IOSR Journal of Mathematics (IOSR-JM) e-ISSN: 2278-0661,
p-ISSN: 2278-8727, Volume 20, Issue 1, Ser. 1 (Jan. – Feb. 2024), PP 56-64
www.iosrjournals.org

Analytical Solution of Nonlinear Differential Equations Convolution Type by using Modified-Laplace Transform with Homotopy Perturbation Method

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Abstract
This work focused on a new semi-analytical method called modified Laplace transform with Homotopy Perturbation Method for solving nonlinear differential equations convolution type. The modified Laplace transform and homotopy perturbation method are used to solve the suggested problems. Some fundamental theorems have been established about the transforms of function convolution type and nth order derivatives. Theorems that had already been established were employed in conjunction with homotopy perturbation method capable of simplifying nonlinear functions of convolution type. The correctness of the suggested method was evaluated by considering some nonlinear differential equations convolution type. The findings were then compared to the referenced solutions, showing a favourable comparison. MLHPM can be implemented to solve nonlinear problems convolution type because it has a straightforward implementation.

***Keywords**: Modified Laplace Transform, Convolution, Homotopy Perturbation Method.*

Date of Submission: 13-01-2024 Date of Acceptance: 23-01-2024

I. Introduction

Nonlinear differential equations are of fundamental importance in various fields of science, technology and engineering. Nonlinear models of real-world problems are still difficult to solve either numerically or analytically. Over years different numerical methods such as Variational Iteration Method [3, 9, 13], Homotopy Perturbation Method [2] have been used to solve various linear and nonlinear Differential Equations (DEs). Several studies have proven that these methods are reliable and efficient for a wide range of scientific problems, linear and nonlinear differential equations with initial or boundary conditions. Homotopy perturbation method has been successfully applied to many initial and boundary value problems, the application of HPM to differential equations usually involve the following steps: applying zeroth order deformation to a given equation; writing the solution of given equation in a power series of p;comparing the coefficient like powers of p [2].

Attention has been paid to the search for better and more reliable approximate or exact, analytical or numerical methods for solving nonlinear models [10]. Integral transformations are one of the most commonly mathematical techniques used to find the solutions of linear advance problems of space, science, technology and engineering.The importance of these integral transforms is that they provide powerful operational methods for solving initial value problems as well as boundary value problems for linear differential and integral equations, Integral transforms play a crucial role in control engineering. There are many standard integral transform methods for solving linear differential equations, for example, the Laplace transform method and its various modifications [10].

Laplace transform is used to solve linear differential equations emerging in various field of Engineering, Science and Technology. During the study of probability, Laplace transform was introduced to solve differential equations with suitable initial and boundary conditions. Modified Laplace transform for a piecewise continuous function of exponential order which reduce to Laplace transform for a = e where $a \neq 1$, $a > 0$ and $t > 0$ was introduced [6].

Researchers have coupled Laplace transform with other numerical methods because of its inability to handle nonlinear equations. Among the schemes are: Homotopy Perturbation Method [2] Variational Iteration Method [7, 8, 12] and Adomian Decomposition Method [1]. This limitation of Laplace transform also affects other integral transforms such as Sawi transform, Sadik transform, mohand transform, Kamal transform and Elzaki transform. Modified Laplace transform [6] is one these integral transform, thus, it has to be coupled with another method to effectively solve nonlinear differential problems and this necessitate its combination with Homotopy Perturbation Method. In this work, a coupled modified Laplace transform with HPM is proposed to solve nonlinear differential equations of convolution type. The proposed scheme yields analytical solutions. The solutions at third iteration were compared with exact solutions and these results revealed that the coupled modified Laplace transform with HPM is semi- analytical method.The obtained results show the reliability, accuracy and efficient of the proposed scheme.

II. Basic of Homotopy perturbation Method (HPM)

Consider the differential equation in an operator form as:

$$Ry(t)+Uy(t)+Ny(t)=g(t) \tag{1}$$

Where R is a linear operator with the highest derivative, U is remaining linear operator with derivative less than R, N is a nonlinear operator, g is nonhomogeneous. Homotopy perturbation method can be defined as [11]

$$(1-p)[Ry(t)-Ry_0(t)]+p[Ry(t)+Uy(t)+Ny(t)-g(t)]=0 \tag{2}$$

or

$$Ry(t)-Ry_0(t)+pRy_0(t)+p[Ry(t)-Ry(t)+Uy(t)+Ny(t)-g(t)]=0 \tag{3}$$

Where p is a homotopy artificial parameter, whose values are within range of 0 and 1. By substituting $p=0$ and $p=1$ in Eqn. (3), gives

$$Ry(t)-Ry_0(t)=0,$$

and

$$Ry(t)+Uy(t)+Ny(t)-g(t)=0, \text{ respectively.}$$

And $y(0)$ is initial approximation for the solution of eqn. (1) that satisfies initial conditions given.

Assuming, solution for eqn. (2) or eqn. (3) can be written as a power series of p, then

$$y(t)=\sum_{n=0}^{\infty} p^n v_n \tag{4}$$

Substituting eqn. (4) into eqn. (3) and using the initial conditions and equating terms with identical powers of p led to set of equations as

$$\left\{\begin{array}{l} p^0 : Rv_0 = R(y_0), \\ p^1 : Rv_1 = -Ry_0 - N(v_0) - U(v_0) + g(t) \\ p^2 : Rv_2 = -U(v_0, v_1) - N(v_0, v_1) \\ . \\ . \\ . \\ p^n : Rv_n = -U(v_0, v_1, v_{n-1}) - N(v_0, v_1, v_{n-1}) \end{array}\right\} \tag{5}$$

Therefore, the solution of eqn. (1) can be obtained as $p=1$, gives

$$y(t)=\lim_{p\to 1} y\ (t)=v_0+v_1+v_2+... \tag{6}$$

Definition 1

Laplace transform of a function f (t)) for a piecewise continuous functions of exponential order is defined as

$$L(f(t))=\int_0^{\infty} e^{-st} f(t)dt\ ,(R(s)>0), \tag{7}$$

Definition 2
Modified Laplace transform of f (t)) for a piecewise continuous functions of exponential order is defined as

$$L(f(t))=\int_0^{\infty} a^{-st} f(t)dt \ ,(R(s)>0), \tag{8}$$

Where eqn. (8) reduces to simple Laplace transform for $e=a$ [6].

Definition 3
Convolution of two functions, $f(t)$ and $g(t)$ for a piecewise continuous functions of exponential order is defined as

$$f(t)*g(t)=\int_0^{\infty} f(q)g(t-q)dq . \text{ [6]}$$

Table1: Some Fundamental Properties of Modified-Laplace Transform (Faisal *et al*., 2020)

S/N	f(t)	**Modified-Laplace transform;** $L_a(f(t))$
1	1	$\frac{1}{s\log a}$
2	$t^n, n>0$	$\frac{n!}{s^{n+1}(\log a)^{n+1}}$
3	$\sin bt$	$\frac{s}{s^2(\log\ a)^2+b^2},(s\log\ a>0)$
4	$\cos bt$	$\frac{b}{s^2\ (\log\ a)^2+b^2},\ (s\log\ a>0)$
5	$\cosh bt$	$\frac{s}{s^2(\log\ a)^2-b^2},(s\log\ a>0)$
6	$\sinh bt$	$\frac{b}{s^2(\log\ a)^2-b^2},(s\log\ a>0)$
7	e^{bt}	$\frac{1}{s\log\ a-b},(s\log\ a>\|b\|)$

Theorem 1:
If $f(t)$ and $g(t)$ are piecewise continuous functions of exponential order, where

$$L_a(f(t))=f(s,a) \tag{9}$$

And

$$L_a(g(t))=g(s,a) \tag{10}$$

then, modified-Laplace transform of convolution of two functions is given as

$$L_a(f(t)*g(t))=f(s,a)g(s,a) \tag{11}$$

Proof:
convolution of two functions f (t) and g (t) is defined as

$$f(t)*g(t)=\int_0^{\infty} f(q)g(t-q)dq . \tag{12}$$

Taking modified-Laplace transform of convolution f (t) and g (t), gives

$$L_a(f(t)*g(t))=\int_0^{\infty} f(t)*g(t)a^{-st}dt . \tag{13}$$

Substituting Equation (12) into Equation (13), gives

$$L_a\left(f(t)*g(t)\right)=\int_0^\infty\left(\int_0^\infty f(q)g(t-q)dq\right)a^{-st}dt\ . \qquad (14)$$

Simplifying Equation (14), gives

$$L_a\left(f(t)*g(t)\right)=\int_0^\infty\int_0^\infty f(q)g(t-q)a^{-st}dtdq \qquad (15)$$

Let z= t – q, dz = dt and t = z + q, Equation (15) becomes

$$L_a\left(f(t)*g(t)\right)=\int_0^\infty\int_0^\infty f(q)g(z)a^{-s(z+q)}dzdq\ . \qquad (16)$$

Simplifying Equation (16), gives

$$L_a\left(f(t)*g(t)\right)=\int_0^\infty f(q)a^{-sq}dq\int_0^\infty g(z)a^{-s(z)}dz\ . \qquad (17)$$

Where

$$f(s,a)=\int_0^\infty f(q)a^{-sq}dq\ . \qquad (18)$$

And

$$g(s,a)=\int_0^\infty g(z)a^{-sz}dz\ . \qquad (19)$$

Substituting Equation (18) and Equation (19) into equation (17), gives

$$L_a\left(f(t)*g(t)\right)=f(s,a)g(s,a)\ . \qquad (20)$$

Proof complete.

Theorem 2

If $f(t)$ is a piecewise continuous function of exponential order, where Equation (8) is modified-Laplace transform of $f(t)$ and $f^z(t)$ is the z-th order derivative of a function $f(t)$, then modified-Laplace transform of z-th order derivative is

$$L_a\left(f^z\right)=s^z\left(\log_e a\right)^z L_a f(t)-\sum_{k=1}^{z}s^{z-k}\left(\log_e a\right)^{z-k}f^{k-1}(0) \qquad (21)$$

Where f^z is the z-th order derivative.

Proof: Applying the principle of mathematical induction, when z=1 in Equation (21),

$$L_a\left(f^{'}\right)=s\left(\log_e a\right)L_a\left(f(t)\right)-f\ \ (0)\quad . \qquad (22)$$

To achieve Equation (22),

$$L_a\left(f^{'}(t)\right)=\int_0^\infty a^{-st}\left(f^{'}(t)\right)dt \qquad (23)$$

Simplifying Equation (23), using the method of integration by parts, gives

$$L_a\left(f^{'}(t)\right)=-f(0)+s\log_e a\int_0^\infty f(t)\,a^{-st}dt\ ; \qquad (24)$$

Hence,

$$L_a\left(f^{'}(t)\right)=s\left(\log_e a\right)L_a\left(f(t)\right)-f(0)\ . \qquad (25)$$

And so z =1 is true.

Assuming z=m in Equation (21) is true, then

$$L_a\left(f^m(t)\right)=s^m\left(\log_e a\right)^m L_a\left(f(t)\right)-\sum_{k=1}^{m}s^{m-k}\left(\log_e a\right)^{m-k}f^{k-1}(0)\ , \qquad (26)$$

is assumed to be true. It has to be shown that z=m+1 in Equation (21) is true whenever z=m is true. That is

$$L_a\left(f^{m+1}(t)\right)=s^{m+1}\left(\log_e a\right)^{m+1}L_a\left(f(t)\right)-\sum_{k=1}^{m+1}s^{m+1-k}\left(\log_e a\right)^{m+1-k}f^{k-1}(0), \qquad (27)$$

is true.

to show Equation (27), let

$$L_a\left(f^{m+1}(t)\right)=\int_0^\infty a^{-st}\left(f^{(m+1)}(t)\right)dt \qquad (28)$$

Simplifying Equation (28) with method of integration by part, then

$$L_a\left(f^{m+1}(t)\right) = -f^m(0) + s\log_e a\int_0^\infty a^{-st} f^m(t)dt \tag{29}$$

Since

$$L_a\left(f^m(t)\right) = \int_0^\infty a^{-st}\left(f^{(m)}(t)\right)dt \tag{30}$$

Substituting Equation (30) into Equation (29), gives

$$L_a\left(f^{m+1}(t)\right) = -f^m(0) + s\log_e a\ L_a\left(f^m(t)\right) \tag{31}$$

Substituting Equation (26) into Equation (31), gives

$$L_a\left(f^{m+1}(t)\right) = s\log_e a\left(s^m(\log_e a)^m L_a(f(t)) - \sum_{k=1}^{m} s^{m-k}(\log_e a)^{m-k} f^{k-1}(0)\right) - f^m(0), \tag{32}$$

expanding Equation (32), gives

$$L_a\left(f^{m+1}(t)\right) = s^{m+1}(\log_e a)^{m+1} L_a(f(t)) - \sum_{k=1}^{m+1} s^{m+1-k}(\log_e a)^{m+1-k} f^{k-1}(0)\ . \tag{33}$$

This completes the proof. The essence of theorem 2 is to find the modified-Laplace transform of derivative of a function which has order z. This derivative will be used to derive the scheme for solving nonlinear differential equation of classical order z.

III. The proposed scheme of MLHPM.

Applying zeroth order deformation to Equation (1), gives

$$(1-p)[Ry(t) - Ry_0(t)] + p[Ry(t) + Uy(t) + Ny(t) - f(t)] = 0 \tag{34}$$

Taking modified-Laplace transform of Equation (34), gives

$$L_a(R\,y(t) - R\,y_0(t) + pR\ y_0\ (t) + P[U\,y\,(t) + N\,y\,(t) - f\,(t)]) = 0 \tag{35}$$

Simplifying Equation (35), gives

$$\begin{aligned} &s^m(\log\ a)^m\, Y_n(s\log\ a) - s^{m-1}(\log\ a)^{m-1} y(0) \\ &\quad -\ldots - y^{m-1}(0) = L_a\begin{pmatrix} R\,y_0\ (t) - P\ (R\,y_0\ (t)) \\ +P\ [-U\ y\ (t) - N\ y\ (t) + f(t)] \end{pmatrix} \end{aligned} \tag{36}$$

Isolating $y_n(s\log\ a)$ in Equation (36), gives

$$y_n(s\log a) = \frac{1}{s^m(\log a)^m}\begin{pmatrix} s^{m-1}(\log a)^{m-1} y(0) + \ldots + y^{m-1}(0) \\ + L_a(Ry_0(t) - P(Ry_0(t)) + P[-Uy\ (t) - Ny\ (t) + f(t)]) \end{pmatrix} \tag{37}$$

Applying inverse modified-Laplace transforms to both sides of Equation (37) gives

$$y(t) = L_a^{-1}\left(\frac{1}{s^m(\log a)^m}\begin{bmatrix} s^{m-1}(\log a)^{m-1} y(0) + \ldots + y^{m-1}(0) \\ + L_a(Ry_0(t) - P(Ry_0\ (t)) + P[-Uy\ (t) - Ny\ (t) + f(t)]) \end{bmatrix}\right) \tag{38}$$

Assuming that the solution of Equation (1) can be expressed as a power series of p, gives

$$y(t) = \sum_{n=0}^{\infty} p^n v_n\ , \tag{39}$$

substituting Equation (39) into Equation (38), this gives

$$\sum_{n=0}^{\infty} p^n v_n = L_a^{-1}\left(\frac{1}{s^m(\log a)^m}\begin{bmatrix} s^{m-1}(\log a)^{m-1} y(0) + \ldots + y^{m-1}(0) \\ + L_a\left(Ry_0 - P(Ry_0) + P\left[-U\sum_{n=0}^{\infty} p^n v_n - N\sum_{n=0}^{\infty} p^n v_n + f(t)\right]\right) \end{bmatrix}\right), \tag{40}$$

comparing coefficient of p with the same powers in Equation (40), gives

$$\left\{\begin{array}{l} p^0 : v_0 = L_a^{-1}\left(\frac{1}{s^m(\log a)^m}\left[s^{m-1}(\log a)^{m-1} y(0) + \ldots + y^{m-1}(0) + L_a R(y_0)\right]\right) \\ p^1 : v_1 = L_a^{-1}\left[\frac{1}{s^m(\log a)^m} L_a\left[-Ry_0 - N(v_0) - U(v_0) + f(t)\right]\right] \\ p^2 : v_2 = L_a^{-1}\left[\frac{1}{s^m(\log a)^m} L_a\left[-U(v_0, v_1) - N(v_0, v_1)\right]\right] \\ . \\ . \\ p^n : v_n = L_a^{-1}\left[\frac{1}{s^m(\log a)^m} L_a\left[-U(v_0, v_1, v_{n-1}) - N(v_0, v_1, v_{n-1})\right]\right] \end{array}\right\}. \tag{41}$$

Therefore, the solution of Equation (41) can be given as

$$y(t) = \lim_{p \to 1} y\ (t) = v_0 + v_1 + v_2 + \ldots \tag{42}$$

IV. Numerical illustrations.

The above method is applied to obtain solutions of certain nonlinear ordinary differential equations(ODE) of convolution type.

Illustration1: Consider nonlinear differential equation of convolution type given as (Elzaki and Mohmod, 2013).

$$y' - (y')^2 - 2x + y' * (y'')^2 = 0, \qquad y(0) = 1, \tag{43}$$

Equation (43) has exact solution

$$y(t) = 1 + t^2 \tag{44}$$

Applying zeroth order deformation to eqn.(43), gives

$$(1-p)(y' - y_0') + p\left(y' - (y')^2 - 2x + y' * (y'')^2\right) = 0 \tag{45}$$

Simplifying Equation (45), gives

$$y' - y_0' + py_0' + p(-(y')^2 - 2x + y' * (y'')^2) = 0, \tag{46}$$

introduce L_a to both sides of Equation (46) and simplify correctly using the initial conditions gives

$$y(s\log a) = \frac{1}{s(\log_e a)} + \frac{1}{s(\log_e a)} L_a\left(y_0' - py_0' - p\left(-(y')^2 - 2x' + y' * (y'')^2\right)\right), \tag{47}$$

taking inverse modified-Laplace transform of Equation (47), gives

$$y(t) = L_a^{-1}\left(\frac{1}{s(\log_e a)} + \frac{1}{s(\log_e a)} L_a\left(y_0' - py_0' - p\left(-(y')^2 - 2x' + y' * (y'')^2\right)\right)\right), \tag{48}$$

where

$$L_a\{y' * y''\} = L_a(y') \cdot L_a(y''). \tag{49}$$

Equation (48) was obtained by using modified-Laplace transform of convolution of functions which was established in theorem 1. Assuming solution of Equation (43) can be written as

$$\sum_{n=0}^{\infty} p^n v_n = y\ , \tag{50}$$

substituting eqn. (50) into Equation (48), gives

$$\sum_{n=0}^{\infty} p^n v_n = L_a^{-1}\left(\begin{array}{l}\frac{1}{s(\log_e a)} \\ +\frac{1}{s(\log_e a)} La\left(y_0^{'} - py_0^{'} - p\left(\begin{array}{l}2\left(v_0^{'} + pv_1^{'} + p^2 v_2^{1}\right)\left(v_0^{''} + pv_1^{''} + p^2 v_2^{''}\right) - \\ \left(v_0^{'} + pv_1^{'} + p^2 v_2^{'}\right)\left(v_0^{''} + pv_1^{''} + p^2 v_2^{''}\right)^2\end{array}\right)\right)\end{array}\right). \quad (51)$$

By comparing the coefficient of like powers of p in Equation (51), gives

$$\left.\begin{array}{l}p^0 : v_0 = L_a^{-1}\left(\frac{1}{s(\log_e a)} + \frac{1}{s(\log_e a)} La\left(y_0^{'}\right)\right) \\ p^1 : v_1 = L_a^{-1}\left(\frac{-1}{s(\log_e a)} La\left(-y_0^{'} + \left(2v_0^{'}v_0^{''} - \left(v_0^{'}\right)\left(v_0^{''}\right)^2\right)\right)\right) \\ p^2 : v_2 = L_a^{-1}\left(\frac{-1}{s(\log_e a)} La\left(\left(2v_1^{'}v_1^{''} - \left(v_1^{'}\right)\left(2v_0^{''}v_1^{''}\right)\right)\right)\right) \\ p^3 : v_3 = L_a^{-1}\left(\frac{-1}{s(\log_e a)} La\left(\left(2v_2^{'}v_2^{''} - \left(v_2^{'}\right)\left(2v_0^{''}v_2^{''} + \left(v_1^{''}\right)\right)\right)\right)\right)\end{array}\right\}, \quad (52)$$

solving Equation (52), gives

$$\left.\begin{array}{l}v_0 = 1 \\ v_1 = t^2 \\ v_2 = 0 \\ v_3 = 0\end{array}\right\}, \quad (53)$$

substituting Equation (53) into Equation (42), gives solution to Equation (43) as

$$y(t) = 1 + t^2 \quad (54)$$

which is actually the exact solution.

Table 2: Solutions of MLHPM and EXACT for Eqn. (43)

t	MLHPM (Y)	Exact (Y)
0.1	**1.01000000**	**1.01000000**
0.2	**1.04000000**	**1.04000000**
0.3	**1.09000000**	**1.09000000**
0.4	**1.16000000**	**1.16000000**
0.5	**1.25000000**	**1.25000000**
0.6	**1.36000000**	**1.36000000**
0.7	**1.49000000**	**1.49000000**
0.8	**1.64000000**	**1.64000000**
0.9	**1.81000000**	**1.81000000**
1.0	**1.01000000**	**1.01000000**

Illustration 2: Consider nonlinear differential equation of convolution type given as (Elzaki and Mohmod. 2013).

$$y^{''} - 2 + 2y^{'} * y^{''} - y^{'} * \left(y^{''}\right)^2 = 0, \qquad y(0) = y^{'}(0) = 0, \quad (55)$$

Equation (55) has exact solution

$$y(t) = t^2 \quad (56)$$

Applying zeroth order deformation to eqn.(55), gives

$$(1-p)\left(y^{''} - y_0^{''}\right) + p\left(y^{''} - 2 + 2y^{'} * y^{''} - y^{'} * \left(y^{''}\right)^2\right) = 0 \quad (57)$$

Simplifying Equation (57), gives

$$y^{''} - y_0^{''} + py_0^{''} + p(-2 + 2y^{'} * y^{''} - y^{'} * \left(y^{''}\right)^2) = 0, \quad (58)$$

introduce L_a to both sides of Equation (58) and simplify correctly using the initial conditions gives

$$y(s\log a)=\frac{2}{s^2(\log_e a)^2}+\frac{1}{s^2(\log_e a)^2}L_a\left(y_0''-py_0''-p\left(2y'*y''-y'*\left(y''\right)^2\right)\right), \tag{59}$$

taking inverse modified-Laplace transform of Equation (59), gives

$$y(t)=L_a^{-1}\left(\frac{2}{s^2(\log_e a)^2}+\frac{1}{s^2(\log_e a)^2}L_a\left(y_0''-py_0''-p\left(2y'*y''-y'*\left(y''\right)^2\right)\right)\right), \tag{60}$$

where

$$L_a\{y'*y''\}=L_a(y')\cdot L_a(y''). \tag{61}$$

Equation (60) was obtained by using modified-Laplace transform of convolution of functions which was established in theorem 1. Assuming solution of Equation (55) can be written as

$$\sum_{n=0}^{\infty}p^n v_n=y\ , \tag{62}$$

substituting eqn. (62) into Equation (60), gives

$$\sum_{n=0}^{\infty}p^n v_n=L_a^{-1}\left(\begin{matrix}\dfrac{2}{s^2(\log_e a)^2}\\ +\dfrac{1}{s^2(\log_e a)^2}La\left(y_0''-py_0''-p\left(\begin{matrix}2\left(v_0'+pv_1'+p^2v_2^1\right)\left(v_0''+pv_1''+p^2v_2''\right)-\\ \left(v_0'+pv_1'+p^2v_2'\right)\left(v_0''+pv_1''+p^2v_2''\right)^2\end{matrix}\right)\right)\end{matrix}\right). \tag{63}$$

By comparing the coefficient of like powers of p in Equation (63), gives

$$\left.\begin{aligned}
p^0 &: v_0=L_a^{-1}\left(\frac{2}{s^3(\log_e a)^3}+\frac{1}{s^2(\log_e a)^2}La\left(y_0'\right)\right)\\
p^1 &: v_1=L_a^{-1}\left(\frac{-1}{s^2(\log_e a)^2}La\left(-y_0'+\left(2v_0'v_0''-\left(v_0'\right)\left(v_0''\right)^2\right)\right)\right)\\
p^2 &: v_2=L_a^{-1}\left(\frac{-1}{s^2(\log_e a)^2}La\left(\left(2v_1'v_1''-\left(v_1'\right)\left(2v_0''v_1''\right)\right)\right)\right)\\
p^3 &: v_3=L_a^{-1}\left(\frac{-1}{s^2(\log_e a)^2}La\left(\left(2v_2'v_2''-\left(v_2'\right)\left(2v_0''v_2''+\left(v_1''\right)\right)\right)\right)\right)
\end{aligned}\right\}, \tag{64}$$

solving Equation (64), gives

$$\left.\begin{aligned}
v_0&=t^2\\
v_1&=0\\
v_2&=0\\
v_3&=0
\end{aligned}\right\}, \tag{65}$$

substituting Equation (65) into Equation (42), gives solution to Equation (55) as

$$y(t)=t^2 \tag{66}$$

Table 3: Solutions of MLHPM and EXACT for Eqn. (55)

t	MLHPM (Y)	Exact (Y)
0.1	0.01000000	0.01000000
0.2	0.04000000	0.04000000
0.3	0.09000000	0.09000000
0.4	0.16000000	0.16000000
0.5	0.25000000	0.25000000
0.6	0.36000000	0.36000000
0.7	0.49000000	0.49000000
0.8	0.64000000	0.64000000
0.9	0.81000000	0.81000000
1.0	0.01000000	0.01000000

V. Discussion

Modified Laplace Transform of zth order derivative $\left(f^{z}(t)\right)$ and convolution of two functions f(t) and g(t) has been established in theorem 1 and theorem 2, this result was used alongside the scheme of HPM to form a new modified method capable of solving nonlinear differential equations of convolution type. The derived scheme was applied to solve differential problems and the results are shown in tables 2 and 3. It was observed from the tables that the proposed scheme gives the same results when compared with exact solutions. This justify the effectiveness of the proposed method.

VI. Conclusion

A new method for solving nonlinear differential equations convolution type has been derived by coupling modified Laplace transform with HPM. Theorems associated with this scheme were established. The numerical results showed that the method is reliable and effective. The method is used without linearization of functions.

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