where $\varepsilon_k\in\overline{z_kz_{k-1}}$ is any. In particular we will take

$$\|z_k-z_{k-1}\|<min\{\delta,\rho,\eta\};k=1,2,\ldots,n \tag{3}$$

$$\left\|\int_\Gamma f(z)dz-\sum_{k=1}^n f(\xi_k)(z_k-z_{k-1})\right\|<\frac{\varepsilon}{2}. \tag{4}$$

yes $\int_{P_\varepsilon} f(z)dz=\sum_{k=1}^n\int_{\overline{z_{k-1}z_k}}[f(z)-f(z_k)]dz+\sum_{k=1}^n f(z_k)(z_k-z_{k-1})$, then

$$\left\|\int_{P_\varepsilon} f(z)dz-\sum_{k=1}^n f(\xi_k)(z_k-z_{k-1})\right\|\le\sum_{k=1}^n\|z_k-z_{k-1}\|\underset{z\in\overline{z_{k-1}z_k}}{m\acute{a}x}\|f(z_k)-f(z_{k-1})\| \tag{5}$$

Now, for all $k=1,2,\ldots,n$and for all $z\in\overline{z_{k-1}z_k}$, $\|z-z_k\|=\|z_{k-1}-z_k\|<\eta$ by (3), and since $z_k\in E$ and from (3) $\|z-z_k\|<\rho$, then from (1) and $z\in E$, then we have $z,z_k\in E$ and $\|z-z_k\|<\eta$. Therefore, from (2), $\|f(z)-f(z_k)\|<\frac{\varepsilon}{2L}$, $\forall z\in\overline{z_{k-1}z_k}$, $\forall k=1,2,\ldots,n$, in (5), $\left\|\int_{P_\varepsilon} f(z)dz-\sum_{k=1}^n f(\xi_k)(z_k-z_{k-1})\right\|<\frac{\varepsilon}{2L}\sum_{k=1}^n\|z_k-z_{k-1}\|$. The sum $\sum_{k=1}^n\|z_k-z_{k-1}\|$ gives an approximate value for the length of Γ, as L is the supremum of all such sums $\sum_{k=1}^n\|z_k-z_{k-1}\|\le L$. Therefore,

$$\left\|\int_{P_\varepsilon} f(z)dz-\sum_{k=1}^n f(\xi_k)(z_k-z_{k-1})\right\|<\frac{\varepsilon}{2}. \tag{6}$$

By the triangular inequality, from (4) and (6), $\left\|\int_\Gamma f(z)dz-\int_{P_\varepsilon} f(z)dz\right\|<\varepsilon$.

Now Cauchy's theorem is proved, first for Γ a triangle, then for convex polygons, we continue for polygonal Jordan curves and finally for closed polygonal curves [7], [8].

Theorem 6. Let $f:\mathbb{C}\longrightarrow\mathbb{C}$, be analytic in $\Omega\subset\mathbb{C}$ a simply connected domain, then for every Γ piecewise regular Jordan curve, $\Gamma\subset\Omega,\int_\Gamma f(z)dz=0$.

Proof. Case 1: For triangles, let be a Γ counterclockwise $D,E\ y\ F$ triangle and be ABC the midpoints of the sides $AB,ACy\ BC$ respectively. Joining these points gives four triangles $\Delta_1,\Delta_2,\Delta_3\ y\ \Delta_4$,

$$\int_\Gamma f(z)dz=\int_{\Delta_2} f(z)dz+\int_{\Delta_2} f(z)dz+\int_{\Delta_3} f(z)dz+\int_{\Delta_4} f(z)dz.$$

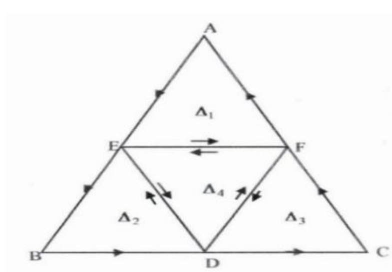

Figure 1. Counterclockwise triangle.

Because the integral over $ED, EF\ y\ DF$ vanishes in pairs (having opposite directions). Let $M = \left\|\int_{\Gamma} f(z)dz\right\| \geq 0 \Rightarrow M \leq \sum_{k=1}^{4}\left\|\int_{\Delta_k} f(z)dz\right\|$, in at least one of the triangles Δ_k, say Δ_1: $\left\|\int_{\Delta_1} f(z)dz\right\| \geq \frac{M}{4}$. Starting now from Δ_1 (instead of Γ) and proceeding in exactly the same way, we obtain a triangle Δ_2, $\Delta_2 \subset \Delta_1$ such that $\left\|\int_{\Delta_2} f(z)dz\right\| \geq \frac{M}{4^2}$ and so on, resulting in a sequence of triangles $\Gamma \supset \Delta_1 \supset \Delta_2 \supset \ldots \supset \Delta_n \supset \ldots$ whose area tends to zero as $(n \to \infty)$ and

$\left\|\int_{\Delta_n} f(z)dz\right\| \geq \frac{M}{4^n}$, there exists such $z_0 \in \Gamma \cup Int(\Gamma)$ that $z_0 \in \Delta_n, \forall n = 1,2,\ldots$, be $\eta(z) = \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0), z \neq z_0, z \in \Omega$, and $\lim_{z \to z_0} \eta(z) = 0, z \neq z_0, z \in \Omega$ then, given $\varepsilon > 0$ any $\exists\delta > 0$ such that

$$\|\eta(z)\| < \varepsilon, \forall 0 < \|z - z_0\| < \delta. \qquad (7)$$

Now $\left\|\int_{\Delta_n} f(z)dz\right\| = \left\|\int_{\Delta_n} [f(z_0) + f'(z_0)(z - z_0)\eta(z)]dz\right\|$,

$$= \left\|\int_{\Delta_n} (z - z_0)\eta(z)dz\right\| \leq [perimetro] \max_{z \in \Delta} \|\eta(z)(z - z_0)\|. \qquad (8)$$

Let $l =$ Perímetro de (Γ). Then, perímetro de $(\Delta_n) = \frac{l}{2^n}$. If n is large enough $\frac{l}{2^n} < \delta$, for all $z_0 \in \Delta_n$, $\|z - z_0\| < \frac{l}{2^n} < \delta$, from (7): $\|\eta(z)\| < \varepsilon$, $\forall z \in \Delta_n$, perímetro de $(\Delta_n) = \frac{l}{2^n}$ and $\|(z - z_0)\eta(z)\| < \varepsilon\left(\frac{l}{2^n}\right)$; and $\forall z \in \Delta_n$ n is large enough. Therefore, in (8) $\left\|\int_{\Delta_n} f(z)dz\right\| < \left(\frac{l}{2^n}\right)\left(\frac{\varepsilon l}{2^n}\right) = \frac{\varepsilon l^2}{4^n}$, n is large, replacing $\frac{M}{4^n} \leq \frac{\varepsilon l^2}{4^n}$, n is large, $M \leq \varepsilon l^2, \forall \varepsilon > 0$. Therefore, $M = \left\|\int_{\Gamma} f(z)dz\right\| = 0$. From here: $\int_{\Gamma} f(z)dz = 0$.

Case 2. For convex polygons. Let $\Gamma = A_0A_1\ldots A_{n-1}A_n(A_0 = A_n), n \geq 4$ is a convex polygon , then we take a vertex of Γ, which can be A_0 and join it with the other vertices, thus obtaining $(n-2)$ triangles, all of which are parameterized in the counterclockwise direction ($\Delta_1, \ldots, \Delta_{n-2}$), from the first step: $\int_{\Delta_k} f(z)dz = 0, \forall k = 1,2,\ldots,n-2$. Therefore, $\int_{\Gamma} f(z)dz = \sum_{k=1}^{n}\int_{\Delta_k} f(z)dz = 0$

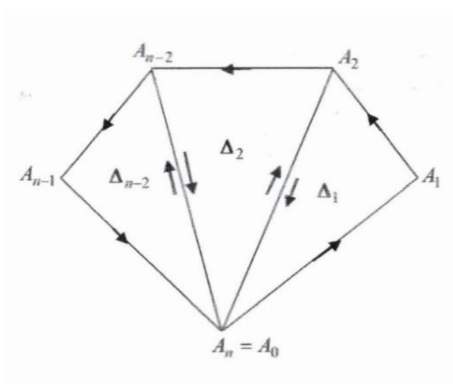

Figure 2: case 2

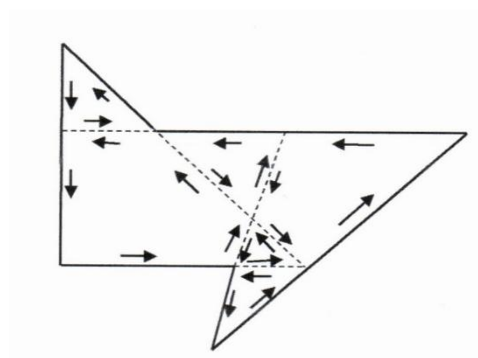

Figure 3. Case 3

Case 3: For polygonal Jordan curves. As shown in the figure, all sides of the polygon extend in one direction or the other (or perhaps both). This breaks down the integral over convex polygons, all of which vanish in step 2. Therefore $\int_{\Gamma} f(z)dz = 0$, [9], [10].

Case 4: For polygonal curves. The integral over is subdivided Γ into integrals over polygonal (simple) Jordan curves. Some may overlap, then, from the previous step $\int_{\Gamma} f(z)dz = 0$.

DOI: 10.9790/5728-2105023446

Case 5: (piecewise regular Jordan curves). Let be Γ a piecewise regular Jordan curve $\Gamma \subset \Omega$ and $\varepsilon > 0$, from the lemma, there exists P_ε a closed polygonal curve $P_\varepsilon \subset \Omega$ such that $\left\|\int_\Gamma f(z)dz - \int_{P_\varepsilon} f(z)dz\right\| < \varepsilon$. From the fourth step: $\left\|\int_\Gamma f(z)dz\right\| < \varepsilon$, as $\varepsilon > 0$ is any, then: $\left\|\int_\Gamma f(z)dz\right\| = 0$. From here $\int_\Gamma f(z)dz = 0$.

Theorem 7. Cauchy's integral formula**.** Let be $f: \mathbb{C} \longrightarrow \mathbb{C}$ analytic in a domain $\Omega \subset \mathbb{C}$. Let be ℓ a counterclockwise rectifiable Jordan curve such that $\ell \cup Int(\ell) \subset \Omega$, then $f(z) = \frac{1}{2\pi i}\int_\ell \frac{f(w)}{w-z}dw, \forall z \in Int(\ell)$.

3.3 Consequences of Cauchy's integral formula

In Cauchy's formula we obtained $f(z_0) = \frac{1}{2\pi i}\int_\ell \frac{f(z)}{z-z_0}dz$, where f is analytic in $\ell \cup Int(\ell)$ and $z_0 \in Int(\ell)$. An integral of this type is called a Cauchy-type integral, the function $f(z)$ is called the derivative and $\frac{1}{z-z_0}$ is the kernel of the integral. This theorem is very important because it proves that if f is known only over some Jordan curve ℓ, then the values of f can be found throughout the interior of ℓ; one would only need to evaluate the kernel of the integral. Furthermore, it will be seen that all the derivatives of f can also be found from this formula.

Theorem 8. Let $f: \mathbb{C} \longrightarrow \mathbb{C}$ analytic in a domain $\Omega \subset \mathbb{C}$, then f is infinitely differentiable in Ω. Furthermore, $\forall z_0 \in \Omega$ if $\ell \subset \Omega$ is any counterclockwise Jordan curve such that $Int(\ell) \subset \Omega$ and $z_0 \in Int(\ell)$ then $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\ell \frac{f(z)dz}{(z-z_0)^{n+1}}; \forall n = 0,1,2,\ldots$

Proof: It is clear that it is enough to verify the above formula. We will prove it by mathematical induction. For $n = 0$, we obtain the already proven Cauchy formula. Assuming that it holds for, $n > 0$ it is proved that it also holds for $n + 1$. Let $\varepsilon > 0, \exists\, \delta > 0$ such that $B(z_0, \rho) \subset Int(\ell)$ and $\delta_0 = dist(\ell, \Gamma_\rho) > 0(\ \Gamma_\delta: \|z - z_0\| = \rho)$. We have

$\delta = min\left\{\rho, \frac{(4\pi\delta_0^{2n+3})\varepsilon}{(n+2)!M(2R)^n l}\right\} > 0$, where $M = \underset{z\in\ell}{máx}|f(z)|$, $l = long(\ell)$and $R > 0$ is such that $\ell \subset B(0, R)$. It is enough to prove that if

$$0 < |h| < \delta, \text{ then } \left\|\frac{f^{(n)}(z_0+h)-f^{(n)}(z_0)}{h} - \frac{(n+1)!}{2\pi i}\int_\ell \frac{f(z)dz}{(z-z_0)^{n+1}}\right\| < \varepsilon \qquad (9)$$

Let $0 < |h| < \delta$, then $z_0 + h \in B(z_0, \rho) \subset Int(\ell)$, then $z_0\ y\ z_0 + h \in Int(\ell)$.

Then by hypothesis

$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\ell \frac{f(z)dz}{(z-z_0)^{n+1}}$ and $f^{(n)}(z_0 + h) = \frac{n!}{2\pi i}\int_\ell \frac{f(z)dz}{(z-(z_0+h))^{n+1}}$. $\frac{f^{(n)}(z_0+h)-f^{(n)}(z_0)}{h} = \frac{n!}{2\pi i}\int_\ell \frac{f(z)[(z-z_0)^{n+1}-(z-(z_0+h))^{n+1}]}{(z-z_0)^{n+1}[z-(z_0+h)]^{n+1}h}dz$.

If $z - z_0 = t$, then $z - (z_0 + h) = t - h$and by the identity

$$a^{n+1} - b^{n+1} = (a - b)\sum_{k=0}^n a^k b^{n-k} \qquad (10)$$

$$\frac{f^{(n)}(z_0+h)-f^{(n)}(z_0)}{h} = \frac{n!}{2\pi i}\int_\ell \frac{f(z)[\sum_{k=0}^n t^k(t-h)^{n-k}]}{t^{n+1}(t-h)^{n+1}}dz$$

$$= \left\|\frac{n!}{2\pi i}\int_\ell \frac{f(z)[t\sum_{k=0}^n t^k(t-h)^{n-k} - (n+1)(t-h)^{n+1}]}{t^{n+2}(t-h)^{n+1}}dz\right\| \qquad (11)$$

and $t[\sum_{k=0}^n t^k(t-h)^{n-k}] - \sum_{k=0}^n (t-h)^{n+1} = h\sum_{k=0}^n (t-h)^{n-k}\left[\sum_{k=0}^k t^l(t-h)^{k-l}\right]$ by (10), as $\delta_0 < |t| < 2R$, $\delta_0 < |t - h| < 2R$, taking modulus

$$\|t[\sum_{k=0}^n t^k(t-h)^{n-k}] - \sum_{k=0}^n (t-h)^{n+1}\| \le |h|\sum_{k=0}^n (2R)^n \frac{(n+1)(n+2)}{2}. \qquad (12)$$

in (11) and (12) $\left\|\frac{f^{(n)}(z_0+h)-f^{(n)}(z_0)}{h}-\frac{(n+1)!}{2\pi i}\int_\ell \frac{f(z)dz}{(z-z_0)^{n+1}}\right\|<\frac{(n+2)!}{2\pi}\frac{ML(2R)^n}{\delta_0^{2n+3}}\delta<\varepsilon$, then $\forall\ 0<|h|<\delta$ is fulfilled (9).

Theorem 9 (Morera**).** Let f be $f:\mathbb{C}\longrightarrow\mathbb{C}$ continuous in a domain $\Omega\subset\mathbb{C}$ and such that $\int_\ell f(z)dz=0$ for every rectifiable Jordan curve, then f is analytic in Ω, [11].

Demonstration: By the (equivalence) theorem of the hypothesis f is integrable in Ω. That is, there exists $F:\mathbb{C}\longrightarrow\mathbb{C}$, $F\in C(\Omega)$ such that $F'(z)=f(z)$; as $F\in C'(\Omega)$, then F is analytic in Ω and from the previous theorem (Cauchy's F), F' it is analytic in Ω, so f it is analytic in Ω.

Theorem 10. (Cauchy Estimation). Let be $f:\mathbb{C}\longrightarrow\mathbb{C}$analytic in a domain $f:\mathbb{C}\longrightarrow\mathbb{C}$. Let $z_0\in\Omega$ and $r>0$ such that $\Gamma_r\subset\Omega(\Gamma_r:\|z-z_0\|=r)$; then $\|f^{(n)}(z_0)\|\leq\frac{n!M(r)}{r^n}$, $n=0,1,2,\ldots$, where $M(r)=\underset{z\in\Gamma_r}{m\acute{a}x}$, [12].

Demonstration: From Cauchy's formula

$$\|f^{(n)}(z_0)\|=\left\|\frac{n!}{2\pi i}\int_{\Gamma_r}\frac{f(z)dz}{(z-z_0)^{n+1}}\right\|\overset{ML}{\leq}\frac{n"}{2\pi}\frac{M(r)}{r^{n+1}}(2\pi r).$$

Theorem 11 (Liouville's**).** Every bounded integer function is a constant function.
Proof. Let $z_0\in C$. Since $\Omega=\mathbb{C}$, then $\forall r>0$, $\Gamma_r\subset\Omega$, then the Cauchy estimate $\|f'(z_0)\|\leq\frac{M}{r}$, $\forall r>0$ (where $M>0$ such that $\|f(z)\|<M,\forall z\in C$, when $r\to\infty$, then $f'(z_0)=0,\forall z\in C$, then $f(z)=Cte$ in C.

3.4 Power series and absolute convergence

Theorem 12. (Cauchy- Hadamard). Let be $\sum_{k=0}^{\infty}a_k(z-z_0)^k$ a power series and $R=\frac{1}{\underset{n\to\infty}{lim}\sqrt[n]{a_n}}$, then a) If $R=0$, the power series converges only for $z=z_0$, b) If $0<R<\infty$, the power series converges absolutely $\forall\|z-z_0\|<R$, and diverges $\forall\|z-z_0\|>R$, c) If $R=\infty$, the power series converges absolutely everywhere $\mathbb{C}$.

Obviously, the theorem says nothing about the behavior of the series on the circle of convergence: $\|z-z_0\|=R$[7], [8].

Theorem 13. Let be $\sum_{k=0}^{\infty}a_k(z-z_0)^k$ a power series with radius of convergence $R>0$. Let $\Omega:\|z-z_0\|<R$ and be $f(z)=\sum_{k=0}^{\infty}a_k(z-z_0)^k$, $z\in\Omega$; then a) f is analytic in Ω, b) $a_k=\frac{f^{(k)}(z_0)}{k!}$ and hence f is equal to the Taylor series in Ω, c) $\forall k=0,1,2,\ldots$, $\forall z\in\Omega$: $f^{(k)}(z)=\sum_{n=k}^{\infty}\frac{n!a_n}{(n-k)!}(z-z_0)^{n-k}$.

Theorem 14 (Taylor**).** Let be $f(z)$ an analytic function in Ω and $z_0\in\Omega$; then $f(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$, for all z in the largest disk around z_0 y contained in Ω.

Definition 3. Let be f analytic in a domain Ω. A point $z_0\in\Omega$ is called a zero of order m of $f(z)$ if: $f^{(k)}(z_0)=0,\forall k=0,1,2,\ldots,m-1$ and $f^{(m)}(z_0)\neq 0$.

Motto. z_0 is a zero of order n of the analytic $f(z)$ function if and only if in a neighborhood of z_0: $f(z)=(z-z_0)^n q(z)$, where $q(z_0)\neq 0$ and $q(z)$ is analytic in z_0, [13].

Proof: ($\Rightarrow$) In a neighborhood of z_0: $f(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k=\sum_{k=n}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k\Rightarrow f(z)=(z-z_0)^n q(z)$, where $q(z)=\sum_{k=0}^{\infty}\frac{f^{(k+n)}(z_0)}{(k+n)!}(z-z_0)^k$ is analytic in a neighborhood of z_0 (because it is a power series) and $q(z_0)=\frac{f^{(n)}(z_0)}{n!}\neq 0$.

($\Leftarrow$) From the previous Theorem: $q(z)=\sum_{k=0}^{\infty}b_k(z-z_0)^k\Rightarrow b_0=q(z_0)\neq 0$, where

$$a_k = \begin{cases} 0 & k = 0,1,2,\dots,n-1 \\ b_{k-n} & k = n, n+1, \dots \end{cases} \Rightarrow a_n = b_0 \neq 0.$$

Therefore, $f^{(k)}(z_0) = k!\,a_k = 0$, $\forall k = 0,1,2,\dots,n-1$ and $f^{(n)}(z_0) = n!\,a_n \neq 0$, where z_0 is a zero of order n of $f(z)$.

Theorem 15. Let be $f(z)$ analytic in Ω and $z_0 \in \Omega$ such that $f(z_0) = 0$, then $\exists\, r > 0$such that $f(z) = 0$, $\forall \|z - z_0\| < r$or $f(z) \neq 0$, $\forall\, 0 < \|z - z_0\| < r$. In fact $r = Dist(z_0, \partial\Omega)$, (That is, the zeros of f occur in balls or else they are isolated zeros).

Theorem 16. Let f be analytic in Ω and be $\{z_n\} \subset \Omega$ a sequence of zeros of f (all distinct) that converge to $z_0 \in \Omega$. Then $f \equiv 0$ in Ω.

Theorem 17 (Parseval 's Identity). Let f analytic in Ω and $z_0 \in \Omega$. Let $R > 0$ such that $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$, $\forall\, \|z - z_0\| < R$; then $\forall\, 0 < r < R$, $\frac{1}{2\pi}\int_0^{2\pi} \left\|f(z_0 + re^{i\theta})\right\|^2 d\theta = \sum_{k=0}^{\infty} \|a_k\|^2 r^{2k}$.

3.5 Regarding singularities and Laurent series

Definition 4. Let be $f$defined in $\Omega \subset \mathbb{C}$ a domain. If f is analytic in Ω, except at one point $z_0 \in \Omega$, then f it has a singularity at z_0, [6]**,** [7]**,** [8].

It only focuses on singularities that are isolated. That is $\exists r > 0 / \forall 0 < \|z - z_0\| < r$, $f(z)$it is analytic in Ω. Let's say $f(z) = \left(sen\frac{1}{z}\right)^{-1}$ has singularities in: $0, \frac{1}{n\pi}$; $n = \pm 1, \pm 2, \dots$, it is easy to see that in $z_0 = 0$ f has a non-isolated singularity. Depending on whether it exists or not, $\lim_{z \to z_0} f(z)$ there are three types of singularities: removable singularity if: $\lim_{z \to z_0} f(z) \in \mathbb{C}$, polar singularity if: $\lim_{z \to z_0} \|f(z)\| = \infty$ and essential $\lim_{z \to z_0} \|f(z)\|$ singularity if: neither exists n or is ∞.

Lemma. Let be z_0 a singularity of f, a) If z_0 is a removable singularity, then redefining $f(z_0) = \lim_{z \to z_0} f(z)$ f is analytic in z_0. b) z_0 is a pole of $f$if and only if z_0 is a zero of $\frac{1}{f}$, [14].

Demonstration: a) Obviously, since $f(z_0) = \lim_{z \to z_0} f(z)$, b) $(\Rightarrow)$ $\lim_{z \to z_0} \left\|\frac{1}{f(z)}\right\| = \frac{1}{\lim_{z \to z_0} \|f(z)\|} = \frac{1}{+\infty} = 0$, $\Rightarrow \frac{1}{f(z_0)} = 0 \Rightarrow \frac{1}{f}$ has a zero in z_0, $(\Leftarrow)$ $\lim_{z \to z_0} \|f(z)\| = \frac{1}{\lim_{z \to z_0} \left\|\frac{1}{f(z)}\right\|} = \frac{1}{0^+} = +\infty$.

Theorem 18. (Laurent). If f is analytic in the ring $R_1 < \|z - z_0\| < R_2$, then $f(z)$ it can be represented in the Laurent series $f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$, which converges to $f(z)$ in the ring. Moreover $c_n = \frac{1}{2\pi i}\int_{\ell} \frac{f(s)ds}{(s - z_0)^{n+1}}$, $n \in \mathbb{Z}$, where ℓ is any circle centered at z_0 and contained in the ring, [15].

Proof: Let $z_1 / R_1 < \|z_1 - z_0\| < R_2$, from Cauchy 's formula for doubly connected domains, $f(z_1) = \frac{1}{2\pi i}\int_{\ell_2} \frac{f(s)}{s - z_1} ds - \frac{1}{2\pi i}\int_{\ell_1} \frac{f(s)}{s - z_1} ds = I_2 + I_1$, where ℓ_i: $\|s - z_0\| = r_i$, $i = 1,2$. ($R_1 < r_1 < \|z - z_0\| < r_2 < R_2$). For I_2: $\frac{1}{s - z_1} = \frac{1}{(s - z_0)\left[1 - \frac{z_1 - z_0}{s - z_0}\right]} = \sum_{n=0}^{\infty} \frac{(z_1 - z_0)^n}{(s - z_0)^{n+1}}$, uniform convergence, since $\|z_1 - z_0\| < \|s - z_0\| = r_2$, $I_2 = \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i}\int_{\ell_2} \frac{f(s)ds}{(s - z_0)^{n+1}}\right](z_1 - z_0)^n$, since both closed curves are in the ring $I_2 = \sum_{n=0}^{\infty} c_n (z_1 - z_0)^n$.

For I_1: $-\frac{1}{s - z_1} = \frac{1}{z_1 - s} = \frac{1}{(z_1 - z_0)\left[1 - \frac{s - z_0}{z_1 - z_0}\right]} = \sum_{n=0}^{\infty} \frac{(s - z_0)^n}{(z_1 - z_0)^{n+1}}$, uniform convergence, since $\|s - z_0\| = r_1 < \|z_1 - z_0\|$, $I_1 = \sum_{n=-1}^{-\infty} c_n (z_1 - z_0)^n$.

Therefore: $f(z_1) = \sum_{n=-\infty}^{-\infty} c_n(z_1 - z_0)^n$, $\forall z_1$, where $c_n = \frac{1}{2\pi i}\int_\ell \frac{f(s)ds}{(s-z_0)^{n+1}}$.

If f is analytic in $R_1 < \|z\| < R_2$and in $\|z\| < R_1$, then its Lawrence series becomes its Taylor series, since $c_n = \frac{1}{2\pi i}\int_\ell \frac{f(s)ds}{(s-z_0)^{n+1}} = 0$, for $n = -1, -2, \ldots$, the integrand being analytic in and on ℓ.

Corollary. Let be f analytic in $\Omega - \{z_0\}$. Then F is analytic in $z_0 \Leftrightarrow \|f(z)\|$ this bounded in a neighborhood of z_0.

Proof: $(\Rightarrow)$ Direct, since $\|f(z)\|$ it is continuous. $(\Leftarrow)$Let Γ_ρ: $\|z - z_0\| = \rho$, $0 < \rho < R$, $R = Dist(z_0, \Omega)$, $\Rightarrow$ $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$; $a_n = \frac{1}{2\pi i}\int_{\Gamma\rho} \frac{f(z)dz}{(z-z_0)^{n+1}}$;$n \in \mathbb{N}$

Let $M > 0/\|f(z)\| \le M$, $\forall z \in B(z_0, R)$, $\|a_n\| < \frac{M}{2\pi}\frac{1}{\rho^{n+1}}(2\pi\rho) = \frac{M}{\rho^n}$. If $n < 0$, making $\rho \to 0 \Rightarrow \|a_n\| = 0 \Rightarrow$ $a_n = 0$, $\forall\, n = -1, -2, \ldots$, Therefore, $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$, its Taylor series. Then f is analytic in z_0.

3.6 Calculation of residues

Definition 5. Let be f analytic in a domain Ω except $z_0 \in \Omega$. Let be the $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ Lawrence series around f. z_0 The residue of f in z is the coefficient a_{-1}. Notation: $a_{-1} = Re\,s\,(f(z), z_0)$, [8], [9], [10]. How to calculate the residue:

1) Directly: Finding the Laurent series of f around z_0, then $Re\,s\,(f(z), z_0)$ will be the coefficient of the term $(z - z_0)^{-1}$.

2) $Res(f, z_0) = \frac{1}{2\pi i}\int_\ell f(z)dz$; where is a ℓ closed and simple curve around. z_0

   Indeed ($n = -1$):$a_{-1} = \frac{1}{2\pi i}\int_\ell \frac{f(z)dz}{(z-z_0)^{n+1}} = \frac{1}{2\pi i}\int_\ell f(z)dz$.

3) $Res(f, z_0) = \frac{1}{(k-1)!}\lim_{z\to z_0}\frac{d^{k-1}}{dz^{k-1}}[(z - z_0)^k f(z)]$, when z_0 is a pole of order k of $f(z)$. If $k = 1$, $Res(f, z_0) = \lim_{z\to z_0}(z - z_0)f(z)$. Indeed, if z_0 is a pole of order k of $f(z)$, $f(z) = \frac{a_{-k}}{(z-z_0)^k} + \ldots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z - z_0) + \ldots$

   $(z - z_0)^k f(z) = a_{-k} + \ldots + a_{-1}(z - z_0)^{k-1} + a_0(z - z_0)^k + a_1(z - z_0)^{k+1} + \ldots$

Differentiating (k-1) times $\frac{d^{k-1}}{dz^{k-1}}[(z - z_0)^k f(z)] = (k-1)!\,a_{-1} + \frac{k!}{1!}a_0(z - z_0) + \frac{(k+1)!}{2!}a_1(z - z_0)^2 + \ldots$, then$(k - 1)!\,a_{-1} = \lim_{z\to z_0}\frac{d^{k-1}}{dz^{k-1}}[(z - z_0)^k f(z)]$

4) If: $f(z) = \frac{p(z)}{q(z)}, p, q$ analytical in z_0, $q(z_0) = 0$, $q'(z_0) \ne 0$ and $p(z_0) \ne 0$, $Res(f(z_0)) = \frac{p(z_0)}{q'(z_0)}$, such that has a simple zero in z_0, then in $k = 1$, $Res(f, z_0) = \lim_{z\to z_0}(z - z_0)\frac{p(z)}{q(z)} = \lim_{z\to z_0}\frac{p(z)}{\frac{q(z)-q(z_0)}{z-z_0}} = \frac{p(z_0)}{q'(z_0)}$.

Theorem 19 (Remainder). Let be f analytic Ω except for singularities, $z_1, z_2, \ldots, z_m$ of Ω, let, be ℓ a Jordan curve in Ω which encloses $z_1, z_2, \ldots, z_m$. Then $\int_\ell f(z)dz = 2\pi i\sum_{k=1}^{m} Res(f, z_k)$.

Proof: Let be $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ curves in the interior of ℓ that enclose and $z_1, z_2, \ldots, z_m$ respectively. Then, by Cauchy's theorem for simply connected domains

$$\int_\ell f(z)dz = \sum_{k=1}^{m}\int_{\Gamma_k} f(z)dz \overset{de(2)}{\cong} \sum_{k=1}^{m} 2\pi i\,Res(f, z_k),$$

$$\int_\ell f(z)dz = 2\pi i\sum_{k=1}^{m} Res(f, z_k).$$

Theorem 20 (From the argument) **.** Let be ℓ a Jordan curve and $f(z)$ analytic on ℓ and in ℓ except for a finite number of poles in ℓ, then $\frac{1}{2\pi i}\int_\ell \frac{f'(z)}{f(z)}dz = n - p$, where p is the number of poles (counted with their multiplicity) within ℓ and n is the number of zeros (counted with their multiplicity) within ℓ).

Proof: Let $g(z) = \frac{f'(z)}{f(z)}$, then the singularities of g are the zeros and poles f inside ℓ. Therefore, by the residue theorem $\frac{1}{2\pi i}\int_\ell \frac{f'(z)}{f(z)}dz$ is equal to the sum of all residues of g. We first calculate the residue of g at z_0 a zero of f. If the zero is of order k, then $f(z) = (z - z_0)^k \phi(z)$, where ϕ is analytic and $\phi(z_0) \neq 0$, then: $g(z) = \frac{f'(z)}{f(z)} = \frac{k}{z-z_0} + \frac{\phi'(z)}{\phi(z)}$, and $Res(g; z_0) = \frac{1}{2\pi i}\int_\ell g(z)dz = k\left[\frac{1}{2\pi i}\int_\ell \frac{dz}{z-z_0}\right] + \int_\ell \frac{\phi'(z)}{\phi(z)}dz = k.$

Therefore, by adding the residues obtained at each zero within f, the ℓtotal number of zeros (counted with their multiplicity) f within ℓ, that is, n clearly results.

On the other hand, if f has a pole of order k in $z_0 \in Int(\ell)$, then $\frac{f'(z)}{f(z)} = \frac{-k}{z-z_0} + \frac{\phi'(z)}{\phi(z)}$. and $Res(g; z_0) = -k$, summing the residues of g at the poles of f gives $-p$, Therefore: $\frac{1}{2\pi i}\int_\ell \frac{f'(z)}{f(z)}dz = n - p$, [4].

Theorem 21 (Rouche)**.** Let $f(z)$and be $g(z)$ analytic functions on and inside a Jordan curve ℓ. If $\|g(z)\| < \|f(z)\|, \forall z \in \ell$, then $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside ℓ.

Proof: Let: $F(z) = \frac{g(z)}{f(z)}$ and be n_1 and n_2 the number of zeros of $(f+g)$ and f respectively within ℓ, both functions do not have poles within ℓ (since $f + g$ and g are analytic), then

$$n_1 = \frac{1}{2\pi i}\int_\ell \frac{f'+g'}{f+g}dz \text{ and } n_2 = \frac{1}{2\pi i}\int_\ell \frac{f'}{f}dz,\ n_1 - n_2 = \frac{1}{2\pi i}\int_\ell \frac{F'}{1+F}dz,$$

as: $\|F(z)\| < 1$on ℓ, then $\frac{1}{1+F} = 1 - F + F^2 - F^3 + \ldots$(and the convergence is uniform), therefore, $n_1 - n_2 = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(0)dz = 0$, that is, $n_1 = n_2$.

IV. Discussion

The discussion is focused on the application and evaluation of Integrals classified into several groups:

Group 1: $\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}dx$, where p and q are relatively prime polynomials and the degree of q is at least 2 greater than the degree of p and q has no real zeros (p/q must be even). In this case: $\lim_{R\to\infty}\int_{-R}^{R}\frac{p(x)}{q(x)}dx = 2\pi i\sum_{Im(z)>0} Res(f, z)$, where $f(z) = \frac{p(z)}{q(z)}$.

Proof: Let $\ell = \ell_R \cup [-R, R]$, where ℓ_R: $\|z\| = R$, $Im(z) > 0$, be the residue theorem $\int_{-R}^{R}\frac{p(x)}{q(x)}dx + \int_{\ell_R} f(z)dz = 2\pi i\sum_{Im(z)>0} Res(f, z)$, where $R > 0$ is large enough to ℓ contain all singularities of f with $Im(z) > 0$. In this case, it suffices to prove that $\int_{\ell_R} f(z)dz \xrightarrow{R\to\infty} 0$, $\left\|\int_{\ell_R} f(z)dz\right\| \leq 2\pi R \underset{z\in\ell_R}{máx}\{\|f(z)\|\} \leq 2\pi\left(\frac{M}{R^2}\right) \xrightarrow{R\to\infty} 0$, $M > 0$ is a constant.

Application 1. Analyze the problem $\int_{-\infty}^{\infty}\frac{dx}{x^2+1} = \lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{x^2+1} = 2\pi i\left[Res\left(\frac{1}{z^2+1}, i\right)\right]$ and $\int_{-\infty}^{\infty}\frac{dx}{x^2+1} = 2\pi i\left[\lim_{z\to i}(z - i)\left(\frac{1}{z^2+1}\right)\right] = \pi$.

Application 2. Evaluate the integral $I = \int_0^{\infty}\frac{dx}{x^6+1} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{x^6+1}$, since $x^6 + 1$ it is an even function then, $\int_0^{\infty}\frac{dx}{x^6+1} = \frac{1}{2}\oint_C \frac{dz}{z^6+1}$, $z^6 + 1 = 0, z = \sqrt[6]{-1} = 1\left(cos\left(\frac{\pi+2k\pi}{6}\right) + isen\left(\frac{\pi+2k\pi}{6}\right)\right)$; $k = 0,1,2,3,4,5$. Only the roots for $k = 0,1,2$

which are in the upper half-plane, these are: $z_0 = e^{i\pi/6}$, $z_1 = e^{3i\pi/6}$and $z_2 = e^{5i\pi/6}$, for $z_0 = e^{i\pi/6}$, $Res(f(z))|_{z_0} = \lim_{z\to e^{\frac{i\pi}{6}}} \left(z - e^{\frac{i\pi}{6}}\right)\frac{1}{z^6+1} = \frac{0}{0}$. We can apply L' Hospital's rule, and we have $\lim_{z\to e^{\frac{i\pi}{6}}} \frac{1}{6z^5} = \frac{1}{6}e^{-\frac{5\pi i}{6}} = -\frac{\sqrt{3}}{12} - \frac{i}{12}$. For: $z_1 = e^{3i\pi/6}$ $Res(f(z))|_{z_1} = \lim_{z\to e^{\frac{3\pi i}{6}}} \left(z - e^{\frac{3\pi i}{6}}\right)\frac{1}{z^6+1} = \frac{0}{0}$, when applying L' Hospital's rule, $Res(f(z))|_{z_1} = \frac{1}{6}[i(-1)] = -\frac{i}{6}$.

For: $z_2 = e^{5i\pi/6}$, $Res(f(z))|_{z_2} = \lim_{z\to e^{\frac{5\pi i}{6}}} \left(z - e^{\frac{5\pi i}{6}}\right)\frac{1}{z^6+1} = \frac{0}{0}$. By applying L' Hospital's rule, we have $Res(f(z))|_{z_2} = \frac{\sqrt{3}}{12} - \frac{i}{12}$with,

$$\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = 2\pi i\left[Res(f(z))_{z_0} + Res(f(z))_{z_1} + Re\,s\,(f(z))_{z_2}\right], \int_{-\infty}^{\infty} \frac{dx}{x^6+1} = 2\pi i\left(-\frac{i}{3}\right) = \frac{2\pi}{3}.$$

Then $I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{1}{2}\left(\frac{2\pi}{3}\right) = \frac{\pi}{3}$.

Application 3. Analyze the problem $\int_{-\infty}^{\infty} \frac{dx}{1+x^{10}}$. Using real analysis, it is required to have an antiderivative of this function, which is very complicated. The most efficient way is to solve this integral via complex analysis, using the function $f(z) = \frac{1}{1+z^{10}}$, first determining the singularities of this function. In this case, they are the complex numbers $z_0, z_1, \ldots, z_9$of the form $z_k = e^{i\pi\left(\frac{2k+1}{10}\right)}$, $k = 0,1,2,\ldots,9$, and these are poles of order 1 (simple poles). Only the roots $k = 0,1,2,3,4$ are in the upper half-plane [7]. These are $z_0 = e^{\frac{i\pi}{10}}$, $z_1 = e^{\frac{3i\pi}{10}}$; $z_2 = e^{\frac{5i\pi}{10}}$; $z_3 = e^{\frac{7i\pi}{10}}$; $z_4 = e^{\frac{9i\pi}{10}}$, using the Cauchy residue theorem $\int_{-\infty}^{\infty} \frac{dx}{1+x^{10}} = 2\pi i \sum_{z_i\in Int(\Gamma)} Res(f, z_i)$, as

$$Res(f, z_i) = \left.\frac{1}{(1+z^{10})^1}\right|_{z=z_i} = \frac{1}{10z_i^9}$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^{10}} = 2\pi i \sum_{i=1}^{5} Res(f, z_i) = 2\pi i \sum_{n=0}^{4} \frac{1}{10\left(e^{i\pi\left(\frac{2n+1}{10}\right)}\right)^9}$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^{10}} = \frac{\pi i}{5}\sum_{n=0}^{4} \frac{1}{e^{i\pi\left(\frac{18n+9}{10}\right)}} = \frac{\pi i}{5}\left(\frac{1}{r_0+r_1+r_2+r_3+r_4}\right)$$

$$r_0 = -\cos(18^0) + isen(18^0) = -\frac{\sqrt{10+2\sqrt{5}}}{4} + i\frac{\sqrt{5}-1}{4}$$

$$r_1 = \cos\left(\frac{7\pi}{10}\right) + isen\left(\frac{7\pi}{10}\right) = -\frac{\sqrt{10-2\sqrt{5}}}{4} + i\frac{\sqrt{5}+1}{4}$$

$$r_2 = e^{i\pi\left(\frac{45}{10}\right)} = \cos\left(\frac{45\pi}{10}\right) + isen\left(\frac{45\pi}{10}\right) = i$$

$$r_3 = e^{i\pi\left(\frac{63}{10}\right)} = \cos\left(\frac{63\pi}{10}\right) + isen\left(\frac{63\pi}{10}\right) = \frac{\sqrt{10-2\sqrt{5}}}{4} + i\frac{\sqrt{5}+1}{4}$$

$$r_4 = e^{i\pi\left(\frac{81}{10}\right)} = \cos\left(\frac{81\pi}{10}\right) + isen\left(\frac{81\pi}{10}\right) = \frac{\sqrt{10+2\sqrt{5}}}{4} + i\frac{\sqrt{5}-1}{4}$$

Adding these expressions we have $r_0 + r_1 + r_2 + r_3 + r_4 = \frac{\pi(\sqrt{5}-1)}{20}$.

Group 2: $\int_{-\infty}^{\infty} g(x)\cos(kx)dx$, or $\int_{-\infty}^{\infty} g(x)sen(kx)dx$where $g(x) = \frac{p(x)}{q(x)}$ is as in the first case. Either of the two integrals is calculated by considering the integral $\int_{-\infty}^{\infty} g(x)e^{ikx}dx$, and equating the respective real and imaginary parts. We have $\int_{-\infty}^{\infty} g(x)e^{ikx}dx = 2\pi i \sum_{Im(z)>0} Res(g(z)e^{ikz}, z)$, as $\|g(z)e^{ik}\| = \|g(z)\|e^{-yk} \leq \|g(z)\|$, for $y > 0$ ($k > 0$ without loss of generality). The proof is similar.

Application 4. Analyze the problem $\int_{-\infty}^{\infty} \frac{cos(kx)dx}{x^2+a^2}$, $k > 0$ and $a > 0$.

$$\int_{-\infty}^{\infty} \frac{e^{ilx}dx}{x^2+a^2} = 2\pi i \left[Res\left(\frac{e^{ikz}}{x^2+a^2}, +ai\right)\right] = 2\pi i \lim_{z\to ai}(z-ai)\frac{e^{ikz}}{(z-ai)(z+ai)},$$

$$\int_{-\infty}^{\infty} \frac{cos(kx)dx}{x^2+a^2} = Re\left(\int_{-\infty}^{\infty} \frac{e^{ilx}dx}{x^2+a^2}\right) = \frac{\pi}{2a}e^{-ka} \text{and} \int_{-\infty}^{\infty} \frac{sen(kx)dx}{x^2+a^2} = Im\left(\int_{-\infty}^{\infty} \frac{e^{ilx}dx}{x^2+a^2}\right) = 0.$$

Application 5. Analyze the problem $\int_{-\infty}^{\infty} \frac{cos(x)dx}{(x^2+a^2)(x^2+b^2)}$.

$$I = \int_{-\infty}^{\infty} \frac{cos(x)dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right), a > 0$$

$$I = Re\left[\int_{-\infty}^{\infty} \frac{e^{ix}dx}{(x^2+a^2)(x^2+b^2)}\right] = 2\pi i \left[Res\left(\frac{e^{ikz}}{(x^2+a^2)(x^2+b^2)}, +ai\right)\right]$$

$$I = 2\pi i \left[\frac{e^{-a}}{(-a^2+b^2)(2ai)} + \frac{e^{-b}}{(-b^2+a^2)(2bi)}\right] = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right).$$

Group 3: Analyze the problem $\int_0^{2\pi} R(sen(\theta), cos(\theta))d\theta$. This integral is equal to the integral of a certain complex function $\phi(z)$ over the circle.

$$e^{i\theta} = cos(\theta) + isen(\theta) = z, e^{-i\theta} = cos(\theta) - isen(\theta) = \frac{1}{z}$$

$$cos(\theta) = \frac{1}{2}\left(z + \frac{1}{z}\right), sen(\theta) = \frac{1}{2i}\left(z - \frac{1}{z}\right),$$

By replacing in R, we obtain $\int_{\|z\|=1} \phi(z)dz$, this integral can be calculated by the residue theorem.

Application 6. Analyze the problem $\int_0^{2\pi} \frac{d\theta}{3+2sen(\theta)}$.

$$I = \int_0^{2\pi} \frac{d\theta}{3+2sen(\theta)} = \frac{2\pi}{\sqrt{5}}, \quad z = cos(\theta) + isen(\theta) = e^{i\theta}, dz = izdz$$

$$z^2 + 3iz - 1 = 0 \text{ si sólo si } z = \left(\frac{-3\pm\sqrt{5}}{2}\right)i, z_0 = \frac{-3+\sqrt{5}}{2}i \in Int(\ell)$$

$$I = 2\pi i\left(\frac{1}{3i+(-3+\sqrt{5}i)}\right) = \frac{2\pi}{\sqrt{5}}.$$

Application 7. Analyze the problem $I = \int_0^{2\pi} \frac{cos(3\theta)d\theta}{5-4sen(\theta)}$.

$$cos(\theta) = \frac{1}{2}\left(z + \frac{1}{z}\right), cos(3\theta) = \frac{1}{2}(z^3 + z^{-3}), dz = izd\theta$$

$$I = \int_{\|z\|=1} \frac{\frac{1}{2}(z^3+z^{-3})}{5-2(z+z^{-1})}\left(\frac{dz}{iz}\right) = \frac{1}{2i}\int_{\|z\|=1} \frac{(z^6+1)dz}{z^3(5z-1)(z-2)}.$$

Inside θ: $z = 0$, pole of order 3, $z = \frac{1}{2}$ pole of order 1

$$I = -\pi\left\{\lim_{z\to 0} \frac{1}{2i}\frac{d^2}{dz^2}\left(\frac{z^6+1}{z^3(2z-1)(z-2)}\right)\right\} - \pi\left\{\lim_{z\to\frac{1}{2}} \frac{\left(z-\frac{1}{2}\right)(z^6+1}{z^3(2z-1)(z-2)}\right\} = \frac{\pi}{12}.$$

Group 4: Integration around branch points. Evaluate $\int_0^\infty \frac{x^{-k}dx}{1+x}, k \in \langle 0,1\rangle$ (for convergence). In this case $f(z) = \frac{exp(-kLn(z))}{1+z}$, is a multivalued function; its principal value is taken. The function $f(z)$ is analytic in all $\mathbb{C}$ but the simple pole $z=-1$ and the branching line $y=0, x>0$ (i.e. $\theta = 0$). When $\theta = 0$, $r = x, z = x$, $f(z) = \frac{exp(-kLn(x))}{1+x} = \frac{x^{-k}}{1+x}$, i integrating f around the closed contour ℓ, which consists of two circular arcs: $\ell_0 : \|z\| = r$, $\ell_1 : \|z\| = R$ ($0 < r < 1, R = 1$) and two segments L_1 and L_2 of the rays $\theta = \varepsilon$ and $\theta = -\varepsilon$ ($\varepsilon > 0$) respectively, $\ell = \ell_0 \cup \ell_1 \cup L_1 \cup L_2$, as ℓ contains within itself the singularity $z=-1$,

$$\int_{L_1} f(z)dz + \int_{\ell_1} f(z)dz + \int_{L_2} f(z)dz + \int_{L_0} f(z)dz = 2\pi i[Res(f(z), -1)]$$

$$= 2\pi i \, exp[-k(Ln(1) + \pi i)] = 2\pi i e^{-k\pi i}, \tag{13}$$

about $L_1 : z = te^{i\varepsilon} \Rightarrow z^{-k} = t^{-k}e^{-ki\varepsilon}, t \in [r, R]$,

about $L_2 : z = te^{i(2\pi-\varepsilon)} \Rightarrow z^{-k} = t^{-k}e^{-ki(2\pi-\varepsilon)}, t \in [R, r]$,

$$\int_{L_1} f(z)dz + \int_{L_2} f(z)dz = \int_r^R \frac{t^{-k}e^{-ik\varepsilon}e^{i\varepsilon}dt}{1+te^{i\varepsilon}} + \int_R^r \frac{t^{-k}e^{-ik(2\pi-\varepsilon)}e^{i(2\pi-\varepsilon)}dt}{1+te^{i(2\pi-\varepsilon)}}$$

$$= e^{i(1-k)} \int_r^R \frac{t^{-k}dt}{1+te^{i\varepsilon}} - e^{i(1-k)(2\pi-\varepsilon)} + \int_r^R \frac{t^{-k}dt}{1+te^{i(2\pi-\varepsilon)}}$$

$$\lim_{\varepsilon\to 0}\left[\int_{L_1} f(z)dz + \int_{L_2} f(z)dz\right] = (1 - e^{2k\pi}) \int_r^R \frac{t^{-k}dt}{1+t}. \tag{14}$$

About $\ell_0 : z = Re^{i\theta}\theta \in [\varepsilon, 2\pi - \varepsilon]$,

about $\ell_1 : z = re^{i\theta}\theta \in [2\pi - \varepsilon, \varepsilon]$

$$\int_{\ell_0} f(z)dz + \int_{\ell_1} f(z)dz = \int_\varepsilon^{2\pi-\varepsilon} \frac{R^{-k}e^{-ik\theta}}{1+Re^{i\theta}} Rie^{i\theta}d\theta - \int_\varepsilon^{2\pi-\varepsilon} \frac{r^{-k}e^{-ik\theta}rie^{i\theta}d\theta}{1+re^{i\theta}}$$

$$I = \lim_{\varepsilon\to 0}\left[\int_{\ell_0} f(z)dz + \int_{\ell_1} f(z)dz\right] = iR^{((1-k)} \int_0^{2\pi} \frac{R^{-k}e^{-i\theta(1-k)}}{1+Re^{i\theta}} d\theta -$$

$$ir^{(1-k)} \int_0^{2\pi} \frac{r^{-k}e^{-i\theta(1-k)}dt}{1+re^{i\theta}}, \|I\| \le \frac{R^{1-k}}{R-1}(2\pi) + \frac{r^{1-k}}{r-1}(2\pi), \text{as} 0 < 1-k < 1$$

$$\lim_{\substack{r\to 0\\ R\to\infty}} \|I\| = 0 \Rightarrow \lim_{\substack{r\to 0\\ R\to\infty}} I = 0, \tag{15}$$

from (13), (14) and (15) it follows that $\lim_{\substack{r\to 0\\ R\to\infty}} (1 - e^{-2k\pi i}) \int_r^R \frac{t^{-k}}{1+t} dt + 0 = 2\pi i e^{-k\pi i}$,

$$\int_0^\infty \frac{t^{-k}}{1+t} dt + 0 = \frac{2\pi i e^{-k\pi i}}{1 - \pi i e^{-2k\pi i}} = \frac{2\pi i}{e^{k\pi i} - \pi i e^{-k\pi i}} = \frac{\pi}{sen(k\pi)},$$

Whenever there are branch points, circles around them should be avoided.

V. Conclusion

It was necessary to use the result of Goursat 's lemma, and that Cauchy's theorem is fulfilled for multiply connected regions, that is, $f : \mathbb{C} \longrightarrow \mathbb{C}$ an analytic function in $\Omega \subset C$ a domain and is $\Re \subset \Omega$ a multiply connected set whose boundary $Fr(\Re) = \ell \cup \ell_1 \cup \ldots \cup \ell_n$, where $\ell_k \subset Int(\ell)$ and $\ell_k, \ell, k = 1,2, \ldots, n$ are piecewise regular and disjoint Jordan curves oriented counterclockwise, $\int_\ell f(z)dz = \sum_{k=1}^n \int_{\ell_k} f(z)dz$.

It is evident that it satisfies the Cauchy integral formula, $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ analytical in a domain $\Omega \subset C$, with ℓ a counterclockwise rectifiable Jordan curve such that $\ell \cup Int(\ell) \subset \Omega$, that is $f(z) = \frac{1}{2\pi i}\int_{\ell} \frac{f(w)}{w-z} dw$, $\forall z \in Int(\ell)$; furthermore, for the definition of residue, f analytical Ω except at singularities, $z_1, z_2, \ldots, z_m$ of Ω and ℓ a Jordan curve in Ω which encloses $z_1, z_2, \ldots, z_m$, therefore, $\int_{\ell} f(z)dz = 2\pi i \sum_{k=1}^{m} Res(f, z_k)$.

Finally, to evaluate the integral $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} dx$, where p and are q mutually prime polynomials and the degree of q is at least two more than the degree of p and q has no real zeros (p/q must be even), we used that $\lim_{R\to\infty} \int_{-R}^{R} \frac{p(x)}{q(x)} dx = 2\pi i \sum_{Im(z)>0} Res(f, z)$, where $f(z) = \frac{p(z)}{q(z)}$.

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