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On Numerical Examples Of Boundary Knot Method

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ABSTRACT
The boundary knot method is a simple boundary-type meshless method. Due to the use of non-singular general solutions rather than singular fundamental solutions, there is no need to consider the artificial boundary. It has the merits of purely meshless, easy to program, high solution accuracy and so on. In this paper, with some new classes of numerical experiments, we make some conclusions concerning the effectiveness of solving Helmholtz-type problems with the boundary knot method.

Key Words: *Boundary knot method, meshless method, non-singular general solution, Helmholtz equation*

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I. Introduction

As is known to all, meshless methods are mature in dealing with many boundary value problems[1-4]. One of the popular boundary-type meshless collocation methods, which was named the boundary knot method (BKM), was pioneered by Kang and his coworkers[5].

The BKM has been applied to many problems, including two-dimensional three-dimensional and inverse cases [6-9]. For the traditional ill-conditioned interpolation matrix, the effective condition number is introduced to scale the BKM[10], and some regularization methods [11] are considered in dealing with direct problems by using the BKM. An early work made an overview of this method[12].

As a complimentary work, this paper will give some classes of numerical experiments to show the effectiveness of the BKM in solving Helmholtz-type problems.

II. Problem description

The Helmholtz-type partial differential equation has the following mathematical formulation

$$\nabla^2 u(X) + \lambda^2 u(X) = 0, \quad X = (x, y) \in \Omega \qquad (1)$$

where ∇^2 is the laplacian operator, λ the wave number, Ω the physical domain. Eq. (1) is the so-called Helmholtz equation.

To get the solution of Eq. (1), one has to give certain boundary conditions on the physical boundary $\partial\Omega$. There are three types of commonly-used boundary conditions. More specifically, the Dirichlet boundary condition

$$u(X) = \bar{u}(X), \quad X \in \partial\Omega \qquad (2)$$

the Neumann boundary condition

$$\frac{\partial u(X)}{\partial n} = \bar{q}(X), \quad X \in \partial\Omega \qquad (3)$$

and the Robin boundary condition

$$u(X) + \frac{\partial u(X)}{\partial n} = \bar{p}(X), \quad X \in \partial\Omega \qquad (4)$$

where $\bar{u}(X)$, $\bar{q}(X)$, $\bar{p}(X)$ are the known boundary data at point X. For different problems, the boundary conditions are always different.

The governing equation (1) and boundary conditions lead to boundary value problems. This can be solved by using numerical methods.

III. The boundary knot method

The basic theory of the BKM is the same as the other collocation numerical methods. More specifically, the numerical solution for $u(X)$ is given by a linear combination of radial basis functions which is expressed by

$$\tilde{u}(X) = \sum_{j=1}^{N} c_j G(\lambda r_j), \qquad (5)$$

where N is the number of boundary collocation knots, and $c_j, (j = 1,2,...,N)$ are the unknown coefficients, $r = \|X - Y\|$ is the Euclidean norm distance between points X and Y. The non-singular general solutions of Helmholtz equations are written as

$$G(\lambda r) = \begin{cases} J_0(\lambda r), & R^2 \\ \dfrac{\sin(\lambda r)}{r}, & R^3 \end{cases} \qquad (6)$$

with J_0 and I_0 denoting the Bessel and modified Bessel function of the first kind, respectively. By collocating Dirichlet boundary condition on boundary collocation knots, i.e., substitute Eq. (5) into Eq. (2), we have

$$\sum_{j=1}^{N} c_j G(\lambda r_{ij}) = \overline{u}(X_i),\ X_i \in \partial\Omega. \qquad (7)$$

The same procedure can be applied to Neumann boundary condition and Robin boundary conditions. Equation (7) can be directly solved by using the backslash operation in MATLAB.

IV. Numerical examples

Case 1

As is known to all, the traditional way to construct numerical solutions is using a function which satisfies the government equation and the corresponding boundary conditions. Here, we consider the exact solution $u = \sin x \cos y$ in a circular domain with only Dirichlet boundary conditions.

Figures (1)-(3) provide the picture of exact solution, numerical solution and PDE solution from MATLAB toolbox. We can see that the three types of solutions are the same. To see the differences, we consider error distribution between each two solutions against the test points which are shown in Figs. (4)-(6). It should be noted that the average error between exact solution and numerical solution is err_two = 3.49e-009. The average error between exact solution and PDE solution is err_two = 0.18. The average error between numerical solution and PDE solution is also err_two = 0.18. This reveals that the BKM numerical solution is more accurate than the PDE solution from MATLAB toolbox.

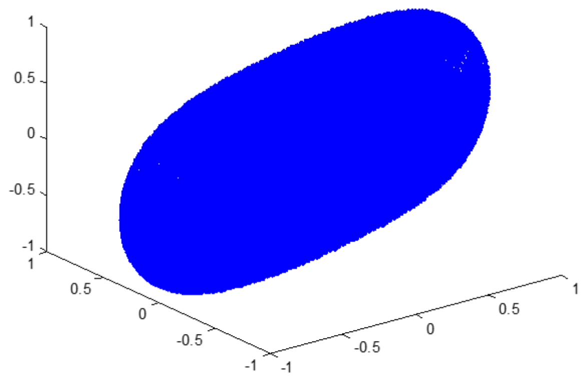

Fig. 1 Exact solution for Case 1

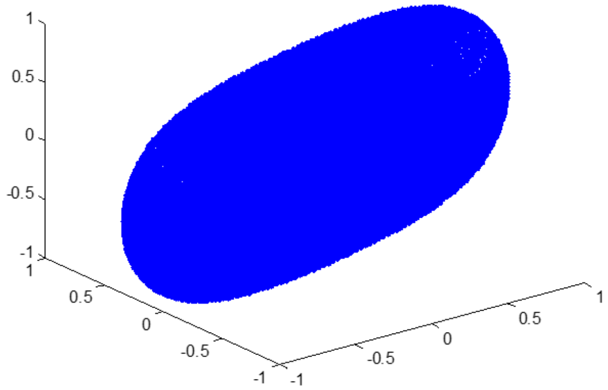

Fig. 2 Numerical solution for Case 1

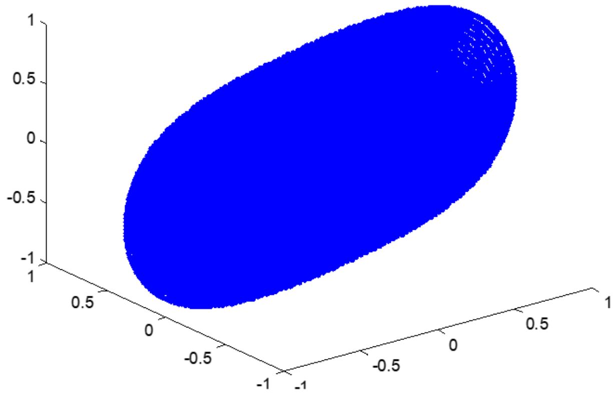

Fig. 3 PDE solution for Case 1

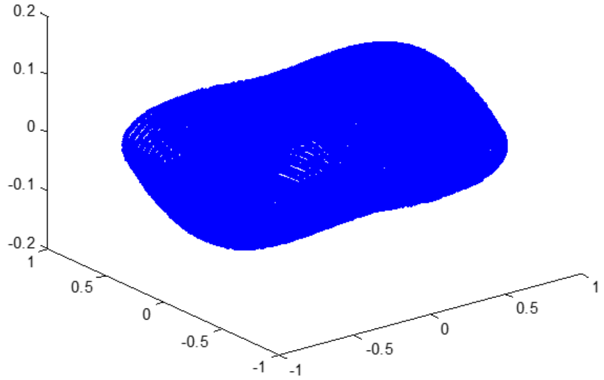

Fig. 4 Error distribution (PDE - Numer) against the test points

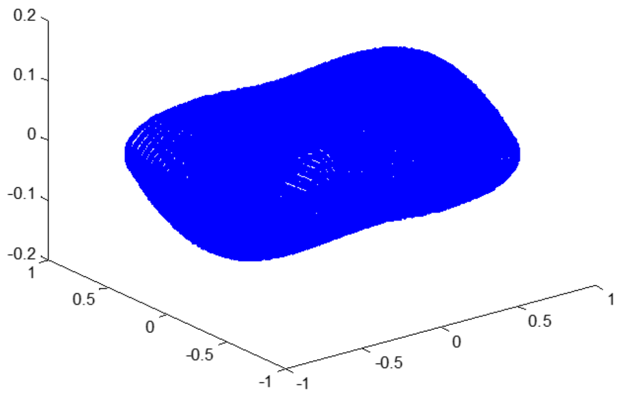

Fig. 5 Error distribution (PDE - Exact) against the test points

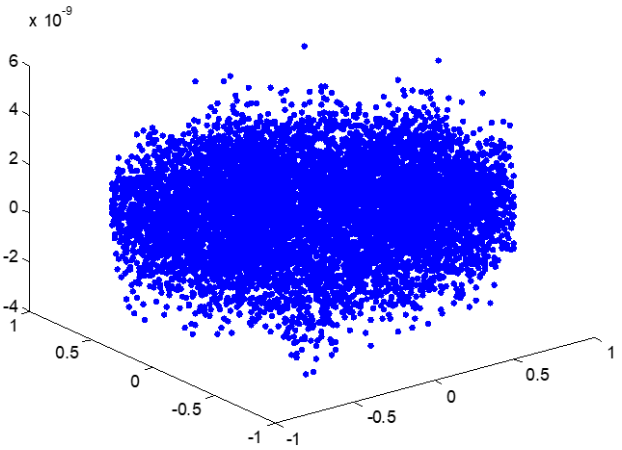

Fig. 6 Error distribution (Exact - Numer) against the test points

Case 2

Here, the boundary data function is chosen as $u = x^2 y^3$ in a circular domain with only Dirichlet boundary conditions. We note that there is no exact solutions for this case. The boundary collocation number is N=50 and the tested knot number is Nt=8385.

The PDE solution from MATLAB toolbox, numerical solution and error distribution are shown in Figs. (7)-(9), where the average errors between numerical solution and PDE solution is err_two = 0.10188. We can see that the result is similar to the previous case 1.

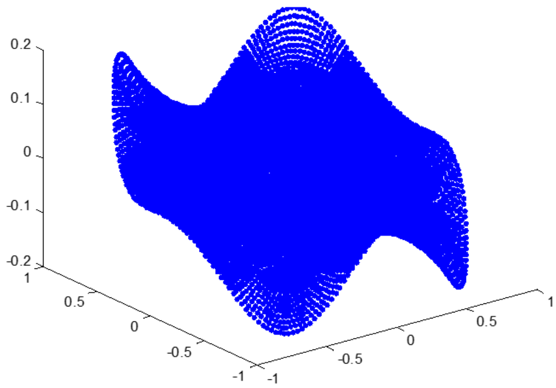

Fig. 7 PDE solution for Case 1

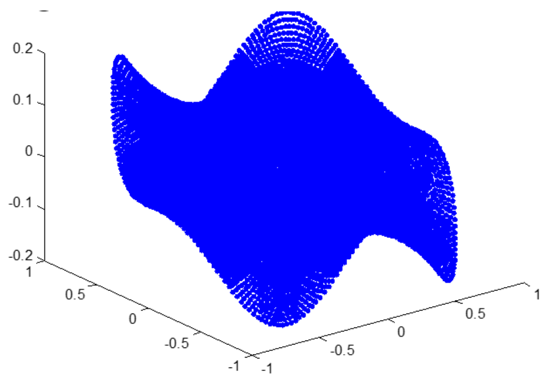

Fig. 8 Numerical solution for Case 1

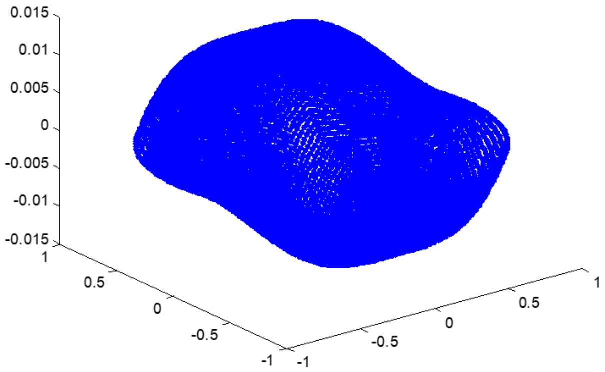

Fig. 9 Error distribution (PDE - Numer) against the test points

Case 3

The third case considered boundary data function $u = 1$ on the top semicircle and $u = -1$ for the rest semicircle in a circular domain with only Dirichlet boundary conditions. The PDE solution from MATLAB toolbox, numerical solution and error distribution are shown in Figs. (10)-(12), where the average errors err_two = 3.335. we can see that the PDE solution is different with the BKM solution. This reveals that the BKM solution may be not accurate for this case.

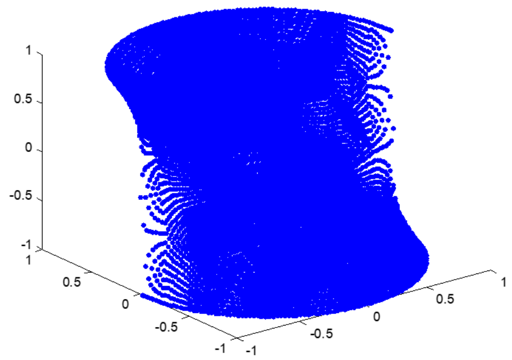

Fig. 10 PDE solution for Case 3

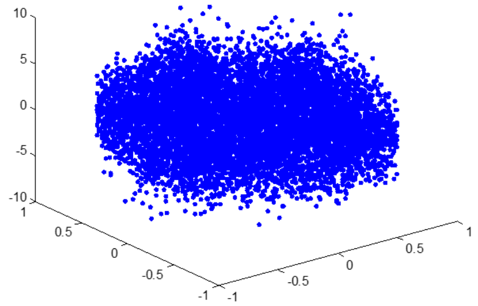

Fig. 11 Numerical solution for Case 3

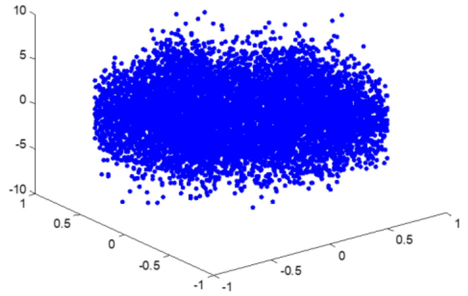

Fig. 12 Error distribution (PDE - Numer) against the test points

V. Conclusions

In this paper, by solving some different types of numerical experiments, we re-considered the effectiveness of solving Helmholtz-type problems with the BKM. It is shown that the BKM is applicable to problems with smooth/continuous boundary conditions. For problems with discontinuous boundary conditions, the BKM may be failed.

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