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Strong Liouville's Theorem Based Conjecture on Nonelementary Integrals

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Abstract
The introduction of special functions like hypergeometric function, Bessel function, Legendre function, etc. and accepting the new functions beyond the scope of elementary functions like the error function, Fried Conte function, Fresnel function, exponential integral, sine and cosine integrals, logarithmic integral, etc. have ended the study of searching special types of nonelementary integrals. Due to the less research on it has terminated the development of new properties in integration on elementary and nonelementary functions. It is a big drawback that the students are not taught the basic properties of integration like Bernoulli Conjecture, Laplace Theorem, Abel Theorem, Liouville Theorem, Chebyshev theorem, Liouville Hardy theorem, Inverse function theorem, etc., which make it more informative. The aim of this article is to determine those elementary functions (integrands), which are either integrable or nonintegrable in context of antiderivatives. Such functions have been formed from the composition of exponential function, polynomial function of degree one and two, and the inverse trigonometric functions. It is an attempt to proffer a conjecture on nonelementary integrals proved by strong Liouville theorem. The article ends with some notes on limitations and its future scope of research.

Key-Words: *Conjecture, Nonelementary Integral, Strong Liouville Theorem.*

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I. Introduction

A conjecture means to guess about something without a real proof. In mathematics, it states a mathematical statement that is propounded on a tentative basis without a proof (Conjecture – Wikipedia). In some cases, it may have many examples for the reason to propound the statement such as in Goldbach's conjecture 'every even natural number greater than 2 is the sum of two prime numbers', we have 4 = 2 + 2, 6 = 3 + 3, 8 = 3 + 5, 10 = 3 + 7, 12 = 5 + 7, etc. (Goldbach's conjecture - Wikipedia). Some well known conjectures like Riemann hypothesis 'all non trivial zeros of the zeta function lie along the critical line' and Fermat's Last theorem 'no three positive integers a, b, and c can satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than two' (a conjecture until proven in 1995 by Andrew Wiles) have developed new areas of mathematics in order to prove them (Weisstein - Mathworld). In the present paper, a new conjecture related to nonelementary integrals has been proffered, which is also known as nonelementary function originated from non-antiderivative of elementary functions.

Elementary and nonelementary functions were introduced by Joseph Liouville in a series of papers from 1833 to 1841 and the algebraic treatment of elementary functions was started by Joseph Fels Ritt in 1930s. An elementary function is a single variable (real or complex) function that is expressed as sums, differences, products, divisions, roots and composition of finitely many polynomials, rational, trigonometric, hyperbolic, exponential, and their inverse functions. For example, $x^2 + 1$, $\sqrt{x^2 - 1}$, e^x, $\log(x)$, $sinx + x^2 + x$, $\int x\, sinx\, dx$, $\int x\, e^{x^2}\, dx$, π, e, 4, $sinhx$, $arcsinx$, $|x|$, etc. are elementary functions. But every function is not an elementary function. For example, the integral $\int e^{-t^2} dt$ is not an elementary function (Anton, 2014; Cherry, 1985, 1986; Elementary Function – Wikipedia; Hardy, 2018; Kasper, 1980; Lutfi, 2016; Marchisotto et al., 1994; Risch, 1969; 1970, 2022; Ritt, 2022; Rosenlicht, 1972; Thompson, 2021; Yadav et al., 2012; Yadav, 2023).

Every elementary function can always be written in a closed form. A closed form expression is a mathematical expression that uses a finite number of mathematical standard operations. It may contain constants, variables, arithmetic operations (e.g., +, -, x, /) and functions like nth root, exponent, logarithm, trigonometric functions, hyperbolic functions, inverse trigonometric functions, inverse hyperbolic functions, etc. It usually does not contain limit or integral (Closed form expression – Wikipedia; Hardy, 2018; Marchisotto et al., 1994; Risch, 1969; 1970, 2022; Ritt, 2022; Rosenlicht, 1972; Yadav et al., 2012; Yadav, 2023). A special care is needed in talking about elementary and nonelementary functions or closed form expressions and non-closed form expressions. There are expressions which are not in closed form but can be reduced into it after simplification or summation or using other operations. For example, the expression

$$\mathrm{f}(x) = \sum_{n=0}^{\infty} \frac{x}{2^n} = \frac{x}{2^0} + \frac{x}{2^1} + \frac{x}{2^2} + \cdots + \frac{x}{2^n} + \cdots$$

is not in closed form because the summation contains infinite number of terms and elementary operations, however using the summation rule of a geometric series it can be expressed in the closed form as $\mathrm{f}(x) = 2x$. If a function is expressed in closed form expressions, its derivative can be expressed in closed form expression. But its integral may or may not be expressed in closed form. An example of an elementary function whose antiderivative does not have a closed form expression is e^{-x^2}, whose one antiderivative is the error function given by

$$\mathrm{erf}(x) = \frac{1}{\pi} \int_0^x e^{-t^2} dt.$$

We have called such integral as nonelementary integral. In other words, a nonelementary integral or nonelementary antiderivative of a given elementary function is an antiderivative that is not an elementary function and that cannot be expressed in closed form expression. Some well known examples are the elliptic integral $\sqrt{1-x^4}$, logarithmic integral $\frac{1}{\ln x}$, Gaussian integral e^{-x^2}, Fresnel integrals $\sin(x^2)$ and $\cos(x^2)$, Sine integral (or Dirichlet integral) $\frac{\sin x}{x}$, Exponential integral $\frac{e^{-x}}{x}$, etc. (Closed-form expression-Wikipedia; Corliss et al., 1989; Elliptic integral – Wikipedia; Error function – Wikipedia; Exponential integral – Wikipedia; Fresnel integral – Wikipedia; Gaussian integral – Wikipedia; Hardy, 2018; Marchisotto et al., 1994; Nijimbere, 2017, 2018, 2020a, 2020b; Nonelementary integral – Wikipedia; Sao, 2021; Sharma et al., 2020; Singer et al., 1985; Trager, 2022; Trigonometric integral – Wikipedia; Victor, 2017; Yadav et al., 2012; Yadav, 2023).

As far as the previous works on nonelementary integrals are concerned, the first example which leads us beyond the region of elementary functions is the elliptic integrals, which cannot be expressed in terms of elementary functions. The first reported study of such integrals was due to John Wallis in 1655, when he began to study the arc length of an ellipse. Euler also studied elliptic functions and discovered that they were not integrable in terms of the elementary functions. But such integrals cannot be evaluated in terms of elementary functions was proved by Joseph Liouville in 1833. Although many pioneers contributed in the advancement of the subject like John Bernoulli (1702), Laplace (1812), A. M. Legendre (1825), N. H. Abel (1826, 1829), P. L. Chebyshev (1853), E. Hermite (1872), G. H. Hardy (1905), D. D. Mordoukhay Boltovskoy (1906-1910, 1913, 1937), C. Hermite (1912), A. Ostrowski (1940), Joseph F. Ritt (1916, 1948), M. Rosenlicht (1967-68), etc. but in 1833 Joseph Liouville created a framework for constructive integration by finding out when antiderivative of elementary functions are again elementary functions i.e., the main results on functions with nonelementary integrals began with Liouville's works (Cherry, 1985, 1986; Kasper, 1980; Lutfi, 2016; Marchisotto et al., 1994; Risch, 1969; 1970, 2022; Ritt, 2022; Rosenlicht, 1972; Trager, 2022; Yadav et al., 2012; Yadav, 2023).

Liouville first introduced Liouville's First Theorem on Integration in 1833 but in 1835 he generalized this theorem to several variables and gave strong Liouville theorem, and thereby greatly extended the class of functions one can prove to have nonelementary integrals. In the same year 1835, he developed the special case of this theorem, which gives the necessary and sufficient conditions for the existence of elementary function of some special functions (Nonelementary integral - Wikipedia). He showed also that the elliptic integrals of the first and second kinds have no elementary expressions. By 1841, Liouville had developed a theory of integration that settled the question of integration in finite terms for many important cases. Unfortunately no single textbook or research article is available on it having the history of development of the subject. In 1994 E. A. Marchisotto and G. A. Zakeri presented a short note on nonelementary functions, one of the best articles available online (Marchisotto et al., 1994; Risch, 1969; 1970, 2022; Ritt, 2022; Rosenlicht, 1972; Trager, 2022; Yadav et al., 2012; Yadav, 2023). Yadav & Sen (2012) propounded six conjectures on nonelementary integrals and the present paper is the continuation of their work for the exceptional cases.

II. Preliminary Ideas

As stated earlier the present study is an attempt to search some special elementary and nonelementary functions in context of antiderivative, in which inverse trigonometric functions will play a major role as a component in the integrand. We will use well known nonelementary integrals as the standard formulae in case of need to decide the new nonelementary integrals as well as we will also use strong Liouville theorem which states that:

(a) If F is an algebraic function of $x, y_1, y_2, \dots, y_m$ where $y_1, y_2, \dots, y_m$ are functions of x, whose derivatives

$$\frac{dy_1}{dx}, \frac{dy_2}{dx}, \frac{dy_3}{dx}, \dots, \frac{dy_m}{dx}$$

are rational functions of $x, y_1, y_2, \dots, y_m$, then

$$\int F(x, y_1, y_2, \dots, y_m)\, dx$$

is elementary if and only if

$$\int F(x, y_1, y_2, \dots, y_m)\, dx = U_0 + \sum_{j=1}^{n} C_j \log(U_j)$$

where the C_j's are constants, and the U_j's are algebraic functions of $x, y_1, y_2, \dots, y_m$.

(b) If $F(x, y_1, y_2, \dots, y_m)$ is a rational function and

$$\frac{dy_1}{dx}, \frac{dy_2}{dx}, \frac{dy_3}{dx}, \dots, \frac{dy_m}{dx}$$

are rational functions of $x, y_1, y_2, \dots, y_m$, then the U_j's in part (a) must be rational functions of $x, y_1, y_2, \dots, y_m$ (Marchisotto et al., 1994; Risch, 1969; 1970, 2022; Ritt, 2022; Rosenlicht, 1972; Trager, 2022; Yadav et al., 2012; Yadav, 2023).

III. Methodology

Some nonelementary antiderivative functions have been given special names as were mentioned in the abstract section and the expressions involving these functions can express a larger class of nonelementary antiderivatives (Nonelementary integral - Wikipedia). However we will use strong Liouville theorem and well proved nonelementary integrals in case we need to prove that whether some function is elementary antiderivative or nonelementary antiderivative.

IV. Discussion

Yadav & Sen (2012) have propounded six conjectures on nonelementary integrals in which they didn't considered inverse trigonometric functions as a component in the integrands, while considering antiderivative of some special type of functions. Continuing on their based hypothesis and taking the advantage of the exceptional cases, we here proffer the conjecture: "An antiderivative (indefinite integral) of the form

$$\int \frac{e^{g\{f(x)\}}}{g'\{f(x)\}} dx$$

where g(x) is an inverse trigonometric function, f(x) a complete non-perfect square polynomial of degree two, and $g'\{f(x)\}$ a derivative of g with respect to x, is always nonelementary".

Proof: We know that exist six types of inverse trigonometric functions and infinite number of polynomials of degree greater than or equal to one. To prove it for polynomial of all degrees is not possible, so we will prove the statement for some particular cases of polynomials. That's why it has been called a conjecture. In every case, first of all we will prove or disprove the conjecture for a polynomial of degree one, to test whether it is elementary or nonelementary and then will proceed for second degree polynomial. For this first let us assume that

$$\int \frac{e^{g\{f(x)\}}}{g'\{f(x)\}} dx \quad = I \text{ (Let)} \qquad \text{(i)}$$

According to the type of inverse trigonometric functions and particular polynomials of degree one and two, there will be six cases and twelve sub-cases to prove the proffered statement as follows:

Case-I: When $g(x) = \sin^{-1} f(x)$ and f(x) a polynomial in x of degree one or two. Then from (i) we have

$$I = \int \frac{e^{\sin^{-1}\{f(x)\}}\sqrt{1 - \{f(x)\}^2}}{f'(x)} dx$$

where f'(x) is the derivative of f(x) with respect to x. Putting $\sin^{-1}\{f(x)\} = z$ implies that $f(x) = \sin z$, which again implies that $f'(x)dx = \cos z\, dz$. Thus we get

$$I = \frac{1}{2}\int \frac{e^z(1 + \cos 2z)}{\{f'(x)\}^2} dz \qquad \text{(ii)}$$

Both of the antiderivatives (i) and (ii) will either be elementary or nonelementary and it completely depends on f(x). So as per our assumption, let us consider two sub-cases of f(x):

Sub-Case-I: For $f(x) = x + b$ a polynomial of degree one, (ii) implies that

$$I = \frac{1}{2}\int e^z(1 + \cos 2z)dz = \frac{e^z}{2} + \frac{1}{2}\frac{e^z}{5}(\cos 2z + 2\sin 2z)$$

Putting the value of z in it, we get

$$I = \frac{e^{\sin^{-1}(x+b)}}{2}\left[1 + \frac{1}{5}\{\cos\{2\sin^{-1}(x+b)\} + 2\sin\{2\sin^{-1}(x+b)\}\right]$$

which is elementary. Therefore the integral I is elementary for polynomial f(x) of degree one and $g(x) = \sin^{-1}f(x)$.

Sub-Case-II: For $f(x) = x^2 + bx + c$, where b and c are arbitrary, a polynomial of degree two, we have from (ii)

$$I = \frac{1}{2}\int \frac{e^z(1+\cos 2z)}{(2x+b)^2}dz \qquad \text{(iii)}$$

Since $\sin z = x^2 + bx + c$ implies that

$$(2x+b)^2 = 4(\sin z + K), \text{where } K = \frac{b^2-4c}{4}$$

We get from (iii)

$$I = \frac{1}{8}\int \frac{e^z(1+\cos 2z)}{(\sin z + K)}dz \qquad \text{(iv)}$$

The most simple case arise for K = 0 and for this we get from (iv)

$$I = \frac{1}{4}\int \frac{e^z\cos^2 z}{\sin z}dz \qquad \text{(v)}$$

$$= \frac{1}{4}\int F[z, e^z, \sin z, \cos z]dz = \frac{1}{4}\int F[z, y_1, y_2, y_3]dz$$

where F denotes that F is a function of z, $y_1 = e^z, y_2 = \sin z, y_3 = \cos z$ and

$$\frac{dy_1}{dz} = e^z = y_1, \qquad \frac{dy_2}{dz} = \cos z = y_3, \frac{dy_3}{dz} = -\sin z = -y_2$$

Applying strong Liouville's theorem after ignoring the constant factor 1/4, we find that the integral (v) will be elementary if and only if the integrand satisfy the following identity

$$\frac{e^z\cos^2 z}{\sin z} = \frac{dU_0}{dz} + \sum_{i=1}^{n} C_i\frac{U_i'}{U_i}$$

We know that the derivative of e^z is e^z and the derivative of an arbitrary function having e^z as a factor i.e. $e^z\varphi(z)$ will always consist e^z as a factor. If any one of U_j doesn't have e^z as a factor, then that term must be cancelled out after some steps on modifications. Using these concepts and searching for different combinations of $e^z, \sin z, \cos z$, etc., we find that no such U_j exists, which satisfy the above identity. Hence the integral (iv) is nonelementary for K = 0.

Now let us consider that K ≠ 0. Then we have from (iv)

$$I = \frac{1}{4}\int \frac{e^z\cos^2 z}{\sin z + K}dz \qquad \text{(vi)}$$

$$= \frac{1}{4}\int F[z, e^z, \sin z, \cos z]dz = \frac{1}{4}\int F[z, y_1, y_2, y_3]dz$$

where

$$\frac{dy_1}{dz} = e^z = y_1, \frac{dy_2}{dz} = \cos z = y_3, \frac{dy_3}{dz} = -\sin z = -y_2$$

Applying the same logic as have been applied for integral (v), we find that no such U_j exists, which satisfy the above identity. Hence the integral (vi) is nonelementary for non-zero K also. Thus the integral (i) is elementary for linear f(x) and nonelementary for quadratic f(x), when $g(x) = \sin^{-1}f(x)$.

Case-II: When $g(x) = \cos^{-1}f(x)$ and f(x) is a polynomial in x of degree one or two. Then we have from (i)

$$I = -\int \frac{e^{\cos^{-1}\{f(x)\}}\sqrt{1-\{f(x)\}^2}}{f'(x)}dx$$

On putting $\cos^{-1}\{f(x)\} = z$, we get

$$I = -\frac{1}{2}\int \frac{e^z(1-\cos 2z)}{\{f'(x)\}^2}dz \qquad \text{(vii)}$$

whose result of being elementary and nonelementary again depends on f(x). So we consider two following sub-cases:

Sub-Case-III: For $f(x) = x + b$, we get $f'(x) = 1$ and then from (vii) we get

$$I = -\frac{1}{2}\int e^z(1-\cos 2z)dz = -\frac{e^z}{2} + \frac{1}{2}\frac{e^z}{5}(\cos 2z + 2\sin 2z)$$

Putting the value of z in it, we get

$$I = \frac{e^{\cos^{-1}(x+b)}}{2}\left[-1 + \frac{1}{5}\{\cos\{2\cos^{-1}(x+b)\} + 2\sin\{2\cos^{-1}(x+b)\}\right]$$

which is elementary. Therefore the given indefinite integral (i) is elementary for polynomial f(x) of degree one and $g(x) = \cos^{-1}\{f(x)\}$.

Sub-Case-IV: For $f(x) = x^2 + bx + c$, where b and c are arbitrary, we have $f'(x) = 2x + b$ and $\cos^{-1}f(x) = z$ i.e., $\cos z = x^2 + bx + c$, then (vii) implies that

$$I = -\frac{1}{2}\int \frac{e^z(1-\cos 2z)}{(2x+b)^2}dz \qquad \text{(viii)}$$

Since $\cos z = x^2 + bx + c$ gives

$$(2x+b)^2 = 4(\cos z + K), \text{where } K = \frac{b^2-4c}{4}$$

Thus from (viii) we get

$$I = -\frac{1}{8}\int \frac{e^z(1-\cos 2z)}{(\cos z + K)}dz \qquad \text{(ix)}$$

The simple case arises for K = 0 and then we get from (ix)

$$I = -\frac{1}{4}\int \frac{e^z \sin^2 z}{\cos z}dz \quad \text{(ix a)}$$

$$= -\frac{1}{4}\int F[z, e^z, \sin z, \cos z]dz = -\frac{1}{4}\int F[z, y_1, y_2, y_3]dz$$

where

$$\frac{dy_1}{dz} = e^z = y_1, \frac{dy_2}{dz} = \cos z = y_3, \frac{dy_3}{dz} = -\sin z = -y_2$$

Applying strong Liouville's theorem and the similar logic, which have been applied for the integral (v), we find that no such U_j exists. Hence the integral (ix a) is nonelementary.

Again let us consider that K ≠ 0. Then we have from (ix)

$$I = -\frac{1}{4}\int \frac{e^z \sin^2 z}{\cos z + K}dz \quad \text{(ix b)}$$

$$= -\frac{1}{4}\int F[z, e^z, \sin z, \cos z]dz = -\frac{1}{4}\int F[z, y_1, y_2, y_3]dz$$

where

$$\frac{dy_1}{dz} = e^z = y_1, \frac{dy_2}{dz} = \cos z = y_3, \frac{dy_3}{dz} = -\sin z = -y_2$$

Applying the similar logic as have been applied in integral (v) and (vi), we will find that no such U_j exists. Hence the integral (ix b) is nonelementary. Thus the integral (i) is nonelementary for linear f(x) and nonelementary for quadratic f(x), when $g(x) = \cos^{-1}\{f(x)\}$.

Case-III: When $g(x) = \tan^{-1}f(x)$ and f(x) is a polynomial in x of degree one or two. Then we have from (i)

$$I = \int \frac{e^{\tan^{-1}\{f(x)\}}[1 + \{f(x)\}^2]}{f'(x)}dx$$

On putting $\tan^{-1}\{f(x)\} = z$, we get

$$I = \int \frac{e^z(1+\tan^2 z)}{\{f'(x)\}^2}\sec^2 z\, dz \qquad \text{(x)}$$

Again we consider two different sub-cases for f(x) as:

Sub-Case-V: For $f(x) = x + b$, from (x) we get

$$I = \int e^z(1+\tan^2 z)\sec^2 z\, dz \qquad \text{(xi)}$$

$$= \int F[z, e^z, \tan z, \sec z]dz = \int F[z, y_1, y_2, y_3]dz$$

where

$$\frac{dy_1}{dz} = e^z = y_1, \frac{dy_2}{dz} = \sec^2 z = (y_3)^2, \frac{dy_3}{dz} = \sec z \tan z = y_2 y_3$$

Using strong Liouville's theorem, we find that the integral (xi) will be elementary if and only if the integrand satisfy the following identity

$$e^z(1+\tan^2 z)\sec^2 z = \frac{dU_0}{dz} + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

$$\text{i. e.}, e^z \sec^4 z = \frac{dU_0}{dz} + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

Using hit and trial method for all possible cases and the same logic, which was applied for integral (v), we find that no such U_j exists, which satisfy the above identity. Therefore (xi) is nonelementary.

Sub-Case-VI: For $f(x) = x^2 + bx + c$, where b and c are arbitrary constants, from (x) we get

$$I = \int \frac{e^z(1+\tan^2 z)}{(2x+b)^2} \sec^2 z\, dz$$

From $\tan z = x^2 + bx + c$, we get

$$(2x+b)^2 = 4(\tan z + K), \text{where } K = \frac{b^2 - 4c}{4}$$

Thus we get

$$I = \int \frac{e^z(1+\tan^2 z)}{(2x+b)^2} \sec^2 z\, dz = \frac{1}{4}\int \frac{e^z(1+\tan^2 z)\sec^2 z}{(\tan z + K)} dz \qquad \text{(xii)}$$

The simple case arises for K = 0 and for this we get from (xii)

$$I = \frac{1}{4}\int \frac{e^z(1+\tan^2 z)\sec^2 z}{\tan z} dz \qquad \text{(xiii)}$$

$$= \int F[z, e^z, \tan z, \sec z]dz = \int F[z, y_1, y_2, y_3]dz$$

where

$$\frac{dy_1}{dz} = e^z = y_1, \frac{dy_2}{dz} = \sec^2 z = (y_3)^2, \frac{dy_3}{dz} = \sec z \tan z = y_2 y_3$$

By strong Liouville's theorem, we find that the integral (xiii) will be elementary if and only if the integrand satisfy the following identity

$$\frac{e^z(1+\tan^2 z)\sec^2 z}{4\tan z} = \frac{dU_0}{dz} + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

$$\text{i. e.}, \frac{e^z \sec^4 z}{4\tan z} = \frac{dU_0}{dz} + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

Using hit and trial method and the same logic as has been applied in integral (v), we can say that no such U_j exists, which satisfy the above identity. Therefore (xiii) is nonelementary.

Now let us consider that $K \neq 0$, then we have from (xii)

$$I = \frac{1}{4}\int \frac{e^z(1+\tan^2 z)\sec^2 z}{(\tan z + K)} dz \qquad \text{(xiv)}$$

$$= \frac{1}{4}\int F[z, e^z, \tan z, \sec z]dz = \frac{1}{4}\int F[z, y_1, y_2, y_3]dz$$

where

$$\frac{dy_1}{dz} = e^z = y_1, \frac{dy_2}{dz} = \sec^2 z = (y_3)^2, \frac{dy_3}{dz} = \sec z \tan z = y_2 y_3$$

Applying strong Liouville's theorem, we find that the integral (xiv) will be elementary if and only if the integrand satisfy the following identity

$$\frac{e^z(1+\tan^2 z)\sec^2 z}{4(\tan z + K)} = \frac{dU_0}{dz} + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

Using the same logic as has been applied in integral (v), we can find that no such U_j exists which satisfy the above identity. Hence the integral (xiv) is nonelementary. Thus the integral (i) is nonelementary for linear f(x) and nonelementary for quadratic f(x), when $g(x) = \tan^{-1}\{f(x)\}$.

Case-IV: When $g(x) = \cot^{-1} f(x)$ and f(x) a polynomial in x of degree one or two. Then we have from (i)

$$I = -\int \frac{e^{\cot^{-1}\{f(x)\}}[1+\{f(x)\}^2]}{f'(x)} dx$$

On putting $\cot^{-1}\{f(x)\} = z$, we get

$$I = -\int \frac{e^z(1+\cot^2 z)}{\{f'(x)\}^2} \operatorname{cosec}^2 z\, dz \qquad \text{(xv)}$$

Considering two different cases for f(x) for integral (xv) as follows:

Sub-Case-VII: For $f(x) = x + b$, from (xv) we get

$$I = -\int e^z(1+\cot^2 z)\operatorname{cosec}^2 z\, dz \qquad \text{(xvi)}$$
$$= -\int F[z, e^z, \cot z, \operatorname{cosec} z]dz = -\int F[z, y_1, y_2, y_3]dz$$

where

$$\frac{dy_1}{dz} = e^z = y_1, \frac{dy_2}{dz} = -\operatorname{cosec}^2 z = -(y_3)^2, \frac{dy_3}{dz} = -\operatorname{cosec} z \cot z = -y_2 y_3$$

By strong Liouville's theorem, the integral (xvi) will be elementary if and only if the integrand satisfy the identity

$$e^z(1+\cot^2 z)\operatorname{cosec}^2 z = \frac{dU_0}{dz} + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$
$$\text{i.e., } e^z \operatorname{cosec}^4 z = \frac{dU_0}{dz} + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

Considering all possible cases using hit and trial methods, we find that no such U_j exists, which satisfy the above identity. Therefore (xvi) is nonelementary.

Sub-Case-VIII: For $f(x) = x^2 + bx + c$, where b and c are arbitrary constants, from (xv) we get

$$I = -\int \frac{e^z(1+\cot^2 z)}{(2x+b)^2}\operatorname{cosec}^2 z\, dz \qquad \text{(xvii)}$$

From $\cot z = x^2 + bx + c$, we get

$$(2x+b)^2 = 4(\cot z + K), \text{where } K = \frac{b^2-4c}{4}$$

Thus from (xvii) we get

$$I = -\frac{1}{4}\int \frac{e^z(1+\cot^2 z)\operatorname{cosec}^2 z}{(\cot z + K)} dz \qquad \text{(xviii)}$$

Again the simple case arises for K = 0 and for this we get from (xviii),

$$I = -\frac{1}{4}\int \frac{e^z(1+\cot^2 z)\operatorname{cosec}^2 z}{\cot z} dz \qquad \text{(xix)}$$
$$= \int F[z, e^z, \cot z, \operatorname{cosec} z]dz = \int F[z, y_1, y_2, y_3]dz$$

where

$$\frac{dy_1}{dz} = e^z = y_1, \frac{dy_2}{dz} = -\operatorname{cosec}^2 z = -(y_3)^2, \frac{dy_3}{dz} = -\operatorname{cosec} z \cot z = -y_2 y_3$$

Applying strong Liouville's theorem and ignoring the negative sign, we find that this will be elementary if and only if the integrand satisfy the identity

$$\frac{e^z(1+\cot^2 z)\operatorname{cosec}^2 z}{4\cot z} = \frac{dU_0}{dz} + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$
$$\text{i.e., } \frac{e^z \operatorname{cosec}^4 z}{4\cot z} = \frac{dU_0}{dz} + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

Using all possible cases of U_j by hit and trial method, we can say that no such U_j exists, which satisfy the above identity. Therefore (xix) is nonelementary.

Now let us consider that K ≠ 0. Then we have from (xviii),

$$I = -\frac{1}{4}\int \frac{e^z(1+\cot^2 z)\operatorname{cosec}^2 z}{(\cot z + K)} dz \qquad \text{(xx)}$$
$$= -\frac{1}{4}\int F[z, e^z, \cot z, \operatorname{cosec} z]dz = -\frac{1}{4}\int F[z, y_1, y_2, y_3]dz$$

where

$$\frac{dy_1}{dz} = e^z = y_1, \frac{dy_2}{dz} = -\operatorname{cosec}^2 z = -(y_3)^2, \frac{dy_3}{dz} = -\operatorname{cosec} z \cot z = -y_2 y_3$$

Applying strong Liouville's theorem and ignoring the negative sign, we find that this will be elementary if and only if the integrand satisfy the identity

$$\frac{e^z(1+\cot^2 z)\operatorname{cosec}^2 z}{4(\cot z + K)} = \frac{dU_0}{dz} + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

$$\text{i.e.}, \frac{e^z \operatorname{cosec}^4 z}{4(\cot z + K)} = \frac{dU_0}{dz} + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

But no such U_j will exist. Hence the integral (xx) will be nonelementary. Thus the integral (i) is nonelementary for linear f(x) and nonelementary for quadratic f(x), when $g(x) = \cot^{-1}\{f(x)\}$.

Case-V: When $g(x) = \sec^{-1} f(x)$ and f(x) a polynomial in x of degree one or two. Then we have from (i)

$$I = \int \frac{e^{\sec^{-1}\{f(x)\}}\left[f(x)\sqrt{\{f(x)\}^2 - 1}\right]}{f'(x)} dx$$

On putting $\sec^{-1}\{f(x)\} = z$, we get

$$I = \int \frac{e^z \left[\sec z \sqrt{\{\sec z\}^2 - 1}\right]}{\{f'(x)\}^2} \sec z \tan z \, dz \qquad \text{(xxi)}$$

Considering two different cases for polynomial f(x) as follows:

Sub-Case-IX: For $f(x) = x + b$, from (xxi) we get

$$I = \int e^z \sec^2 z \tan^2 z \, dz$$

Which can be proved nonelementary as has been proved earlier in integral (xi) by applying the same logic.

Sub-Case-X: For $f(x) = x^2 + bx + c$, where b and c are arbitrary constants, from (xxi) we get

$$I = \int \frac{e^z \left[\sec z \sqrt{\{\sec z\}^2 - 1}\right]}{(2x + b)^2} \sec z \tan z \, dz$$

From $\sec z = x^2 + bx + c$, we get

$$(2x + b)^2 = 4(\sec z + K), \text{where } K = \frac{b^2 - 4c}{4}$$

Thus we get

$$I = \frac{1}{4}\int \frac{e^z \left[\sec z \sqrt{\{\sec z\}^2 - 1}\right]}{(\sec z + K)} \sec z \tan z \, dz \qquad \text{(xxii)}$$

Again the simple case arise for K = 0 and for this we get

$$I = \frac{1}{4}\int e^z \tan^2 z \sec z \, dz = \frac{(e^z \sec z)(\tan z - 1)}{8}$$

which is elementary.

Not let us consider that $K \neq 0$. Then we have from (xxii)

$$I = \frac{1}{4}\int \frac{e^z \sec^2 z \tan^2 z}{(\sec z + K)} dz \qquad \text{(xxii a)}$$

$$= \frac{1}{4}\int F[z, e^z, \sec z, \tan z] dz = \frac{1}{4}\int F[z, y_1, y_2, y_3] dz$$

where

$$\frac{dy_1}{dz} = e^z = y_1, \frac{dy_2}{dz} = \sec z \tan z = y_2 y_3, \frac{dy_3}{dz} = \sec^2 z = (y_2)^2$$

Applying strong Liouville's theorem, we find that this will be elementary if and only if the integrand satisfy the identity

$$\frac{e^z \sec^2 z \tan^2 z}{4(\sec z + K)} = \frac{dU_0}{dz} + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

But no such U_j will exist. Hence the integral (xxii a) will be nonelementary. Thus the integral (i) is nonelementary for linear f(x) and nonelementary for quadratic f(x), when $g(x) = \sec^{-1}\{f(x)\}$.

Case-VI: When $g(x) = \operatorname{cosec}^{-1} f(x)$ and f(x) a polynomial in x of degree one or two. Then we have from (i)

$$I = -\int \frac{e^{\operatorname{cosec}^{-1}\{f(x)\}}\left[f(x)\sqrt{\{f(x)\}^2 - 1}\right]}{f'(x)} dx$$

On putting $cosec^{-1}\{f(x)\} = z$, we get

$$I = -\int \frac{e^z \left[\operatorname{cosec} z \sqrt{\{\operatorname{cosec} z\}^2 - 1}\right]}{\{f'(x)\}^2} \operatorname{cosec} z \cot z \, dz \qquad \text{(xxiii)}$$

As usual considering two different cases for f(x) as:

Sub-Case-XI: For $f(x) = x + b$, from (xxiii) we get

$$\mathrm{I} = -\int \mathrm{e^z\,cosec^2\,z\,cot^2\,z\,dz}$$

It can be proved nonelementary as have been proved earlier in integral (xvi) by applying the same logic.
Sub-Case-XII: For $\mathrm{f(x)} = x^2 + \mathrm{bx} + \mathrm{c}$, where b and c are arbitrary constants, from (xxiii) we get

$$\mathrm{I} = -\int \frac{\mathrm{e^z}\left[\mathrm{cosec\,z}\sqrt{\{\mathrm{cosec\,z}\}^2 - 1}\right]}{(2\mathrm{x}+\mathrm{b})^2}\mathrm{cosec\,z\,cot\,z\,dz}$$

From $\mathrm{cosec\,z} = x^2 + \mathrm{bx} + \mathrm{c}$, we get

$$(2\mathrm{x}+\mathrm{b})^2 = 4(\mathrm{cosec\,z} + \mathrm{K}), \text{where } \mathrm{K} = \frac{\mathrm{b}^2 - 4\mathrm{c}}{4}$$

Thus we have

$$\mathrm{I} = -\frac{1}{4}\int \frac{\mathrm{e^z}\left[\mathrm{cosec\,z}\sqrt{\{\mathrm{cosec\,z}\}^2 - 1}\right]}{(\mathrm{cosec\,z} + \mathrm{K})}\mathrm{cosec\,z\,cot\,z\,dz} \qquad \text{(xxiv)}$$

Once again the simple case arises for K = 0, and for this we get from (xxiv)

$$\mathrm{I} = -\frac{1}{4}\int \mathrm{e^z\,cosec\,z\,cot^2\,z\,dz} = \frac{(\mathrm{e^z\,cosec\,z})(\mathrm{cot\,z}+1)}{8}$$

which is elementary.
Now let us consider that K ≠ 0. Then from (xxiv) we have

$$\mathrm{I} = -\frac{1}{4}\int \frac{\mathrm{e^z\,cosec^2\,z\,cot^2\,z}}{(\mathrm{cosec\,z}+\mathrm{K})}\mathrm{dz} \qquad \text{(xxiv a)}$$

$$= -\frac{1}{4}\int \mathrm{F[z, e^z, cosec\,z, cot\,z]dz} = -\frac{1}{4}\int \mathrm{F[z, y_1, y_2, y_3]dz}$$

where

$$\frac{d\mathrm{y}_1}{dz} = e^z = y_1, \frac{d\mathrm{y}_2}{dz} = -\mathrm{cosec}\,z\cot z = -y_2 y_3, \frac{d\mathrm{y}_3}{dz} = -\,cosec^2\,z = -(y_2)^2$$

Applying strong Liouville's theorem and ignoring negative sign, we find that this will be elementary if and only if the integrand satisfy the identity

$$\frac{\mathrm{e^z\,cosec^2\,z\,cot^2\,z}}{4(\mathrm{cosec\,z}+\mathrm{K})} = \frac{\mathrm{dU_0}}{\mathrm{dz}} + \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{C_i}\frac{\mathrm{U_i}'}{\mathrm{U_i}}$$

But no such $\mathrm{U_j}$ will exist. Hence the integral (xxiv a) will be nonelementary. Thus the integral (i) is nonelementary for linear f(x), elementary for K = 0 and nonelementary for K ≠ 0 for quadratic f(x), when $\mathrm{g(x) = cosec^{-1}\{f(x)\}}$.

V. Conclusion

From above discussion we conclude that the indefinite integral

$$\int \frac{\mathrm{e^{g\{f(x)\}}}}{\mathrm{g'\{f(x)\}}}\mathrm{dx}$$

where g(x) is an inverse trigonometric function, f(x) a complete non-perfect square polynomial of degree two, and $\mathrm{g'\{f(x)\}}$ a derivative of g with respect to x, is always a nonelementary integral. But it is elementary integral for some particular cases like when $\mathrm{f(x) = x + b}$ and $\mathrm{g(x) = sin^{-1}f(x)}$; $\mathrm{f(x) = x + b}$ and $\mathrm{g(x) = cos^{-1}f(x)}$; $\mathrm{f(x)} = (\mathrm{x} + \sqrt{\mathrm{c}})^2$ and $\mathrm{g(x) = sec^{-1}f(x)}$; and when $\mathrm{f(x)} = (\mathrm{x} + \sqrt{\mathrm{c}})^2$ and $\mathrm{g(x) = cosec^{-1}f(x)}$. We also obtain some amazing characters of the proffered conjecture as indefinite integral that it gives the same result for pair wise inverse trigonometric functions for the pairs $\mathrm{sin^{-1}f(x)}$ and $\mathrm{cos^{-1}f(x)}$, $\mathrm{tan^{-1}f(x)}$ and $\mathrm{cot^{-1}f(x)}$, and $\mathrm{sec^{-1}f(x)}$ and $\mathrm{cosec^{-1}f(x)}$.

VI. Future Scope of Research

The indefinite integral given by proffered conjecture has been discussed for only two cases of the polynomial of linear and quadratic nature. A big scope is available for research for higher degree and its special cases polynomials. There is a possibility that they might be nonelementary for higher degree polynomial f(x) because as the degree of the polynomial increases, the integrand becomes more complex than the previous one. This is the reason that the proffered indefinite integral has been called a conjecture and not a theorem or property, which is the big limitations of the present work.

References

[1]. Anton, H., Bivens, I, & Davis, S. (2014). Calculus (7th Edition), Wiley Student Edition, 318-360.
[2]. Cherry, G. W. (1985). Integration In Finite Terms With Special Functions: The Error Function. Journal Of Symbolic Computation, 1(3), 283-302.

[3]. Cherry, G. W. (1986). Integration In Finite Terms With Special Functions: The Logarithmic Integral. SIAM Journal On Computing, 15(1), 1-21.

[4]. Corliss, G., & Krenz, G. (1989). Indefinite Integration With Validation. ACM Transactions On Mathematical Software (TOMS), 15(4), 375-393.

[5]. Hardy, G. H. (2018). The Integration Of Functions Of A Single Variable, Hawk Press, 1-12.

[6]. Kasper, T. (1980). Integration In Finite Terms: The Liouville Theory. ACM Sigsam Bulletin, 14(4), 2-8.

[7]. Lutfi, M. (2016). Misconceptions In Solving Indefinite Integrals For Nonelementary Functions Using The Taylor Series. Jurnal Ilmiah Matematika Dan Terapan, 13(1).

[8]. Marchisotto, E. A., & Zakeri, G. A. (1994). An Invitation To Integration In Finite Terms. The College Mathematics Journal, 25(4), 295-308.

[9]. Nijimbere, V. (2017). Evaluation Of Some Non-Elementary Integrals Involving Sine, Cosine, Exponential And Logarithmic Integrals: Part I. Arxiv Preprint Arxiv:1703.01907.

[10]. Nijimbere, V. (2018). Evaluation of some non-elementary integrals involving sine, cosine, exponential and logarithmic integrals: Part II. *arXiv preprint arXiv:1807.04125*.

[11]. Nijimbere, V. (2020). Analytical valuation of some non-elementary integrals involving some exponential, hyperbolic and trigonometric elementary functions and derivation of new probability measures generalizing the gamma-type and normal distributions. *arXiv preprint arXiv:2005.06951*.

[12]. Nijimbere, V. (2020). Evaluation of some non-elementary integrals involving the generalized hypergeometric function with some applications. *arXiv preprint arXiv:2003.07403*.

[13]. Risch, R. H. (1969). The problem of integration in finite terms. *Transactions of the American Mathematical Society*, *139*, 167-189.

[14]. Risch, R. H. (1970). The solution of the problem of integration in finite terms.

[15]. Risch, R. H. (2022). On the integration of elementary functions which are built up using algebraic operations. In *Integration in Finite Terms: Fundamental Sources* (pp. 200-216). Cham: Springer International Publishing.

[16]. Ritt, J. F. (2022). Integration in Finite Terms Liouville's Theory of Elementary Methods. In *Integration in Finite Terms: Fundamental Sources* (pp. 31-134). Cham: Springer International Publishing.

[17]. Rosenlicht, M. (1972). Integration in finite terms. *The American Mathematical Monthly*, *79*(9), 963-972.

[18]. Sao, G. S. (2021). Special Functions, 3rd Revised Edition, Shree Shiksha Sahitya Prakashan, Meerut, pp. 1-3, 40-45.

[19]. Sharma, J. N. & Gupta, R. K. (2020). Special Functions, Krishna Prakashan Media (P) Ltd., Meerut, 34th Edition, pp. 70-72.

[20]. Singer, M. F., Saunders, B. D., & Caviness, B. F. (1985). An extension of Liouville's theorem on integration in finite terms. *SIAM Journal on Computing*, *14*(4), 966-990.

[21]. Thompson, S. P. (2021). Calculus Made Easy, G. K. Publications (P) Ltd., 182-192.

[22]. Trager, B. M. (2022). Integration of algebraic functions. In *Integration in Finite Terms: Fundamental Sources* (pp. 230-286). Cham: Springer International Publishing.

[23]. Victor, N. (2017). EVALUATION OF THE NON-ELEMENTARY INTEGRAL∫ EλX^ αDX, α≥ 2, AND OTHER RELATED INTEGRALS. *Ural Mathematical Journal*, *3*(2 (5)), 130-142.

[24]. Weisstein, E. W. Fermat's Last Theorem. Mathworld.wolfram.com. Retrieved 2019-11-12.

[25]. Wikipedia contributors. (2023, April 5). Conjecture. In *Wikipedia, The Free Encyclopedia*. Retrieved 16:48, April 15, 2023, from https://en.wikipedia.org/w/index.php?title=Conjecture&oldid=1148406309

[26]. Wikipedia contributors. (2023, January 1). Elementary function. In *Wikipedia, The Free Encyclopedia*. Retrieved 12:05, April 10, 2023, from https://en.wikipedia.org/w/index.php?title=Elementary_function&oldid=1130910903

[27]. Wikipedia contributors. (2023, March 29). Closed-form expression. In *Wikipedia, The Free Encyclopedia*. Retrieved 12:48, April 10, 2023, from https://en.wikipedia.org/w/index.php?title=Closed-form_expression&oldid=1147215178

[28]. Wikipedia contributors. (2022, August 13). Nonelementary integral. In *Wikipedia, The Free Encyclopedia*. Retrieved 13:20, April 10, 2023, from https://en.wikipedia.org/w/index.php?title=Nonelementary_integral&oldid=1104142356

[29]. Wikipedia contributors. (2023, March 27). Hypergeometric function. In *Wikipedia, The Free Encyclopedia*. Retrieved 15:58, April 15, 2023, from https://en.wikipedia.org/w/index.php?title=Hypergeometric_function&oldid=1146819754

[30]. Wikipedia contributors. (2023, April 5). Trigonometric integral. In *Wikipedia, The Free Encyclopedia*. Retrieved 16:00, April 15, 2023, from https://en.wikipedia.org/w/index.php?title=Trigonometric_integral&oldid=1148310503

[31]. Wikipedia contributors. (2023, April 13). Exponential integral. In *Wikipedia, The Free Encyclopedia*. Retrieved 16:02, April 15, 2023, from https://en.wikipedia.org/w/index.php?title=Exponential_integral&oldid=1149620218

[32]. Wikipedia contributors. (2023, April 11). Error function. In *Wikipedia, The Free Encyclopedia*. Retrieved 16:03, April 15, 2023, from https://en.wikipedia.org/w/index.php?title=Error_function&oldid=1149272166

[33]. Wikipedia contributors. (2022, December 3). Elliptic integral. In *Wikipedia, The Free Encyclopedia*. Retrieved 16:05, April 15, 2023, from https://en.wikipedia.org/w/index.php?title=Elliptic_integral&oldid=1125373865

[34]. Wikipedia contributors. (2023, April 6). Fresnel integral. In *Wikipedia, The Free Encyclopedia*. Retrieved 16:06, April 15, 2023, from https://en.wikipedia.org/w/index.php?title=Fresnel_integral&oldid=1148535186

[35]. Wikipedia contributors. (2023, March 14). Gaussian integral. In *Wikipedia, The Free Encyclopedia*. Retrieved 16:08, April 15, 2023, from https://en.wikipedia.org/w/index.php?title=Gaussian_integral&oldid=1144636147

[36]. Wikipedia contributors. (2023, May 18). Goldbach's conjecture. In *Wikipedia, The Free Encyclopedia*. Retrieved 12:09, May 23, 2023, from https://en.wikipedia.org/w/index.php?title=Goldbach%27s_conjecture&oldid=1155611034

[37]. Yadav, D. K. & Sen, D. K. (2012). Ph.D. Thesis: A Study of Indefinite Nonintegrable Functions, (Vinoba Bhave University, Hazaribag, Jharkhand, India). GRIN Verlag, Open Publishing GmbH, Germany.

[38]. Yadav, D. K. (2023). Nonelementary Integrals: Indefinite Nonintegrable Functions, Notion Press, India, 36-54.