Domination Polynomial of a Corona Graph

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Abstract: Let the graph \( G = (V,E) \) be a simple undirected graph of order \( |V(G)| = n \). A set \( S \subseteq V(G) \) is a dominating set of \( G \), if every vertex in \( V(G) - S \) is adjacent to at least one vertex in \( S \). A domination polynomial of a graph \( G \) is \( D(G,x) = \sum_{i=\gamma(G)}^{\delta(G)} d(G,i)x^i \), where \( d(G,i) \) is the number of all dominating sets of \( G \) with size \( i \) and \( \gamma(G) \) is the minimum domination number of \( G \). In this paper, we construct the dominating set of \( P_n \circ mK_1 \), and obtain the domination polynomial \( D(P_n \circ mK_1,x) = p_i = \sum_{i=\gamma(P_n \circ mK_1)}^{\delta(P_n \circ mK_1)} d(P_n \circ mK_1,i) \), which we call an ordinary generating function for the dominating sets of graph \( P_n \circ mK_1 \).

Keywords: Dominating sets, Domination polynomials, Corona graph, Ordinary generating function

Mathematics subject classification: 05C30, 05C31, 05C69

I. Introduction and preliminary results

All of the graphs examined in this paper are finite and simple graph. In an undirected graph \( G = (V,E) \), let \( V(G) \) denote the set of all vertices of \( G \) and let \( E(G) \) denote the set of all edges of \( G \). The collected works on the subject of domination parameters in graphs has been discussed in this book [8]. A partition of \( V(G) \) such that each class is a dominating in \( G \) is called a dogmatic partition of \( G \). We have invariant polynomials for graphs in graph theory. For any vertex \( v \in V(G) \), the open neighbourhood of \( v \) is the set \( N(v) = \{ u \in V(G) | uv \in E(G) \} \) and the closed neighbourhood of \( v \) is the set \( N[v] = N(v) \cup v \). For a set \( S \subseteq V(G) \), the open neighborhood of \( S \) is \( N(S) = \bigcup_{v \in S} N(v) \) and the closed neighborhood of \( S \) is \( N(S) = N(S) \cup S \). A subset \( S \subseteq V(G) \) is a dominating set if \( N(S) = V \). In a simple graph \( G \) a dominating set \( S \) is a minimal dominating set in \( G \) if no proper subset \( S \subseteq S \) is a dominating set. The domination number \( \gamma(G) \) is the minimum cardinality of a dominating set in \( G \) [7]. We call such a set a \( \gamma(G) \) -set of \( G \). Let \( D(G,i) \) be the family of dominating sets of a graph \( G \) with cardinality \( i \) and \( d(G,i) = |D(G,i)| \). A domination polynomial of a graph \( G \) is \( D(G,x) = \sum_{i=\gamma(G)}^{\delta(G)} d(G,i)x^i \), where \( d(G,i) \) is the number of all dominating sets of \( G \) with size \( i \) and \( \gamma(G) \) is the minimum domination number of \( G \) [1, 6]. Our objective in this paper is to study the domination sets and its polynomial of the graph \( P_n \circ mK_1 \). The domination polynomial \( D(G,x) \) is an ordinary generating function for the dominating sets of undirected graph \( G \) with respect to their cardinalities. Domination sets and its polynomials of the path, cycles, cubic graphs and some of special graphs were studied by Alikhani and Peng [1, 2, 6]. The mean value for the matching and graph products of dominating polynomial were studied in [4, 8]. The corona of two graphs

\( G_1 \) and \( G_2 \) as defined by Frucht and Harary [10] is the graph \( G = G_1 \uplus G_2 \) formed from one copy of \( G_1 \) and \( |V(G_1)| \) copies of \( G_2 \) where the \( i \)th vertex of \( G_1G_2 \) is adjacent to every vertex in the \( i \)th copy of \( G_2 \). In the following section for the graph \( P_n \circ mK_1 \) the dominating set and its polynomial is driven.

**Theorem 1.1.** [1] Let \( S \) be a star graph \( n \). Then the domination polynomial is \( D(S_n,x) = x^n + (1 + x)^n \).

**Theorem 1.2.** [1] If a graph \( G \) consists of \( m \) components \( G_1, G_2, \ldots, G_m \), then \( D(G,x) = D(G_1,x)D(G_2,x)\ldots D(G_m,x) \).

**Theorem 1.3.** [6] For every natural number \( n \), \( D(K_n,x) = (1 + x)^{n-1} \).

**Theorem 1.4.** [2, 6] Let \( G \) be a graph with \( |V(G)| = n \). Then

(i). If \( G \) is connected, then \( d(G,n) = 1 \) and \( d(G,n) = n \), where \( d(G,n) \) is number of dominating set in graph \( G \).

(ii). \( d(G,i) = 0 \) if and only if \( i < \gamma(G) \) or \( i > n \).

(iii). \( D(G,x) \) has no constant term.

(iv). \( D(G,x) \) is a strictly increasing function in \([0,\infty]\).

(v) Zero is a root of \( D(G,x) \), with multiplicity \( \gamma(G) \).
II. Main Results

Definition 2.1. A $P_n \circ mK_1$ is a graph obtained by taking the corona of a path $P_n$ with $mK_1$ and has $n(m + 1)$ vertices, where $m$ is the number of pendant vertices in each vertex of the path $P_n$.

Lemma 2.1. If $P_n \circ mK_1$ be a graph with $m(n + 1)$ vertices, then $\gamma(P_n \circ mK_1) = n$, for all $m, n \in N$.

Proof: Let $S$ be the dominating set of $P_n \circ mK_1$. Then by definition, for all $i, v_i \in S$, where $n \leq i \leq mn$. This implies that $|S| \geq n$. If $S$ is $\gamma$-set of $P_n \circ mK_1$, then $S$ exactly contain all vertices of the path $P_n$. Therefore, $\gamma(P_n \circ mK_1) = n$, for $m, n \in N$.

Theorem 2.1. Let $P_n \circ mK_1$ be a graph with $2(m + 1)$ vertices. Then $D(P_n \circ mK_1, x) = x^2(1 + x)^{2m} + 2x^{m+1}[1 + x((1 + x)^m - 1)] + x^{2m}$.

Proof: Let $P_n \circ mK_1$ be a graph and $D(P_n \circ mK_1, i)$ be a family of dominating set with cardinality $i, 2 \leq i \leq 2(m + 1)$. Let $V = \{v_1, v_2, u_1, u_2, \ldots, u_1m, u_2, u_2, \ldots, u_2m\}$ be the vertex set of $G$, where $v_1$ and $v_2$ are the vertices of $P_2$ and $m$ is the number of pendant vertices to the vertices of path $P_2$. Since every pendant vertices of $P_2 \circ mK_1$ are adjacent to either $v_1$ or $v_2$. Hence, for $i = 2$ or $\{v_1, v_2\}$ the $\gamma(P_2 \circ mK_1) = 2$ and its polynomial is $x^2$. For the cardinality $i = 3$ the family of dominating set $D(P_2 \circ mK_1, 3)$ is obtained by selecting vertices $v_1, v_2$ and one vertex from $\{u_1, u_2, \ldots, u_1m, u_2, u_2, \ldots, u_2m\}$. Thus $D(P_2 \circ mK_1, 3) = \binom{2m}{1}x^3$ and the domination polynomial term $\binom{2m}{1}x^3$. If we proceed like this for the remaining cardinality $i, 4 \leq i \leq 2(m + 1)$. Then the domination polynomial of the graph is

$$D(G_1, x) = x^2 + \binom{2m}{1}x^3 + \ldots + \binom{2m}{1}x^{2m+2}$$

$$= x^2 + \binom{2m}{1}x + \ldots + \binom{2m}{1}x^{2m}$$

$$= x^2(1 + x)^{2m}$$

The family of dominating set $D(P_2 \circ mK_1, i)$ is obtained also by selecting one of vertex from path $P_2$ and the remaining vertices taken from the other of the pendant vertices. Let us take vertex $v_1$, and all pendant vertices to $v_2$, which are dominating the graph $P_2 \circ mK_1$ and choose the remaining dominating vertices from the $\{v_2, u_1, u_2, \ldots, u_{1m}\}$ and similarly we can take for vertex $v_2$ and all pendant vertices to $v_1$ which are dominating the graph $P_2 \circ mK_1$. For the cardinality $i$, such that $m + 1 \leq i \leq 2(m + 1)$ the number of dominating set for $i = m + 1, m + 2$ are 1 and $\binom{m}{1}$ respectively. In general, the domination polynomial of the graph is

$$D(G_2, x) = 2 [x^{m+1} + \binom{m}{1}x^{m+2} + \ldots + \binom{m}{1}x^{2m+2}]$$

$$= 2x^{m+1} + 2x^{m+2}[(1 + x)^m - 1]$$

$$= 2x^{m+1}[1 + x((1 + x)^m - 1)]$$

For none of the vertices of the path $P_2$ are dominating set and for the cardinality $i = 2m$. Hence the domination polynomial of the graph is

$$D(G_3, x) = x^{2m}$$

Therefore, from the equation (3), (6) and (7) we get the domination polynomial of $D(P_2 \circ mK_1) = D(G_1, x) + D(G_2, x) + D(G_3, x)$ and this completes the proof.

Theorem 2.2. Let $P_n \circ mK_1$ be graph with $n(m + 1)$ vertices. Then

$$D(P_n \circ mK_1, x) = x^n(1 + x)^{nm} + (1 + x)x^{m+n-1}\sum_{i=1}^{n}(1 + x)^{n(n-i)} + \ldots + x^{nm}$$

Proof: Let $G = P_n \circ mK_1$ be a graph and $D(G, i)$ be a family of dominating set with cardinality $i, n \leq i \leq n(m + 1)$. Let the vertex set $V(G)$ is
Domination Polynomial of a Corona Graph

\[{v_1,v_2,\ldots,v_n},{u_{1,1},u_{1,2},\ldots,u_{1,m}},{u_{2,1},u_{2,2},\ldots,u_{2,m}},\ldots,{u_{n,1},u_{n,2},\ldots,u_{n,m}}\], where \{{v_1,v_2,\ldots,v_n}\} are the vertices of the path \(P_n\) and \(nm\) is the number of pendant vertices to the vertices of path \(P_n\). Since every \(nm\) number of pendant vertices of \(G\) are adjacent to either vertices of \(P_n\). Hence, for the cardinality \(i = n\) the \(\gamma(G) = n\) and its polynomial is \(x^n\). The next family of dominating set is \(D(G,n+1) = \{v_1,v_2,\ldots,v_n,v_j\}\) such that the vertex \(u_{j,k}\) is chosen from \(nm\) numbers of pendant vertices of the path \(P_n\), where \(1 \leq j \leq n\) and \(1 \leq k \leq m\). Hence \(d(G,n+1) = \binom{nm}{1}\) and its domination polynomial term is \(\binom{nm}{1}x^{n+1}\). By proceeding this way for the remaining cardinality \(i\) such that \(n + 2 \leq i \leq n(m + 1)\), then the domination polynomial of the graph yields,

In the other case one of the vertices of the path \(P_n\) is not contained in the domination set of \(G\). Suppose \(v_1 \in V(P_n)\) is not contained in dominating sets of the graph \(G\). This is provided by combining \({v_2,v_3,\ldots,v_n}\) with \(m\) pendant vertices which has minimum cardinality \(i = m + n - 1\) and hence \(d(g_2,m + n - 1) = 1\), and the domination polynomial of the term \(x^{m+n-1}\). For the proceeding cardinality \(i = m + n - 1\) we choose one vertex from the pendant vertices except the vertices pendant to the vertex of \(v_1\). Thus, the domination polynomial of the term is \(\binom{nm}{1}x^{m+n}\). Therefore, the domination polynomial for \(v_1\) is

\[
D(G_1,x) = x^n + \binom{nm}{1}x^{n+1} + \ldots + \binom{nm}{nm}x^{nm+n}
\]

Hence

\[
D(G_1,x) = x^n\left[1 + \binom{nm}{1}x + \ldots + \binom{nm}{nm}x^{n}\right]
\]

\[
= x^n(1 + x)^{nm}
\]

\[\] (8)

\[\] (9)

\[\] (10)

\(V(P_n)\) is not contained in dominating sets of the graph \(G\). This is provided by combining \({v_2,v_3,\ldots,v_n}\) with \(m\) pendant vertices which has minimum cardinality \(i = m + n - 1\) and hence \(d(g_2,m + n - 1) = 1\), and the domination polynomial of the term \(x^{m+n-1}\). For the proceeding cardinality \(i = m + n\) we choose one vertex from the pendant vertices except the vertices pendant to the vertex of \(v_1\). Thus, the domination polynomial of the term is \(\binom{nm}{1}x^{m+n}\). Therefore, the domination polynomial for \(v_1\) is

\[
D(G_2,x) = x^{m+n-1} + \binom{m(n-1)+1}{1}x^{m+n} + \ldots + \binom{m(n-1)+1}{m(n-1)+1}x^{m(n+1)}
\]

Hence

\[
D(G_2,x) = x^{m+n-1}\left[1 + \binom{m(n-1)+1}{1}x + \ldots + \binom{m(n-1)+1}{m(n-1)+1}x^{m(n+1)+1}\right]
\]

\[
= x^{m+n-1}(1 + x)^{m(n-1)+1}
\]

\[\] (11)

\[\] (12)

\[\] (13)

From equation (13), we generate for the overall domination polynomial cases that are obtained by investigating the situations of each \(n\) vertices of the path \(V(P_n) = \{v_1,v_2,\ldots,v_n\}\). Therefore, the general domination polynomial of the second case is given as,

For the remaining third case we suppose that none of the vertices of \(V(P_n) = \{v_1,v_2,\ldots,v_n\}\) is contained in the dominating set of the graph \(G\). Hence, for cardinality \(i = nm\) the family of dominating set \(D(G,nm) = 1\) and

\[
D(G_3,x) = (1 + x)x^{m+n-1}+\sum_{i=1}^{n}(1 + x)^{m(n-i)}
\]

the domination polynomial term of this case is

\[
D(G_3,x) = x^{nm}
\]

Therefore, the equation (10), (14) and (15) above completes the proof.

Definition: 2.2. [7] A graph \(C_m\) is a simple corona graph with \(m(n + 1)\) order of vertices. The vertex set \({v_1,v_2,v_3,\ldots,v_n}\) of a cycle \(C_m\) is joined with \(m\) number of pendant vertices to each vertex of the cycle \(C_m\), where \(m > 1\) and the sets \({u_{1,1},u_{1,2},\ldots,u_{1,m}},{u_{2,1},u_{2,2},\ldots,u_{2,m}},\ldots,{u_{n,1},u_{n,2},\ldots,u_{n,m}}\) are pendant vertices

Theorem 2.3 If \(C_m\) is the graph with \(3(m + 1)\) vertices, then \(D(C_m)\) is the family of dominating set with cardinality \(i\) such that \(3 \leq i \leq 3(n + 1)\).

\[
\text{Proof: Let } H = \text{ be the family of dominating set with cardinality } i \text{ such that } 3 \leq i \leq 3(n + 1).
\]

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Let the vertex set $V(H)$ of the graph $H$ is $\{v_1, v_2, v_3, \{u_{1,1}, u_{1,2}, \ldots, u_{1,m}\}, \{u_{2,1}, u_{2,2}, \ldots, u_{2,m}\}, \ldots, \{u_{n,1}, u_{n,2}, \ldots, u_{n,m}\}\}$ and $3m$ number of pendant vertices to the graph of $H$. Herewith, we proceed four cases to prove the theorem.

**Case-1.** Let consider the vertices $v_1$, $v_2$, and $v_3$ of the cycle $C_3$ and it belongs to the dominating set of the graph $H$. Hence, for the size $i = 3$ the $d(H, 3) = 1$ and its polynomial term is $x^3$. For the preceding size $i = 4$, the $d(H, 4)$ is obtained by considering all vertices of the cycle $C_3$ and selecting one vertex from $3m$ number of pendant vertices to the vertices of cycle $C_3$. Thus, the dominating sets and its polynomial are $|D(H, 4)| = \left(\frac{3m}{1}\right)$ and $\left(\frac{3m}{1}\right)x^4$ respectively. Similarly, we can proceed for the remaining cardinality $i$ such that $5 \leq i \leq 3(m + 1)$. Therefore, the ordinary generating function of the graph case-1 yields,

$$
x^3 + \left(\frac{3m}{1}\right)x^4 + \ldots + \left(\frac{3m}{3m}\right)x^{3m+3}$$

$$= x^3 \left[ 1 + \sum_{i=1}^{3m} \left(\frac{3m}{i}\right)x^i \right] \quad (16)$$

$$= x^3 (1 + x)^{3m} \quad (17)$$

$$\{v_2, v_3, \{u_{2,1}, u_{2,2}, \ldots, u_{2,m}\}, \ldots, \{u_{3,1}, u_{3,2}, \ldots, u_{3,m}\}\}. \ \text{Hence the } d(H, m + 3) = \left(\frac{2m+2}{1}\right). \ \text{In the same way}$$

$$3 \left[ x^{m+2} + \left(\frac{2m+2}{1}\right)x^{m+3} + \ldots + \left(\frac{2m+2}{2m+2}\right)x^{3(m+1)} \right] \quad (19)$$

$$= 3x^{m+2} \left[ 1 + \sum_{i=1}^{2(m+1)} \left(\frac{2(m+1)}{i}\right)x^i \right] \quad (20)$$

$$= 3x^{m+2}(1 + x)^{2(m+1)} \quad (21)$$

we can find the family dominating sets by avoiding the vertex $v_2$ or $v_3$. Therefore, the domination polynomial given by:

$$3 \left[ x^{2m+1} + \left(\frac{m+1}{1}\right)x^{2m+2} + \ldots + \left(\frac{m+1}{m+1}\right)x^{3m+3} \right]$$

$$= 3x^{2m+1} \left[ 1 + \sum_{i=1}^{m+1} \left(\frac{m+1}{i}\right)x^i \right] \quad (22)$$

$$= 3x^{2m+1}(1 + x)^{m+1} \quad (23)$$

**Case-2.** Let for the vertex set $\{v_1, v_2, v_3\}$ of the cycle $C_3$ one of the vertex is not contained in the dominating sets of the graph $H$. Suppose $v_3$ is not contained in the dominating sets. Now by combining the vertex set $\{v_2, v_3\}$ and the $m$ number of pendant vertices to the vertex $v_1$ to find the dominating set for the cardinality $m + 2 \leq i \leq 3(m + 1)$. Let for $i = m + 2$ the $d(H, m + 2) = 1$ and its polynomial term is $x^{m+2}$. For the size $i = m + 3$ the $d(H, m + 3)$ is computed by choosing one vertex from $2m + 2$ the dominating number is obtained by choosing one vertex from $\{v_3, u_{3,1}, u_{3,2}, \ldots, u_{3,m}\}$. Thus, the $d(H, 2m + 2) = \left(\frac{m+1}{1}\right)$ and its term of polynomial is $\left(\frac{m+1}{1}\right)x^{2m+2}$. Since we have three distinct possible ways to avoid two vertices from the vertex set $\{v_1, v_2, v_3\}$ in the dominating sets. Therefore, the domination polynomial is given as,
Case-4. None of the vertex set \( \{v_1, v_2, v_3\} \) of the cycle \( C_3 \) are contained in the dominating sets of the graph \( H \). Hence, for cardinality \( i = 3m \) the family of dominating set \( D(H, 3m) = 1 \) and the domination polynomial term of this case is;

\[
D(H', x) = x^{3m}
\]  

(25)

Therefore, from the equation (18), (21), (24) and (25) we get the domination polynomial of the graph \( H = C_3 \circ mK_3 \) and this completes the proof.

### III. Conclusion

In this paper, we have shown that for a simple connected graph \( P_n \circ mK_2 \) the dominating sets and its ordinary generating function. Along the way, we found the domination polynomial for the corona product of the graphs \( P_2 \circ mK_1 \) and \( C_3 \circ mK_1 \).

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