IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 19, Issue 3 Ser. I (May. – June. 2023), PP 01-09
www.iosrjournals.org

Exploring The Horizontal And Vertical Mathematization Process In Realistic Mathematics Education To Prepare Students For The Era Of Industrial Revolution 5.0

I Made Surat[1], I Komang Sukendra[2], I Dewa Putu Juwana[3],
Basil Puthupadiyil Chako[4]
Dosen Program Studi Pendidikan Matematika FKIP
[1,2,3] Universitas PGRI Mahadewa Indonesia
[4] SMA Taman Rama Denpasar

ABSTRACT. *Mathematics is a discipline that closely relates to everyday life, making it essential to align its learning with the reality of students. An approach used to bridge the gap between classroom learning and real-life experiences is Realistic Mathematics Learning (RML). By placing the experiences and reality of students at the forefront, RML allows students to learn this subject through activities that develop mathematical tools to solve problems associated with everyday life. The mathematization process, which starts from the real world and progresses to the symbol world, comprises two stages, namely horizontal and vertical. In the horizontal stage, students use their informal experiences to link the concepts of this subject to real-world situations, transforming everyday language into mathematical language. In contrast, the vertical stage involves expanding the symbol world through constructing principles and algorithms that allow for generalization and abstraction. Through this mathematization process, informal experiences are gradually formalized into a system that helps to shape the cognitive abilities, soft skills, and mastery of technology of students. This process prepares them to welcome the Society 5.0 era, where mathematical literacy and problem-solving skills are essential for success.*

Keywords*: Horizontal Mathematization, Vertical Mathematization, Realistic Mathematics Learning*

Date of Submission: 19-04-2023 Date of Acceptance: 02-05-2023

I. INTRODUCTION

Mathematics is often considered a challenging subject by most students, leading to a lack of interest in its lessons. This has a detrimental impact on the quality of lessons and also on the morale of teachers. Despite this, it is widely acknowledged that mathematics is a fundamental science that plays a crucial role in the development of other fields, both within the natural and social sciences. In fact, this subject is often referred to as the servant and king of all sciences, which outlines its essential nature. This means that it is not an exaggeration to suggest that mathematics makes life easier. However, the unfortunate reality is that many students view this subject as a dreadful experience. It is ironic that the benefits of mathematics in daily life are juxtaposed with the difficulties of learning in a classroom setting. Teachers of mathematics are expected to reflect on the pedagogical approaches utilized to motivate students and foster a more positive attitude toward the subject.

In 2018, the National Examination (NE) in Indonesia introduced Higher Order Thinking Skills (HOTS) questions to increase the level of difficulty and complexity, and to habituate students to high-level questions. HOTS questions have the potential to foster critical and creative thinking, which is particularly important in the world today being the fifth industrial era, where technology has penetrated the virtual world, connecting humans, machines, and data beyond the limitations of space and time through the Internet of Things (IoT). One of the distinctive features of Society 5.0 is the application of artificial intelligence, known as Artificial Intelligence (AI). The emergence of the industrial revolution 5.0 era has had a huge impact on education, including mathematics. The aim of mathematics education should be to develop students who possess critical, creative, and technological literacy skills.

Critical and creative thinking skills are important competencies for effective problem-solving. To develop these skills, students need to be allowed to solve real-life problems. One of the approaches that bring the learning situation in schools closer to the real environment is Realistic Mathematics Learning (RML). The main idea behind RML is to provide students with ample opportunities to reinvent mathematical concepts, principles, or algorithms under the guidance of their teachers (Gravemeijer, 1994). This approach was initially developed by Han Freudenthal in the Netherlands. Despite the inherent abstractness of mathematics, the RML process starts with real-life situations before the abstraction process occurs. This method uses contextual problems as a starting point for learning mathematics, allowing students to identify and organize the mathematical aspects of the problem.

The assertion by Freudenthal regarding mathematics as a human activity that is closely related to daily life highlighted the need for mathematics education to be associated with real-life conditions (Widyastuti, 2014). This proximity to the actual environment increases the likelihood of transfer of learning. The role of teachers is to incorporate the real-life experiences of students into the mathematics learning approach to facilitate the **mathematization** process. Notably, mathematization is the process of interpreting or expressing a situation mathematically, and it involves two categories, namely horizontal and vertical. The next challenge is how to apply the mathematization process to mathematics topics in the context of Society 5.0.

II. DISCUSSION

The Nature of Mathematics Learning in the Society 5.0 Era

Mathematics is often perceived as a subject solely related to numbers, but this is an incomplete understanding as it is also a language. The language of mathematics is universal and recognized by educated people globally. Communication occurs through the use of symbols and formulas, both within the field and among other disciplines that employ mathematics as a tool. It is essential to comprehend the meanings and functions of many symbols in mathematics. This is because each symbol and its combination represents a certain concept that facilitates communication. It is even argued that mathematics is the language used by God to create the universe. In this regard, mathematics is reflected in the order of universe, cycles, balance, and harmony, demonstrating its characteristics as a consistent and principled system.

Mathematics encompasses numbers and their operations and also the study of space. However, its quantitative and linguistic designations fail to satisfy the broader goals of this subject, which aim to understand relationships, patterns, shapes, and structures. This means that mathematics emphasizes the study of these elements rather than solely numerical and linguistic matters. In addition, this subject deals with logically structured ideas, making it abstract and reliant on deductive reasoning (Herman Hudoyo, 2005). According to Begle (1979), mathematical objects encompass facts, concepts, principles, and operations. Reasoning in mathematics involves the use of symbols that carry meaning, as each symbol embodies an idea. This characteristic allows the applicability of this subject in other fields or branches of science.

In essence, mathematical thinking is based on a set of agreements called axioms, which are considered correct assumptions. For this reason, mathematics is considered an axiomatic system that is consistent and reliable. General axioms can be derived to obtain special properties, which is called **Deductive**. Although mathematics uses deductive reasoning, a creative process also occurs which sometimes involves inductive reasoning, intuition, and even trial and error. In the end, the discoveries from the creative process should be organized by deductive proof. The theorems obtained deductively are then used to solve problems, including real-life situations. The characteristics of mathematics are complex, as it concerns facts, concepts, principles, and algorithms, as well as relationships, patterns, and abstract thinking processes, such as deductive and inductive reasoning. Therefore, it is necessary to use a learning approach that can maximize interaction for effective learning transfer.

Learning is an interaction between teachers, learning resources, and students. This is expected to enable students to carry out learning activities that cause changes in their behavior, including cognitive, psychomotor, and affective aspects. According to Gagne (1977), learning is a set of external events designed to support learning in students. In learning mathematics, interaction is crucial, and based on E De Corte, the method used in this context and the situation of mathematical material in the real environment affect interaction. Regarding the cognitive domain, the main goal of learning mathematics is the transfer of learning (Hudoyo, 2005). Conceptually, transfer of learning is the ability of students to apply mathematical concepts and theorems mastered to solve problems they face. Therefore, all efforts need to be made to ensure that students succeed in mastering mathematics in the cognitive domain, skills, and positive attitudes. Through this, they can solve problems in mathematics and other fields of science. Innovations in mathematics learning should be continuously developed to address future challenges. It is important to note that mathematics also serves as a tool in developing science and technology. Some of its characteristics include being systematic, having regular interrelationships, being consistent and adhering to principles, being abstract, deductive and Inductive, and reasoning. It has the potential to make students think creatively and connectively, diligently, and thoroughly, which are essential abilities for solving problems in life. These abilities are very useful in answering the challenges of the industrial revolution in the Society 5.0 era.

The history of the Industrial Revolution spanned from 1750 to 1850, marking a significant change in agriculture, trade, mining, transportation, and technology. These changes had an impact on socio-economic and cultural aspects, as human labor in various fields began to be replaced by machines. Industrial Revolution 1.0 started with the invention of the steam engine in the 18th century, which was a major turning point in world history, affecting almost all aspects of life and causing a huge effect on socioeconomic and cultural conditions. This led to the development of electricity, computerization, and Industry 4.0, which allowed all entities involved to communicate with each other in real-time using the internet, making the world more accessible. Despite the progress made with Industry 4.0, the emergence of the Society 5.0 era negated the assumption that no further revolutions could occur. The Society 5.0 concept, developed by the Business Federation in Japan, aims to integrate cyberspace and physical space to make life easier. However, this era prioritizes humanization and technology simultaneously, instead of focusing solely on technology.

With the rapid development of industrial technology, the field of education needs to be adequately equipped to respond to future challenges. It is crucial to project a clear and comprehensive picture of life in the future, hence people can survive in the face of global competition. According to Wefrom.org, there are three main abilities that individuals need in the future, namely 1) cognitive, 2) soft skills, and 3) technology. The cognitive ability goes beyond academic excellence or a high Achievement Index but needs to be equipped with the capacity to solve complex problems. To achieve this, there has to be a transfer of learning from academic-mathematical aspects in schools to real-life situations in society. One way to achieve this is by implementing RML, which enables students to understand and model real-life situations, and can eventually be studied academically. Soft skills are equally important, as they relate to the ability of students to communicate effectively, empathize with others, have a growth mindset, and adapt to different situations. On the other hand, technological ability involves mastering the scientific method to apply science to solve real-world problems. Technology is essential for providing the goods necessary for human life to be sustainable and comfortable. It is worth noting that all three main abilities mentioned above can be trained and developed in students.

Realistic Mathematics Learning

Realistic Mathematics Education (RME) was first introduced in Indonesia in 1998 and was subsequently referred to as RML. The field of mathematics education continued to evolve, with ongoing innovations in teaching methods. Based on this, there was a shift in the way students were perceived In the learning process. Previously, students had been seen as passive objects in the learning process, but they were currently viewed as active subjects. Ivor K. Devis (2000) emphasized the significance of recognizing that the nature of mathematics learning primarily revolved around students rather than teachers. Therefore, it was essential for teachers to understand how students learned mathematics and how they should have been taught. An example of the way to consider students as subjects were by presenting the reality of their experiences as the starting point of learning.

RML is an approach used to improve the quality of mathematics education through mathematization activities. In this context, RML refers to school mathematics, which takes into account the reality and experience of students as the starting point of learning. Real problems are used to introduce mathematical concepts and principles, which serve as triggers for problem-solving activities. Within this framework, students are viewed as having a set of concepts about mathematical ideas that play a crucial role in further learning. In this case, teachers serves as learning facilitator, and if necessary, helps students to mathematize real problems, which in turn leads to interactive learning with dominant activities of students. Teachers actively examine the content of mathematical topics to relate them to the real world. According to Suryanto (2007), RML has some distinctive characteristics. First, mathematical concepts are introduced to students through real contextual problems. Second, students utilize their creativity to identify mathematical concepts in the form of models as part of the mathematization process of real things. Third, there is a discussion about the methods and results obtained in carrying out the mathematization process. Fourth, students reflect on or review what has been carried out and the results. Finally, teachers emphasize the mathematization process and facilitate students to link mathematical topics with the real world to further develop and improve upon more complicated and complex concepts.

RML is based on constructivist learning theory, which prioritizes six principles as stages of mathematics learning, namely Activity, Reality, Understanding, Intertwinement, Interaction, and Guidance. In the Activity phase, students learn mathematics through practical problem-solving. This means that they work on problems specifically designed to help them become active participants in the learning process and develop a comprehensive set of mathematical tools. The Reality phase aims to enable students to apply mathematical principles in order to solve real-world problems. The mathematization process, which involves both horizontal and vertical applications, establishes a connection between mathematics and everyday life. In the Understanding phase, mathematics learning includes various stages of understanding, such as 1) developing the ability to find informal solutions related to the context, 2) finding formulas and schemes, and 3) understanding the principles connecting these concepts. In the Intertwinnement phase, students have the opportunity to solve mathematical problems containing contextual elements by applying various concepts, principles, algorithms, and understanding in an integrated and related manner. In the Interaction phase, mathematics learning is considered a social activity, in which students are given the opportunity to share their experiences, problem-solving strategies, and answers with their peers. This interaction allows them to reflect and gain a complete understanding. Finally, the Guidance phase provides students with the opportunity to find their own principles or mathematical formulas through guided reinvention activities, designed by teachers in accordance with RML principles.

Mathematization Process in School Mathematics Topics

In mathematics learning, two crucial directions of mathematization need to be considered, namely horizontal and vertical (Marpaung, 2001). Horizontal mathematization involves transforming everyday language problems into mathematical language. Meanwhile, the vertical is a process that occurs within the field of mathematics (Marpaung, 2001). According to Yuwono (2001), horizontal mathematization is related to the prior knowledge and intuition that students possess to solve real-world problems, but vertical mathematization involves the conversion of symbols to a more abstract symbolization. The aim of horizontal mathematization is to initiate mathematics learning contextually by linking its concepts with real-world situations. This approach allows students to feel more closely connected and interested in the subject matter. However, horizontal mathematization alone is insufficient, as students need to explore and understand mathematical concepts correctly through vertical mathematization activities.

The following sections describe a mathematization process used to teach certain school mathematics topics by relating them to real-world problems.

Division Operation at the primary education level.

The division is considered to be the most difficult of the three basic arithmetic operations, namely +, -, x. In mathematical operations, division is defined as the inverse of multiplication or as repeated subtraction. For example, consider the problem of dividing 42 guavas among six students. In the horizontal mathematization process, the guavas are grouped into six equal parts, producing 42:6 = ..., in this case, a number is sought which, if multiplied by 6 produces 42. Students rely on their memory of multiplication to determine 7x6 = 42, hence 42:6 = 7. Alternatively, in the repeated subtraction method, the number of times 6 can be subtracted from 42 until it becomes zero is counted. The steps include 42-6=36, 36-6=30, 30-6=24, 24-6=18, 18-6=12, 12-6=6, and finally 6-6=0. Based on this calculation, there are seven times to subtract 6 from 42 until it runs out, which produces the same answer as 42:6=7. However, when the number is large, for example, 1989:13 =? students with poor multiplication skills are likely to encounter difficulties. In such cases, teachers are expected to guide the student to combine multiplication and repeated subtraction. Students are free to use their multiplication skills, such as multiplication by 100, 10, or 1, followed by repeated subtraction until the difference is zero or < 13. The results of each step above are recorded, and the numbers are added up. Basically, the division of small numbers can be demonstrated semi-abstractly, leading to the horizontal mathematization of division. As the multiplication skills of students improve, they can transition to formal division, which is a vertical mathematization process.

```
        100+10+10+10+10+10+1+1+1 = 153
13     /1989
        1300  _
         689
         130  _
         559
         130  _
         429
         130  _
         299
         130  _
         169
         130  _
          39
          13  -
          26
          13  -
          13
          13  -
           0
```

Determining the Value of π (Pi)

One of the most important constants in geometry is π (Pi), which represents the ratio of the circumference of a circle to its diameter. This ratio yields an irrational number that is expressed as a decimal fraction that does not repeat or end, where π (Pi) = 3.14159265358..... While its use in primary and secondary schools is approximated by the value 22/7 or 3.14 to aid students who are yet to grasp formal and abstract thinking, the importance of $\boldsymbol{\pi}$ cannot be overemphasized. Assigning students to determine the ratio of circumference and diameter of several arbitrary circles can lead to different comparison values, with some results coming close to 22/7 or 3.14, which are then represented by the symbol $\boldsymbol{\pi}$ (Pi) through the horizontal mathematization process. The introduction of the constant $\boldsymbol{\pi}$ is very important, as it appears in formulas relating to the field of geometry, particularly in circles, ellipses, spheres, and trigonometry, through the vertical mathematization process. The constant $\boldsymbol{\pi}$ is also found in formulas across other scientific fields such as cosmology, number theory, statistics, fractals, thermodynamics, mechanics, and electromagnetism.

Solving System on Linear Equations of Two Variables (SPLDV)

This topic is highly relatable to real-life scenarios, for example, consider a problem, where Putu, Made, and Nyoman visited the Saraswati bookstore. Putu bought four notebooks and three pencils for IDR12,500, and Made bought two notebooks and a pencil for IDR5,500 from the same shop. If Nyoman wants to buy six notebooks and two pencils, what would be his total cost?

From this real-world problem, a mathematical model can be created through horizontal mathematization. For example, suppose the price of 1 notebook = x, and the price of 1 pencil = y, an equation to represent the purchase by Putus can be written as 4x + 3y = 12,500. Made groceries can be expressed as 2x + y = 5,500. By combining these two equations, a system of linear equations (SPLDV) is obtained as follows:

$$\begin{cases} 4x + 3y = 12500 \quad \dots\dots (1) \\ 2x + y = 5500 \dots\dots. \ (2) \end{cases}$$

Using the Elimination/substitution method, the final price of a notebook (x) and a pencil (y) are determined as IDR2,000 and IDR1,500, respectively. This means that Nyoman is expected to pay IDR15,000 for the purchase of 6 notebooks and two pencils. Furthermore, the vertical mathematization process involves exploring the properties of SPLDV, such as single, plural, or no solution, the number of variables, and alternative solving methods. This process can also be extended to incorporate the concept of matrices. To demonstrate the horizontal mathematization process, the SPLDV from the previous example can be arranged accordingly, to form the following:

$\begin{bmatrix} 4x + 3y \\ 2x + y \end{bmatrix} = \begin{bmatrix} 12500 \\ 5500 \end{bmatrix}$ until it becomes $\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12500 \\ 5500 \end{bmatrix}$

The vertical mathematization process occurs in the development of the properties of matrices and their operations.

Determining Market Equilibrium

The economic concepts of demand and supply are critical contextual factors that impact the market. **Demand** can be expressed as a mathematical function that correlates the price with the number of goods/services demanded by buyers. In contrast, **supply** is a function that relates the price to the number of goods/services **sold** by sellers. This implies that demand is the business of buyers, while supply is the responsibility of sellers. Buyers typically prefer cheaper goods, indicating that as prices decrease, more goods are purchased, and vice versa. Conversely, sellers prefer to sell more goods at higher prices, which means the number of goods sold is directly proportional to the price, causing an upward graph. In order for a transaction to occur, the desires of the buyer and seller need to be met. The meeting point of the demand and supply curves is known as the market equilibrium, where the appropriate price and quantity of goods are available for sale. To graphically represent the supply and demand data, a horizontal mathematization process is used to plot the two curves on a graph, and the intersection point denotes the market equilibrium. Meanwhile, the vertical mathematization process involves the development of various types of graphs, from linear to non-linear, and their properties.

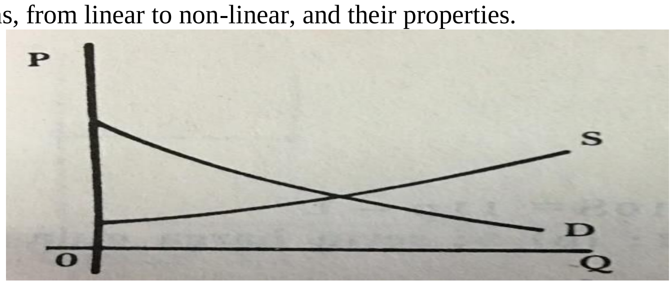

Figure.01: Demand-Supply Curve and Market Equilibrium Point

Determining the Probability of Occurrence

The probability of occurrence problem is closely related to real-life situations. For example, consider the case of students taking a true-false quiz with five questions. The probability that a student can answer a question correctly is 0.6. In such a scenario, what is the probability that students can answer four questions correctly?

In this problem, students are analogous to conducting an experiment five times. In each trial, they can either answer correctly (success) or incorrectly (failure). The probability of answering correctly is fixed over repeated trials because the occurrences are mutually independent. Where the probability of answering correctly is denoted as p, which is equal to 0.6. The probability of failure is also fixed at q = 1 - p = 1 - 0.6 = 0.4. One possible outcome of this experiment is that students answer four questions correctly out of the five. This can occur when they answer the first four questions correctly and the last (fifth) incorrectly. If the correct answer is denoted as "success" abbreviated as S, and the wrong answer as "failure", abbreviated as G, then the result can be expressed in the order S1 S2 S3 S4 G5. Since each event is independent, the probability of this co-occurrence can be determined by the probability of intersection:

$P(S_1 \cap S_2 \cap S_3 \cap S_4 \cap G_5) = P(S_1) \times P(S_2) \times P(S_3) \times P(S_4) \times P(G_5) = 0.6 \times 0.6 \times 0.6 \times 0.6 \times 0.4 = (0,6)^4 \times (0,4)^1$.
The probability of the other four possible answer formations can be expressed as (S_1 S_2 S_3 G_4 S_5), (S_1 S_2 G_3 S_4 S_5), (S_1 S_2 G_3 S_4 S_5), and (G_1 S_2 S_3 S_4 S_5). Therefore, the probability of the second formation is $0.6 \times 0.6 \times 0.6 \times 0.4 \times 0.6 = (0.6)^4 \times (0.4)^1$, and the third is expressed as $0.6 \times 0.6 \times 0.4 \times 0.6 \times 0.6 = (0.6)^4 \times (0.4)1$. The probability of the fourth formation is $0.6 \times 0.4 \times 0.6 \times 0.6 \times 0.6 \times 0.6= (0.6)^4 \times (0.4)1$, while the fifth is $0.4 \times 0.6 \times 0.6 \times 0.6 \times 0.6 \times 0.6 = (0.6)^4 \times (0.4)^1$. From the above calculations, the total probability of a student answering four questions correctly is $5 \times \{(0.6)^4 \times (0.4)^1\}$. The vertical mathematization process is carried out by inducing from several cases as mentioned above in order to obtain a generalization in the form of a deductive formula.

Geometric Rows and Arithmetic Rows

To understand the dynamics of population growth and food growth, one can consider the issue of their imbalance. It is a well-known fact that all living beings have two essential needs, namely sustenance, and reproduction. The population growth rate far exceeds the food growth, which can be attributed to the fact that population growth follows a geometric progression, while food growth follows an arithmetic sequence. The data on population growth and food growth are as follows:

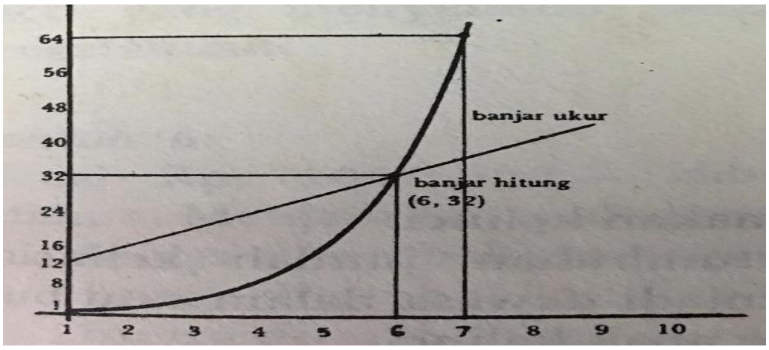

Figure .02: Growth Chart

Foods	12 16 20 24 28 32 36
Population	1 2 4 8 16 32 64

The growths of population and food can be depicted graphically, as shown in Figure 02, to demonstrate how they differ. This process represents a horizontal mathematization process that involves converting real-life examples of growth into graphs. Moreover, a vertical mathematization process is required to develop formulas for both arithmetic and geometric series.

The graph in Figure 02 can also be utilized to illustrate the progression of COVID-19, which is currently a global pandemic. The growth of the virus follows a geometric sequence, which is particularly concerning due to its exponential nature. Conversely, the development of healthcare resources, such as hospitals, medical personnel, and drugs, follows an arithmetic sequence. Delay in addressing the growth of the virus can lead to severe consequences, necessitating prompt intervention.

Exponential Function

A series can be formed from a sequence, and these two entities are closely related to exponential functions. An example of this concept in real-life is the story of Pan Balang Tamak. According to the story, after winning a war, the king planned to reward his advisors and the leaders of the army with wealth and territories. However, one advisor, Pan Balang Tamak, who had played a minor role, was not given such rewards. When asked by the king what he desired, Pan Balang Tamak responded with a clever request, saying “My Lord, I do not ask for luxury items, but only for rice that I will use to fill the chessboard squares. Starting with 4 grains of rice in the first square, I will then add its multiples to each subsequent square, filling it with 4^2 in the second, 4^3 in the third, etc., until the last square of the chessboard”. Pan Balang Tamak chose the chessboard as his unit of measurement because he frequently played chess with the king. Upon hearing this request, the king laughed and mocked him, saying, "You are very foolish! Your request is only that little?" However, the king promised to fulfill his request the next day. Can you estimate how many tons or kilograms of rice were needed to fulfill Pan Balang Tamak's request? Problems of this form can be presented to students to teach the concept of exponential numbers and their relation to geometric series.

When converting the number of grains of rice in each of the 64 boxes on the chessboard into a geometric series, a horizontal mathematical process occurs. The series begins with 4 grains of rice in the first box and increases in a pattern of adding 1 to the exponent with each subsequent box, such as $4 + 4^2 + 4^3 + 4^4 + \ldots . + 4^{64}$ =……. grains of rice. Once the total amount of rice is calculated, it can be calculated, it can be converted into tons/kg. The result is astounding as the request made the king bankrupt for seven generations. Meanwhile, a vertical mathematical process occurs by exploring the properties of exponential numbers and series.

Vector Matrix

One way to introduce the concepts of matrices and vectors is by using a real-life example. Consider a wooden block with dimensions of 3 dm (length), 1 dm (width), and 2 dm (height). At the bottom of the block, the corner points can be labeled as A, starting from the back left point, then forward to B, to the right side as C, and ending at the back point as D. Similarly, the corresponding points on the top face can be labeled as E, F, G, and H, respectively. To present this block in a three-dimensional coordinate system, XYZ is created based on the drawing, leading to the following image.

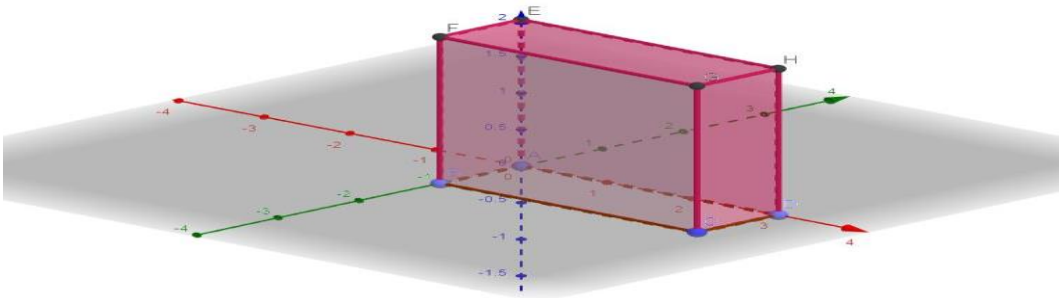

Figure 03: Coordinate the points of block ABCD.EFGH

To represent the physical shape of a wooden block with length = 3 dm, width = 1 dm, and height = 2 dm, it can be transformed into the R^3 coordinate system, where point A serves as the origin A(0,0,0). To simplify the block, it can be represented by only three main points, namely points B, D, and E located on the X-axis, Y-axis, and Z-axis with coordinates (1,0,0), (0,3,0), and (0,0,2), respectively. The other corner points can be obtained based on the positions of these three points. Therefore, the matrix form of the block shape is expressed as $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. This represents a horizontal mathematical simplification process. For vertical mathematical simplification, various matrices and their operating systems can be further developed.

Extreme Value Problem of Function

The problem of extreme values of a function is a topic formally studied in derivative applications. However, before applying formal rules, this concept can be approached through real-world problems. For example, consider four rectangular cardboard sheets, each with a length of 12 cm and a width of 9 cm. To make a box without a lid, cut off the four corners of the cardboard sheets in the shape of squares with side lengths of 1 cm, 2 cm, 3 cm, and 4 cm. Subsequently, plaster/glue was used to attach the sides, as shown in the image below.

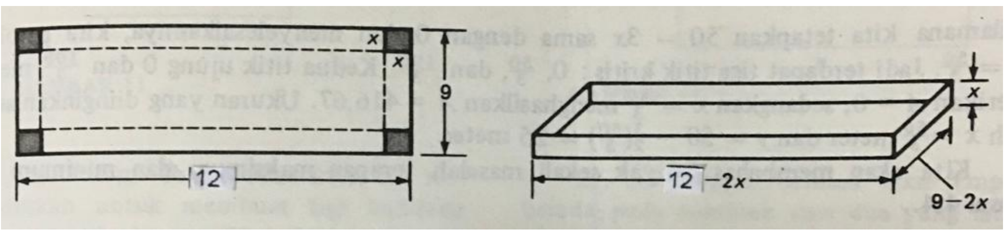

Figure 04. Rectangular cardboard turned into an open box.

To begin, students can be asked to estimate which of the boxes has the largest volume. This estimation can be compared to the final calculation result. By performing the necessary calculations, the volume of each box can be determined and recorded in the following table.

Table 01: Box Dimensions

Box	Cutting	Size l,w,h	Volume
A	1 cm	10, 7, 2	70
B	2 cm	………	80
C	3 cm	………	54
D	4 cm	………	16

In order to gain a clearer understanding of the current problem, it is advisable to draw a graph that depicts the relationship between the volume of the box and the length of the square side cut from each corner of the cardboard. By connecting the various data points on the graph, a general trend can be observed. This information can then be used to estimate the square size that should be cut to obtain the maximum volume of the box. Upon analyzing Table 01, it can be observed that the volume of the box increase from point A to B, while it decreases from point B to C. Therefore, the maximum volume can only be achieved between A and B or B and C. To determine the exact square size that can yield the maximum volume, students need to recalculate the measurements between 1 cm and 2 cm (for example, 1.5 cm) and between 2 cm and 3 cm (for example, 2.5 cm), as shown in Table 02.

Table 02: Box Dimensions

Cutting	Size l,w,h	Volume
2 cm	8, 5, 2	80
2.5 cm	7, 4, 2.5	70
1.5 cm	9, 6, 1.5	81 (>80)

According to Table 02, students can determine the shape of the graph as shown in Figure 06. From the graph, it is apparent that the box has maximum volume when the side length of the square ranges between 1.5 cm and 2.0 cm. To refine their estimation, students attempt various additional values, as shown in Table 03.

.Table 03: Box Dimensions

Cutting	Size l,w,h	Volume
1.5	9,6,1.5	81
1.6	……..	81.664
1.7	8.6,5.6,1.7	81.872
1.8	………	81.648
1.75	……….	81.8125

From Table 03, it is evident that a cut-out square with a size of 1.7cm provides the most accurate approximation to the maximum volume of the box, which is 81.872 cm3. It is important to note that the dimensions of the box are 8.6 cm x 5.6 cm x 1.7 cm. As stated earlier, students can only provide an estimated answer for the problem presented, making it a horizontal mathematization process. In contrast, the exact answer is achieved through vertical mathematization, which involves using differential calculus to determine extreme functions.

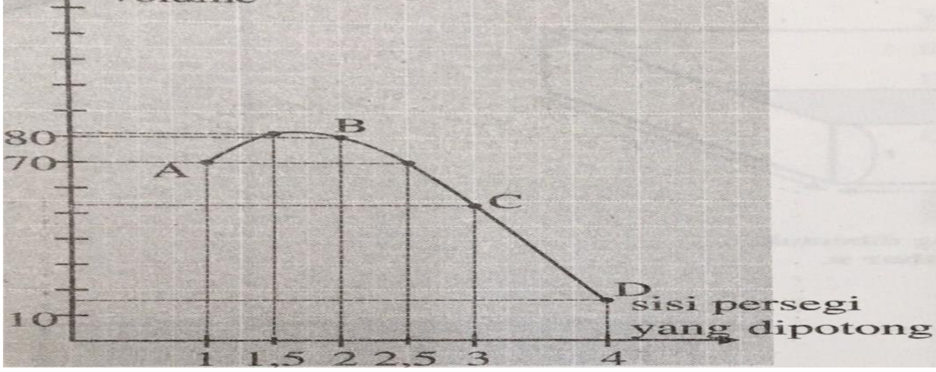

Figure 05: Volume Change Rate Graph

Addition of numbers

Number games can be utilized to enhance the addition skills of students and cultivate their ability to identify successful strategies. This game is specifically designed for two people or groups, while the other students observe and think about potential winning strategies. Any numbers can be used because various numbers require different strategies. For example, if the number 50 is chosen, referred to as the 50-game, the cumulative total can be reached using numbers 1, 2, 3, 4, 5, and 6. The rule is that players take turns choosing numbers, and the first person to accumulate a total of 50 is declared the winner. Each time a new number is selected, its value is added to the total value of the numbers chosen previously by both players. Suppose player A begins and chooses the number 6 (reaching a score of 6), player B can choose 5, bringing their score to 6+5=11. Subsequently, it becomes the turn of player A, and they choose 4, increasing the score to 15+4=19, etc. Based on the calculation, the score always represents the cumulative total of both players, and the winner is the first person to reach 50. This game can be adapted for subtraction and other arithmetic operations, and played multiple times, especially during leisure time or when students are bored. Eventually, students are able to develop successful strategies, which can be applied to other arithmetic operations.

This game can also illustrate problem-solving strategies by working backward. For example, analysis of the game showed that a player can always win by reaching 50 if they reached 43, regardless of the choice of their opponent. The winning strategy involves selecting the numbers 1, 8, 15, 22, 29, 36, 43, and 50. Once this strategy is mastered, the game can be adapted to different numbers and ranges, allowing students to continue developing their strategies.

These contextual and realistic mathematical topics can be presented alongside other topics that cannot be covered in limited time and space.

III. CONCLUSION

Conclusion

RMEn facilitated the mathematization process of real-life problems encountered by students. This process comprised two stages, namely horizontal and vertical. In the horizontal stage, students started by linking mathematical topics to real-world situations, thereby enabling contextual mathematics learning. By leveraging their existing knowledge, students identified, connected, visualized, or formulated real-life problems expressed in everyday language using mathematical language. It is worth noting that mathematical language took the form of declarative sentences in either verbal or symbolic notation.

In the vertical stage, students reorganized the information obtained during the horizontal stage using mathematical concepts and principles. They gradually progressed from combining simple to more complex symbols, from individual concepts to constructing principles and algorithms, ultimately leading to expansion or generalization.

Suggestion

Many mathematical topics relate to the real world and are part of the students life experiences. These topics serve as entry points to introduce school-level mathematical concepts, which can facilitate the transfer of learning and prepare students for the Society 5.0 era.

REFERENCES

[1]. Begle, E.C. Critical Variables in Mathematics Education. Washington DC: MAA & NCTM (1979)
[2]. De Lange, J. Mathematics Insight and Meaning. CW & OC, Utrecht. (1987)
[3]. Gravemeijer. Developing Realistic Mathematics Education. Utrecht: Freudenthal Institute. (1994)
[4]. Gagne, RM. The Conditions of Learning and Theory of Instruction. New York: Hott, Reinhart and Winston. (1985)
[5]. Herman Hudoyo. Heboh tentang Pengajaran Matematika di SD. Makalah yang disajikan dalam Seminar Regional Matematika kota Malang, 20 September (1990).
[6]. Pengantar Beberapa Sistem Matematika untuk Guru. Malang: IKIP Malang. (2005)
[7]. Ivor K Devis. (2000). Pengelolaan Belajar. Jakarta: CV Rajawali. (2000)
[8]. Kurniati. Peningkatan Kemampuan Berpikir Kritis dan Kreatif Matematis serta Soft Skill Mahasiswa Pendidikan Guru Sekolah Dasar melalui Pendekatan Pembelajaran Kontekstual. Unpublished Disertation. Bandung SPs Universitas Pendidikan Indonesia. (2014)
[9]. Marpaung, Y. Implementasi PMR di Indonesia. Makalah disampaikan pada Seminar Nasional Sehari: Penerapan PMR pada Sekolah dan Madrasah. Medan, 5 November (2021).
[10]. Munandar, U. Pengembangan Kreativitas Anak Berbakat. Jakarta: PT Rineka Cipta. (2012)
[11]. Rohaeti. E.E. Critical and Creative Mathematical Thinking of Junior High School Student. Educational Journal, 4(2), 99-106. (2010)
[12]. Suryanto. Pendidikan Matematika Realistik Indonesia (PMRI) dalam PMRI, Vol.V, No.1 Januari 2007
[13]. Sujadi, R. Kiat Pendidikan Matematika di Indonesia. Jakarta: Direktorat DIKTI DEpDikNas. (2000)