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A Note on Fifth Order Jacobsthal Numbers

Yüksel Soykan, EvrenEyicanPolatlı

Department of Mathematics, Art and Science Faculty, ZonguldakBülent Ecevit University, 67100, Zonguldak, Turkey

Abstract. In this paper, we investigate the generalized fifth order Jacobsthal sequences and we deal with, in detail, six special cases which we call them as the fifth order Jacobsthal, fifth order Jacobsthal-Lucas, modified fifth order Jacobsthal, fifth order Jacobsthal Perrin, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas sequences.

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1. Introduction and Preliminaries

The Jacobsthal sequence (sequence A001045 in [25]) $\{J_n\}$ is defined recursively by the equation, for $n \geq 0$

$$J_{n+2} = J_{n+1} + 2J_n$$

in which $J_0 = 0$ and $J_1 = 1$. Then Jacobsthal sequence (second order Jacobsthal sequence) is

$$0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, 1365, 2731, 5461, 10923, \ldots$$

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [1,2,3,10,11,13,14,15,16,17,19,20,21]. For higher order Jacobsthal sequences, see [4,5,6,7,8,9,12,29].

In this paper, we introduce the generalized fifth order Jacobsthal sequences and we investigate, in detail, six special cases which we call them fifth order Jacobsthal, fifth order Jacobsthal-Lucas, modified fifth order Jacobsthal, fifth order Jacobsthal Perrin, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas sequences. First, we recall the definition and some properties of generalized Pentanacci sequence.

The generalized Pentanacci sequence (or the generalized (r, s, t, u, v) sequence or 5-step Fibonacci sequence)

$$\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3, W_4; r, s, t, u, v)\}_{n\geq 0}$$

is defined by the fifth order recurrence relations

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5}, \qquad W_0 = a, W_1 = b, W_2 = c, W_3 = d, W_4 = e \quad (1.1)$$

where the initial values W_0, W_1, W_2, W_3, W_4 are arbitrary complex (or real) numbers and r, s, t, u, v are real numbers. Generalized Pentanacci sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [27,22,23,24]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{u}{v}W_{-(n-1)} - \frac{t}{v}W_{-(n-2)} - \frac{s}{v}W_{-(n-3)} - \frac{r}{v}W_{-(n-4)} + \frac{1}{v}W_{-(n-5)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

As $\{W_n\}$ is a fifth order recurrence sequence (difference equation), it's characteristic equation is

$$x^5 - rx^4 - sx^3 - tx^2 - ux - v = 0 \quad (1.2)$$

whose roots are $\alpha, \beta, \gamma, \delta, \lambda$. Note that we have the following identities:

$$\begin{aligned} \alpha+\beta+\gamma+\delta+\lambda &= r, \\ \alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\alpha\delta+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta &= -s, \\ \alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\gamma+\alpha\gamma\delta+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta &= t, \\ \alpha\beta\lambda\gamma+\alpha\beta\lambda\delta+\alpha\beta\gamma\delta+\alpha\lambda\gamma\delta+\beta\lambda\gamma\delta &= -u \\ \alpha\beta\gamma\delta\lambda &= v. \end{aligned}$$

Generalized Pentanacci numbers can be expressed, for all integers n, using Binet's formula.

THEOREM 1. *[27] (Binet's formula of generalized (r, s, t, u, v) numbers (generalized Pentanacci numbers))*

$$\begin{aligned} W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} \\ &\quad + \frac{p_5\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}, \end{aligned} \quad (1.3)$$

where

$$\begin{aligned}
p_1 &= W_4-(\beta+\gamma+\delta+\lambda)W_3+(\beta\lambda+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta)W_2 \\
&\quad -(\beta\lambda\gamma+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta)W_1+(\beta\lambda\gamma\delta)W_0, \\
p_2 &= W_4-(\alpha+\gamma+\delta+\lambda)W_3+(\alpha\lambda+\alpha\gamma+\alpha\delta+\lambda\gamma+\lambda\delta+\gamma\delta)W_2 \\
&\quad -(\alpha\lambda\gamma+\alpha\lambda\delta+\alpha\gamma\delta+\lambda\gamma\delta)W_1+(\alpha\lambda\gamma\delta)W_0, \\
p_3 &= W_4-(\alpha+\beta+\delta+\lambda)W_3+(\alpha\beta+\alpha\lambda+\beta\lambda+\alpha\delta+\beta\delta+\lambda\delta)W_2 \\
&\quad -(\alpha\beta\lambda+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\delta)W_1+(\alpha\beta\lambda\delta)W_0, \\
p_4 &= W_4-(\alpha+\beta+\gamma+\lambda)W_3+(\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\beta\gamma+\lambda\gamma)W_2 \\
&\quad -(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\beta\lambda\gamma)W_1+(\alpha\beta\lambda\gamma)W_0, \\
p_5 &= W_4-(\alpha+\beta+\gamma+\delta)W_3+(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)W_2 \\
&\quad -(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)W_1+(\alpha\beta\gamma\delta)W_0.
\end{aligned}$$

Usually, it is customary to choose r,s,t,u,v so that the Equ. (1.2) has at least one real (say α) solutions.

Note that the Binet form of a sequence satisfying (1.2) for non-negative integers is valid for all integers n, (see [18], this result of Howard and Saidak [18] is even true in the case of higher-order recurrence relations).

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

LEMMA 2. *[27] Suppose that $f_{W_n}(x)=\sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized (r,s,t,u,v) sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{\Omega}{1-rx-sx^2-tx^3-ux^4-vx^5} \tag{1.4}$$

where

$$\Omega = W_0+(W_1-rW_0)x+(W_2-rW_1-sW_0)x^2+(W_3-rW_2-sW_1-tW_0)x^3+(W_4-rW_3-sW_2-tW_1-uW_0)x^4.$$

We next give Binet formula of generalized (r,s,t,u,v) numbers $\{W_n\}$ by the use of generating function for W_n.

THEOREM 3. *[27] (Binet's formula of generalized (r,s,t,u,v) numbers)*

$$\begin{aligned}
W_n &= \frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\
&\quad +\frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}+\frac{q_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} \\
&\quad +\frac{q_5\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}
\end{aligned} \tag{1.5}$$

where

$$\begin{aligned} q_1 &= W_0\alpha^4 + (W_1 - rW_0)\alpha^3 + (W_2 - rW_1 - sW_0)\alpha^2 \\ &\quad +(W_3 - rW_2 - sW_1 - tW_0)\alpha + (W_4 - rW_3 - sW_2 - tW_1 - vW_0), \\ q_2 &= W_0\beta^4 + (W_1 - rW_0)\beta^3 + (W_2 - rW_1 - sW_0)\beta^2 \\ &\quad +(W_3 - rW_2 - sW_1 - tW_0)\beta + (W_4 - rW_3 - sW_2 - tW_1 - vW_0), \\ q_3 &= W_0\gamma^4 + (W_1 - rW_0)\gamma^3 + (W_2 - rW_1 - sW_0)\gamma^2 \\ &\quad +(W_3 - rW_2 - sW_1 - tW_0)\gamma + (W_4 - rW_3 - sW_2 - tW_1 - vW_0), \\ q_4 &= W_0\delta^4 + (W_1 - rW_0)\delta^3 + (W_2 - rW_1 - sW_0)\delta^2 \\ &\quad +(W_3 - rW_2 - sW_1 - tW_0)\delta + (W_4 - rW_3 - s3W_2 - tW_1 - vW_0), \\ q_5 &= W_0\lambda^4 + (W_1 - rW_0)\lambda^3 + (W_2 - rW_1 - sW_0)\lambda^2 \\ &\quad +(W_3 - rW_2 - sW_1 - tW_0)\lambda + (W_4 - rW_3 - sW_2 - tW_1 - vW_0). \end{aligned}$$

In this paper we consider the case $r = s = t = u = 1, v = 2$ and in this case we write $V_n = W_n$. A generalized fifth order Jacobsthal sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2, V_3, V_4)\}_{n\geq 0}$ is defined by the fifth order recurrence relations

$$V_n = V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} + 2V_{n-5} \tag{1.6}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2, V_3 = c_3, V_4 = c_4$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{1}{2}V_{-(n-1)} - \frac{1}{2}V_{-(n-2)} - \frac{1}{2}V_{-(n-3)} - \frac{1}{2}V_{-(n-4)} + \frac{1}{2}V_{-(n-5)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.6) holds for all integer n.

As $\{V_n\}$ is a fifth order recurrence sequence (difference equation), it's characteristic equation is

$$x^5 - x^4 - x^3 - x^2 - x - 2 = (x-2)(x^4 + x^3 + x^2 + x + 1) = 0. \tag{1.7}$$

The roots $\alpha, \beta, \gamma, \delta$ and λ of Equation (1.7) are given by:

$$\alpha = 2,$$

$$\beta = \frac{1}{4}(\sqrt{5} - 1) + \frac{\sqrt{2\sqrt{5} + 10}}{4}i,$$

$$\gamma = \frac{1}{4}(\sqrt{5} - 1) - \frac{\sqrt{2\sqrt{5} + 10}}{4}i,$$

$$\delta = -\frac{1}{4}(\sqrt{5} + 1) + \frac{\sqrt{-2\sqrt{5} + 10}}{4}i,$$

$$\lambda = -\frac{1}{4}(\sqrt{5} + 1) - \frac{\sqrt{-2\sqrt{5} + 10}}{4}i.$$

Note that we have the following identities:

$$\begin{aligned}\alpha+\beta+\gamma+\delta+\lambda &= 1,\\ \alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\alpha\delta+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta &= -1,\\ \alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\gamma+\alpha\gamma\delta+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta &= 1,\\ \alpha\beta\lambda\gamma+\alpha\beta\lambda\delta+\alpha\beta\gamma\delta+\alpha\lambda\gamma\delta+\beta\lambda\gamma\delta &= -1\\ \alpha\beta\gamma\delta\lambda &= 2.\end{aligned}$$

The first few generalized fifth order Jacobsthal numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized fifth order Jacobsthal numbers

n	V_n	V_{-n}
0	V_0	...
1	V_1	$\frac{1}{2}V_4-\frac{1}{2}V_1-\frac{1}{2}V_2-\frac{1}{2}V_3-\frac{1}{2}V_0$
2	V_2	$\frac{3}{4}V_3-\frac{1}{4}V_1-\frac{1}{4}V_2-\frac{1}{4}V_0-\frac{1}{4}V_4$
3	V_3	$\frac{7}{8}V_2-\frac{1}{8}V_1-\frac{1}{8}V_0-\frac{1}{8}V_3-\frac{1}{8}V_4$
4	V_4	$\frac{15}{16}V_1-\frac{1}{16}V_0-\frac{1}{16}V_2-\frac{1}{16}V_3-\frac{1}{16}V_4$
5	$2V_0+V_1+V_2+V_3+V_4$	$\frac{31}{32}V_0-\frac{1}{32}V_1-\frac{1}{32}V_2-\frac{1}{32}V_3-\frac{1}{32}V_4$
6	$2V_0+3V_1+2V_2+2V_3+2V_4$	$\frac{31}{64}V_4-\frac{33}{64}V_1-\frac{33}{64}V_2-\frac{33}{64}V_3-\frac{33}{64}V_0$
7	$4V_0+4V_1+5V_2+4V_3+4V_4$	$\frac{95}{128}V_3-\frac{33}{128}V_1-\frac{33}{128}V_2-\frac{33}{128}V_0-\frac{33}{128}V_4$
8	$8V_0+8V_1+8V_2+9V_3+8V_4$	$\frac{223}{256}V_2-\frac{33}{256}V_1-\frac{33}{256}V_0-\frac{33}{256}V_3-\frac{33}{256}V_4$
9	$16V_0+16V_1+16V_2+16V_3+17V_4$	$\frac{479}{512}V_1-\frac{33}{512}V_0-\frac{33}{512}V_2-\frac{33}{512}V_3-\frac{33}{512}V_4$
10	$34V_0+33V_1+33V_2+33V_3+33V_4$	$\frac{991}{1024}V_0-\frac{33}{1024}V_1-\frac{33}{1024}V_2-\frac{33}{1024}V_3-\frac{33}{1024}V_4$
11	$66V_0+67V_1+66V_2+66V_3+66V_4$	$\frac{991}{2048}V_4-\frac{1057}{2048}V_1-\frac{1057}{2048}V_2-\frac{1057}{2048}V_3-\frac{1057}{2048}V_0$
12	$132V_0+132V_1+133V_2+132V_3+132V_4$	$\frac{3039}{4096}V_3-\frac{1057}{4096}V_1-\frac{1057}{4096}V_2-\frac{1057}{4096}V_0-\frac{1057}{4096}V_4$
13	$264V_0+264V_1+264V_2+265V_3+264V_4$	$\frac{7135}{8192}V_2-\frac{1057}{8192}V_1-\frac{1057}{8192}V_0-\frac{1057}{8192}V_3-\frac{1057}{8192}V_4$

Now we define six special case of the sequence $\{V_n\}$. Fifth-order Jacobsthal sequence $\{J_n^{(5)}\}_{n\geq 0}$, fifth order Jacobsthal-Lucas sequence $\{j_n^{(5)}\}_{n\geq 0}$, modified fifth order Jacobsthal sequence $\{K_n^{(5)}\}_{n\geq 0}$, fifth order Jacobsthal Perrin sequence $\{Q_n^{(5)}\}_{n\geq 0}$, adjusted fifth order Jacobsthal sequence $\{S_n^{(5)}\}_{n\geq 0}$ and modified fifth order Jacobsthal-Lucas sequence $\{R_n^{(5)}\}_{n\geq 0}$ are defined, respectively, by the fifth order recurrence relations

$$\begin{aligned}J_{n+5}^{(5)} &= J_{n+4}^{(5)}+J_{n+3}^{(5)}+J_{n+2}^{(5)}+J_{n+1}^{(5)}+2J_n^{(5)}, \qquad (1.8)\\ J_0^{(5)} &= 0, J_1^{(5)}=1, J_2^{(5)}=1, J_3^{(5)}=1, J_4^{(5)}=1,\end{aligned}$$

$$\begin{aligned}j_{n+5}^{(5)} &= j_{n+4}^{(5)}+j_{n+3}^{(5)}+j_{n+2}^{(5)}+j_{n+1}^{(5)}+2j_n^{(5)}, \qquad (1.9)\\ j_0^{(5)} &= 2, j_1^{(5)}=1, j_2^{(5)}=5, j_3^{(5)}=10, j_4^{(5)}=20,\end{aligned}$$

$$\begin{aligned} K_{n+5}^{(5)} &= K_{n+4}^{(5)} + K_{n+3}^{(5)} + K_{n+2}^{(5)} + K_{n+1}^{(5)} + 2K_n^{(5)}, \\ K_0^{(5)} &= 3, K_1^{(5)} = 1, K_2^{(5)} = 3, K_3^{(5)} = 10, K_4^{(5)} = 20, \end{aligned} \tag{1.10}$$

$$\begin{aligned} Q_{n+5}^{(5)} &= Q_{n+4}^{(5)} + Q_{n+3}^{(5)} + Q_{n+2}^{(5)} + Q_{n+1}^{(5)} + 2Q_n^{(5)}, \\ Q_0^{(5)} &= 3, Q_1^{(5)} = 0, Q_2^{(5)} = 2, Q_3^{(5)} = 8, Q_4^{(5)} = 16, \end{aligned} \tag{1.11}$$

$$\begin{aligned} S_{n+5}^{(5)} &= S_{n+4}^{(5)} + S_{n+3}^{(5)} + S_{n+2}^{(5)} + S_{n+1}^{(5)} + 2S_n^{(5)}, \\ S_0^{(5)} &= 0, S_1^{(5)} = 1, S_2^{(5)} = 1, S_3^{(5)} = 2, S_4^{(5)} = 4, \end{aligned} \tag{1.12}$$

$$\begin{aligned} R_{n+5}^{(5)} &= R_{n+4}^{(5)} + R_{n+3}^{(5)} + R_{n+2}^{(5)} + R_{n+1}^{(5)} + 2R_n^{(5)}, \\ R_0^{(5)} &= 5, R_1^{(5)} = 1, R_2^{(5)} = 3, R_3^{(5)} = 7, R_4^{(5)} = 19. \end{aligned} \tag{1.13}$$

The sequences $\{J_n^{(5)}\}_{n\geq 0}$, $\{j_n^{(5)}\}_{n\geq 0}$, $\{K_n^{(5)}\}_{n\geq 0}$, $Q_n^{(5)}\}_{n\geq 0}$, $\{S_n^{(5)}\}_{n\geq 0}$ and $\{R_n^{(5)}\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$J_{-n}^{(5)} = -\frac{1}{2}J_{-(n-1)}^{(5)} - \frac{1}{2}J_{-(n-2)}^{(5)} - \frac{1}{2}J_{-(n-3)}^{(5)} - J_{-(n-4)}^{(5)} + \frac{1}{2}J_{-(n-5)}^{(5)}, \tag{1.14}$$

$$j_{-n}^{(5)} = -\frac{1}{2}j_{-(n-1)}^{(5)} - \frac{1}{2}j_{-(n-2)}^{(5)} - \frac{1}{2}j_{-(n-3)}^{(5)} - \frac{1}{2}j_{-(n-4)}^{(5)} + \frac{1}{2}j_{-(n-5)}^{(5)}, \tag{1.15}$$

$$K_{-n}^{(5)} = -\frac{1}{2}K_{-(n-1)}^{(5)} - \frac{1}{2}K_{-(n-2)}^{(5)} - \frac{1}{2}K_{-(n-3)}^{(5)} - \frac{1}{2}K_{-(n-4)}^{(5)} + \frac{1}{2}K_{-(n-5)}^{(5)}, \tag{1.16}$$

$$Q_{-n}^{(5)} = -\frac{1}{2}Q_{-(n-1)}^{(5)} - \frac{1}{2}Q_{-(n-2)}^{(5)} - \frac{1}{2}Q_{-(n-3)}^{(5)} - \frac{1}{2}Q_{-(n-4)}^{(5)} + \frac{1}{2}Q_{-(n-5)}^{(5)}, \tag{1.17}$$

$$S_{-n}^{(5)} = -\frac{1}{2}S_{-(n-1)}^{(5)} - \frac{1}{2}S_{-(n-2)}^{(5)} - \frac{1}{2}S_{-(n-3)}^{(5)} - \frac{1}{2}S_{-(n-4)}^{(5)} + \frac{1}{2}S_{-(n-5)}^{(5)}, \tag{1.18}$$

$$R_{-n}^{(5)} = -\frac{1}{2}R_{-(n-1)}^{(5)} - \frac{1}{2}R_{-(n-2)}^{(5)} - \frac{1}{2}R_{-(n-3)}^{(5)} - \frac{1}{2}R_{-(n-4)}^{(5)} + \frac{1}{2}R_{-(n-5)}^{(5)}, \tag{1.19}$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences (1.8)-(1.13) hold for all integer n.

In the rest of the paper, for easy writing, we drop the superscripts and write J_n, j_n, K_n, Q_n, S_n and R_n for $J_n^{(5)}, j_n^{(5)}$, $K_n^{(5)}, Q_n^{(5)}, S_n^{(5)}$ and $R_n^{(5)}$, respectively. Note that J_n and j_n are the sequences A226310, A226311 in [25], respectively.

Next, we present the first few values of the fifth order Jacobsthal, fifth order Jacobsthal-Lucas, modified fifth order Jacobsthal , fifth order Jacobsthal Perrin, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas numbers with positive and negative subscripts:

Table 2. The first few values of the special fifth order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$J_n^{(5)}$	0	1	1	1	1	4	9	17	33	65	132	265	529	1057
$J_{-n}^{(5)}$		-1	0	$\frac{1}{2}$	$\frac{3}{4}$	$-\frac{1}{8}$	$-\frac{17}{16}$	$-\frac{1}{32}$	$\frac{31}{64}$	$\frac{95}{128}$	$-\frac{33}{256}$	$-\frac{545}{512}$	$-\frac{33}{1024}$	$\frac{991}{2048}$
$j_n^{(5)}$	2	1	5	10	20	40	77	157	314	628	1256	2509	5021	10042
$j_{-n}^{(5)}$		1	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{11}{8}$	$\frac{13}{16}$	$\frac{13}{32}$	$\frac{13}{64}$	$\frac{13}{128}$	$-\frac{371}{256}$	$\frac{397}{512}$	$\frac{397}{1024}$	$\frac{397}{2048}$	$\frac{397}{4096}$
$K_n^{(5)}$	3	1	3	10	20	40	75	151	306	612	1224	2443	4887	9778
$K_{-n}^{(5)}$		$\frac{3}{2}$	$\frac{3}{4}$	$-\frac{13}{8}$	$-\frac{21}{16}$	$\frac{59}{32}$	$\frac{59}{64}$	$\frac{59}{128}$	$-\frac{453}{256}$	$-\frac{709}{512}$	$\frac{1851}{1024}$	$\frac{1851}{2048}$	$\frac{1851}{4096}$	$-\frac{14533}{8192}$
$Q_n^{(5)}$	3	0	2	8	16	32	58	118	240	480	960	1914	3830	7664
$Q_{-n}^{(5)}$		$\frac{3}{2}$	$\frac{3}{4}$	$-\frac{13}{8}$	$-\frac{29}{16}$	$\frac{67}{32}$	$\frac{67}{64}$	$\frac{67}{128}$	$-\frac{445}{256}$	$-\frac{957}{512}$	$\frac{2115}{1024}$	$\frac{2115}{2048}$	$\frac{2115}{4096}$	$-\frac{14269}{8192}$
$S_n^{(5)}$	0	1	1	2	4	8	17	33	66	132	264	529	1057	2114
$S_{-n}^{(5)}$		0	0	0	$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{1}{16}$	$-\frac{1}{32}$	$\frac{31}{64}$	$-\frac{33}{128}$	$-\frac{33}{256}$	$-\frac{33}{512}$	$-\frac{33}{1024}$
$R_n^{(5)}$	5	1	3	7	15	36	63	127	255	511	1028	2047	4095	8191
$R_{-n}^{(5)}$		$-\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{7}{8}$	$-\frac{15}{16}$	$\frac{129}{32}$	$-\frac{63}{64}$	$-\frac{127}{128}$	$-\frac{255}{256}$	$-\frac{511}{512}$	$\frac{4097}{1024}$	$-\frac{2047}{2048}$	$-\frac{4095}{4096}$	$-\frac{8191}{8192}$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

LEMMA 4. *Suppose that $f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized fifth order Jacobsthal sequence $\{V_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} V_n x^n$ is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0+(V_1-V_0)x+(V_2-V_1-V_0)x^2+(V_3-V_2-V_1-V_0)x^3+(V_4-V_3-V_2-V_1-V_0)x^4}{1-x-x^2-x^3-x^4-2x^5}$$

Proof. Take $r = s = t = u = 1, v = 2$ in Lemma 2. □

The previous Lemma gives the following results as particular examples.

COROLLARY 5. *Generated functions of fifth order Jacobsthal, fifth order Jacobsthal-Lucas, modified fifth order Jacobsthal , fifth order Jacobsthal Perrin, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas numbers are*

$$\sum_{n=0}^{\infty} J_n x^n = \frac{x-x^3-2x^4}{1-x-x^2-x^3-x^4-2x^5},$$

$$\sum_{n=0}^{\infty} j_n x^n = \frac{2-x+2x^2+2x^3+2x^4}{1-x-x^2-x^3-x^4-2x^5},$$

$$\sum_{n=0}^{\infty} K_n x^n = \frac{3-2x-x^2+3x^3+3x^4}{1-x-x^2-x^3-x^4-2x^5},$$

$$\sum_{n=0}^{\infty} Q_n x^n = \frac{3-3x-x^2+3x^3+3x^4}{1-x-x^2-x^3-x^4-2x^5},$$

$$\sum_{n=0}^{\infty} S_n x^n = \frac{x}{1-x-x^2-x^3-x^4-2x^5},$$

$$\sum_{n=0}^{\infty} R_n x^n = \frac{5-4x-3x^2-2x^3-x^4}{1-x-x^2-x^3-x^4-2x^5},$$

respectively.

We next give Binet formula of generalized fifth order Jacobsthal numbers $\{V_n\}$ by the use of Theorems 1 and 3.

THEOREM 6. *(Binet formula of generalized fifth order Jacobsthal numbers)*

$$\begin{aligned} V_n &= \frac{c_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{c_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad + \frac{c_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} \\ &\quad + \frac{c_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{c_5\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)} \end{aligned} \tag{1.20}$$

where

$$\begin{aligned} c_1 &= V_4-(\beta+\gamma+\delta+\lambda)V_3+(\beta\lambda+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta)V_2 \\ &\quad -(\beta\lambda\gamma+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta)V_1+(\beta\lambda\gamma\delta)V_0 \\ &= V_0\alpha^4+(V_1-V_0)\alpha^3+(V_2-V_1-V_0)\alpha^2 \\ &\quad +(V_3-V_2-V_1-V_0)\alpha+(V_4-V_3-V_2-V_1-V_0), \end{aligned}$$

$$\begin{aligned} c_2 &= V_4-(\alpha+\gamma+\delta+\lambda)V_3+(\alpha\lambda+\alpha\gamma+\alpha\delta+\lambda\gamma+\lambda\delta+\gamma\delta)V_2 \\ &\quad -(\alpha\lambda\gamma+\alpha\lambda\delta+\alpha\gamma\delta+\lambda\gamma\delta)V_1+(\alpha\lambda\gamma\delta)V_0 \\ &= V_0\beta^4+(V_1-V_0)\beta^3+(V_2-V_1-V_0)\beta^2 \\ &\quad +(V_3-V_2-V_1-V_0)\beta+(V_4-V_3-V_2-V_1-V_0), \end{aligned}$$

$$\begin{aligned} c_3 &= V_4-(\alpha+\beta+\delta+\lambda)V_3+(\alpha\beta+\alpha\lambda+\beta\lambda+\alpha\delta+\beta\delta+\lambda\delta)V_2 \\ &\quad -(\alpha\beta\lambda+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\delta)V_1+(\alpha\beta\lambda\delta)V_0 \\ &= V_0\gamma^4+(V_1-V_0)\gamma^3+(V_2-V_1-V_0)\gamma^2 \\ &\quad +(V_3-V_2-V_1-V_0)\gamma+(V_4-V_3-V_2-V_1-V_0), \end{aligned}$$

$$\begin{aligned} c_4 &= V_4-(\alpha+\beta+\gamma+\lambda)V_3+(\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\beta\gamma+\lambda\gamma)V_2 \\ &\quad -(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\beta\lambda\gamma)V_1+(\alpha\beta\lambda\gamma)V_0 \\ &= V_0\delta^4+(V_1-V_0)\delta^3+(V_2-V_1-V_0)\delta^2 \\ &\quad +(V_3-V_2-V_1-V_0)\delta+(V_4-V_3-V_2-V_1-V_0), \end{aligned}$$

$$\begin{aligned} c_5 &= V_4-(\alpha+\beta+\gamma+\delta)V_3+(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)V_2 \\ &\quad -(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)V_1+(\alpha\beta\gamma\delta)V_0 \\ &= V_0\lambda^4+(V_1-V_0)\lambda^3+(V_2-V_1-V_0)\lambda^2 \\ &\quad +(V_3-V_2-V_1-V_0)\lambda+(V_4-V_3-V_2-V_1-V_0). \end{aligned}$$

Next, using Theorem 6, we present the Binet formulas of fifth order Jacobsthal, fifth order Jacobsthal-Lucas, modified fifth order Jacobsthal , fifth order Jacobsthal Perrin, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas sequences.

COROLLARY 7. *Binet formulas of fifth order Jacobsthal, fifth order Jacobsthal-Lucas, modified fifth order Jacobsthal , fifth order Jacobsthal Perrin, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas numbers are*

$$\begin{aligned} J_n &= \frac{(\alpha^3-\alpha-2)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{(\beta^3-\beta-2)\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad +\frac{(\gamma^3-\gamma-2)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}+\frac{(\delta^3-\delta-2)\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} \\ &\quad +\frac{(\lambda^3-\lambda-2)\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}, \end{aligned}$$

$$\begin{aligned} j_n &= \frac{(\alpha^4+4\alpha^3+4\alpha^2+4\alpha+4)\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{(\beta^4+4\beta^3+4\beta^2+4\beta+4)\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad +\frac{(\gamma^4+4\gamma^3+4\gamma^2+4\gamma+4)\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}+\frac{(\delta^4+4\delta^3+4\delta^2+4\delta+4)\delta^{n-1}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} \\ &\quad +\frac{(\lambda^4+4\lambda^3+4\lambda^2+4\lambda+4)\lambda^{n-1}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}, \end{aligned}$$

$$\begin{aligned} K_n &= \frac{(\alpha^4+2\alpha^3+6\alpha^2+6\alpha+6)\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{(\beta^4+2\beta^3+6\beta^2+6\beta+6)\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad +\frac{(\gamma^4+2\gamma^3+6\gamma^2+6\gamma+6)\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}+\frac{(\delta^4+2\delta^3+6\delta^2+6\delta+6)\delta^{n-1}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} \\ &\quad +\frac{(\lambda^4+2\lambda^3+6\lambda^2+6\lambda+6)\lambda^{n-1}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}, \end{aligned}$$

$$\begin{aligned} Q_n &= \frac{2(\alpha^3+3\alpha^2+3\alpha+3)\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{2(\beta^3+3\beta^2+3\beta+3)\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad +\frac{2(\gamma^3+3\gamma^2+3\gamma+3)\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}+\frac{2(\delta^3+3\delta^2+3\delta+3)\delta^{n-1}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} \\ &\quad +\frac{2(\lambda^3+3\lambda^2+3\lambda+3)\lambda^{n-1}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}, \end{aligned}$$

$$
\begin{aligned}
S_n &= \frac{\alpha^{n+3}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{\beta^{n+3}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\
&\quad + \frac{\gamma^{n+3}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} + \frac{\delta^{n+3}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} \\
&\quad + \frac{\lambda^{n+3}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)},
\end{aligned}
$$

$$R_n = \alpha^n + \beta^n + \gamma^n + \delta^n + \lambda^n$$

respectively.

Binet formulas of fifth order Jacobsthal, fifth order Jacobsthal-Lucas, modified fifth order Jacobsthal , fifth order Jacobsthal Perrin, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas numbers can be given in the following forms:

$$
\begin{aligned}
J_n &= \frac{4}{31}\alpha^n - \frac{1}{155}((6\sqrt{5}+5) + 2\sqrt{2}\sqrt{\sqrt{5}+5}(6+\sqrt{5})i)\beta^n \\
&\quad + \frac{1}{155}(-(6\sqrt{5}+5) + 2\sqrt{2}\sqrt{\sqrt{5}+5}(6+\sqrt{5})i)\gamma^n \\
&\quad + \frac{1}{155}((6\sqrt{5}-5) + 2\sqrt{2}\sqrt{5-\sqrt{5}}(\sqrt{5}-6)i)\delta^n \\
&\quad + \frac{1}{155}((6\sqrt{5}-5) + 2\sqrt{2}\sqrt{5-\sqrt{5}}(-\sqrt{5}+6)i)\lambda^n,
\end{aligned}
$$

$$
\begin{aligned}
j_n &= \frac{38}{31}\alpha^n + \frac{1}{1240}(12(20-7\sqrt{5}) + \sqrt{\sqrt{5}+5}(111\sqrt{2}+3\sqrt{10})i)\beta^n \\
&\quad + \frac{1}{1240}(12(20-7\sqrt{5}) - \sqrt{\sqrt{5}+5}(111\sqrt{2}+3\sqrt{10})i)\gamma^n \\
&\quad + \frac{1}{1240}(12(20+7\sqrt{5}) + \sqrt{5-\sqrt{5}}(111\sqrt{2}-3\sqrt{10})i)\delta^n \\
&\quad + \frac{1}{1240}(12(20+7\sqrt{5}) + \sqrt{5-\sqrt{5}}(-111\sqrt{2}+3\sqrt{10})i)\lambda^n,
\end{aligned}
$$

$$
\begin{aligned}
K_n &= \frac{37}{31}\alpha^n + \frac{1}{1240}(4(13\sqrt{5}+140) + \sqrt{2}\sqrt{\sqrt{5}+5}(73i+69i\sqrt{5}))\beta^n \\
&\quad + \frac{1}{1240}(4(13\sqrt{5}+140) - \sqrt{2}\sqrt{\sqrt{5}+5}(73i+69i\sqrt{5}))\gamma^n \\
&\quad + \frac{1}{1240}(4(-13\sqrt{5}+140) + \sqrt{2}\sqrt{5-\sqrt{5}}(73i-69i\sqrt{5}))\delta^n \\
&\quad + \frac{1}{1240}(4(-13\sqrt{5}+140) + \sqrt{2}\sqrt{5-\sqrt{5}}(-73i+69i\sqrt{5}))\lambda^n,
\end{aligned}
$$

$$
\begin{aligned}
Q_n &= \frac{29}{31}\alpha^n + \frac{1}{1240}(8(80+3\sqrt{5}) + 10\sqrt{2}\sqrt{\sqrt{5}+5}(11+7\sqrt{5})i)\beta^n \\
&\quad + \frac{1}{1240}(8(80+3\sqrt{5}) - 10\sqrt{2}\sqrt{\sqrt{5}+5}(11+7\sqrt{5})i)\gamma^n \\
&\quad + \frac{1}{1240}(8(80-3\sqrt{5}) + 10\sqrt{2}\sqrt{5-\sqrt{5}}(11-7\sqrt{5})i)\delta^n \\
&\quad + \frac{1}{1240}(8(80-3\sqrt{5}) + 10\sqrt{2}\sqrt{5-\sqrt{5}}(-11+7\sqrt{5})i)\lambda^n,
\end{aligned}
$$

$$
\begin{aligned}
S_n &= \frac{8}{31}\alpha^n + \frac{1}{1240}(4(-20+7\sqrt{5}) - \sqrt{2}\sqrt{\sqrt{5}+5}(37i+i\sqrt{5}))\beta^n \\
&+\frac{1}{1240}(4(-20+7\sqrt{5}) + \sqrt{2}\sqrt{\sqrt{5}+5}(37i+i\sqrt{5}))\gamma^n \\
&+\frac{1}{1240}(-4(20+7\sqrt{5}) + \sqrt{2}\sqrt{\sqrt{5}+5}(21-19\sqrt{5})i)\delta^n \\
&+\frac{1}{1240}(-4(20+7\sqrt{5}) + \sqrt{2}\sqrt{\sqrt{5}+5}(-21+19\sqrt{5})i)\lambda^n,
\end{aligned}
$$

$$R_n = \alpha^n + \beta^n + \gamma^n + \delta^n + \lambda^n$$

2. Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following Theorem gives generalization of this result to the generalized Pentanacci sequence $\{W_n\}$.

THEOREM 8 (Simson Formula of Generalized Pentanacci Numbers). *For all integers n we have*

$$\begin{vmatrix} W_{n+4} & W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+3} & W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} & W_{n-3} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} & W_{n-4} \end{vmatrix} = v^n \begin{vmatrix} W_4 & W_3 & W_2 & W_1 & W_0 \\ W_3 & W_2 & W_1 & W_0 & W_{-1} \\ W_2 & W_1 & W_0 & W_{-1} & W_{-2} \\ W_1 & W_0 & W_{-1} & W_{-2} & W_{-3} \\ W_0 & W_{-1} & W_{-2} & W_{-3} & W_{-4} \end{vmatrix}. \tag{2.1}$$

Proof. (2.1) is given in Soykan [26, Theorem 5]. □

A special case of the above theorem is the following Theorem which gives Simson formula of the generalized fifth order Jacobsthal sequence $\{V_n\}$.

THEOREM 9 (Simson Formula of Generalized Fifth-Order Jacobsthal Numbers). *For all integers n we have*

$$\begin{vmatrix} V_{n+4} & V_{n+3} & V_{n+2} & V_{n+1} & V_n \\ V_{n+3} & V_{n+2} & V_{n+1} & V_n & V_{n-1} \\ V_{n+2} & V_{n+1} & V_n & V_{n-1} & V_{n-2} \\ V_{n+1} & V_n & V_{n-1} & V_{n-2} & V_{n-3} \\ V_n & V_{n-1} & V_{n-2} & V_{n-3} & V_{n-4} \end{vmatrix} = 2^n \begin{vmatrix} V_4 & V_3 & V_2 & V_1 & V_0 \\ V_3 & V_2 & V_1 & V_0 & V_{-1} \\ V_2 & V_1 & V_0 & V_{-1} & V_{-2} \\ V_1 & V_0 & V_{-1} & V_{-2} & V_{-3} \\ V_0 & V_{-1} & V_{-2} & V_{-3} & V_{-4} \end{vmatrix}. \tag{2.2}$$

The previous Theorem gives the following results as particular examples.

COROLLARY 10. *Simson's formulas of fifth order Jacobsthal, fifth order Jacobsthal-Lucas, modified fifth order Jacobsthal , fifth order Jacobsthal Perrin, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas are given as*

$$\begin{vmatrix} J_{n+4} & J_{n+3} & J_{n+2} & J_{n+1} & J_n \\ J_{n+3} & J_{n+2} & J_{n+1} & J_n & J_{n-1} \\ J_{n+2} & J_{n+1} & J_n & J_{n-1} & J_{n-2} \\ J_{n+1} & J_n & J_{n-1} & J_{n-2} & J_{n-3} \\ J_n & J_{n-1} & J_{n-2} & J_{n-3} & J_{n-4} \end{vmatrix} = 11 \times 2^{n-2}$$

$$\begin{vmatrix} j_{n+4} & j_{n+3} & j_{n+2} & j_{n+1} & j_n \\ j_{n+3} & j_{n+2} & j_{n+1} & j_n & j_{n-1} \\ j_{n+2} & j_{n+1} & j_n & j_{n-1} & j_{n-2} \\ j_{n+1} & j_n & j_{n-1} & j_{n-2} & j_{n-3} \\ j_n & j_{n-1} & j_{n-2} & j_{n-3} & j_{n-4} \end{vmatrix} = 1539 \times 2^{n-3}$$

$$\begin{vmatrix} K_{n+4} & K_{n+3} & K_{n+2} & K_{n+1} & K_n \\ K_{n+3} & K_{n+2} & K_{n+1} & K_n & K_{n-1} \\ K_{n+2} & K_{n+1} & K_n & K_{n-1} & K_{n-2} \\ K_{n+1} & K_n & K_{n-1} & K_{n-2} & K_{n-3} \\ K_n & K_{n-1} & K_{n-2} & K_{n-3} & K_{n-4} \end{vmatrix} = 17057 \times 2^{n-4}$$

$$\begin{vmatrix} Q_{n+4} & Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n \\ Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n & Q_{n-1} \\ Q_{n+2} & Q_{n+1} & Q_n & Q_{n-1} & Q_{n-2} \\ Q_{n+1} & Q_n & Q_{n-1} & Q_{n-2} & Q_{n-3} \\ Q_n & Q_{n-1} & Q_{n-2} & Q_{n-3} & Q_{n-4} \end{vmatrix} = 1595 \times 2^{n}$$

$$\begin{vmatrix} S_{n+4} & S_{n+3} & S_{n+2} & S_{n+1} & S_n \\ S_{n+3} & S_{n+2} & S_{n+1} & S_n & S_{n-1} \\ S_{n+2} & S_{n+1} & S_n & S_{n-1} & S_{n-2} \\ S_{n+1} & S_n & S_{n-1} & S_{n-2} & S_{n-3} \\ S_n & S_{n-1} & S_{n-2} & S_{n-3} & S_{n-4} \end{vmatrix} = 2^{n-1}$$

$$\begin{vmatrix} R_{n+4} & R_{n+3} & R_{n+2} & R_{n+1} & R_n \\ R_{n+3} & R_{n+2} & R_{n+1} & R_n & R_{n-1} \\ R_{n+2} & R_{n+1} & R_n & R_{n-1} & R_{n-2} \\ R_{n+1} & R_n & R_{n-1} & R_{n-2} & R_{n-3} \\ R_n & R_{n-1} & R_{n-2} & R_{n-3} & R_{n-4} \end{vmatrix} = 120125 \times 2^{n-4}$$

respectively.

3. Some Identities

In this section, we obtain some identities of fifth order Jacobsthal, fifth order Jacobsthal-Lucas, modified fifth order Jacobsthal, fifth order Jacobsthal Perrin, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas numbers. First, we can give a few basic relations between $\{J_n\}$ and $\{j_n\}$.

LEMMA 11. *The following equalities are true:*

$$\begin{aligned}
57J_n &= -9j_{n+4}+10j_{n+3}+29j_{n+2}-9j_{n+1}-28j_n, \qquad (3.1)\\
57J_n &= j_{n+3}+20j_{n+2}-18j_{n+1}-37j_n-18j_{n-1},\\
57J_n &= 21j_{n+2}-17j_{n+1}-36j_n-17j_{n-1}+2j_{n-2},\\
57J_n &= 4j_{n+1}-15j_n+4j_{n-1}+23j_{n-2}+42j_{n-3},\\
57J_n &= -11j_n+8j_{n-1}+27j_{n-2}+46j_{n-3}+8j_{n-4},
\end{aligned}$$

and

$$\begin{aligned}
22j_n &= -7J_{n+4}+41J_{n+3}-13J_{n+2}+23J_{n+1}-J_n,\\
11j_n &= 17J_{n+3}-10J_{n+2}+8J_{n+1}-4J_n-7J_{n-1},\\
11j_n &= 7J_{n+2}+25J_{n+1}+13J_n+10J_{n-1}+34J_{n-2},\\
11j_n &= 32J_{n+1}+20J_n+17J_{n-1}+41J_{n-2}+14J_{n-3},\\
11j_n &= 52J_n+49J_{n-1}+73J_{n-2}+46J_{n-3}+11J_{n-4}.
\end{aligned}$$

Proof. Note that all the identities hold for all integers n. We prove (3.1). To show (3.1), writing

$$J_n = a\times j_{n+4}+b\times j_{n+3}+c\times j_{n+2}+d\times j_{n+1}+e\times j_n$$

and solving the system of equations

$$\begin{aligned}
J_0 &= a\times j_4+b\times j_3+c\times j_2+d\times j_1+e\times j_0\\
J_1 &= a\times j_5+b\times j_4+c\times j_3+d\times j_2+e\times j_1\\
J_2 &= a\times j_6+b\times j_5+c\times j_4+d\times j_3+e\times j_2\\
J_3 &= a\times j_7+b\times j_6+c\times j_5+d\times j_4+e\times j_3\\
J_4 &= a\times j_8+b\times j_7+c\times j_6+d\times j_5+e\times j_4
\end{aligned}$$

we find that $a=-\frac{3}{19}, b=\frac{10}{57}, c=\frac{29}{57}, d=-\frac{3}{19}, e=-\frac{28}{57}$. The other equalities can be proved similarly. □

Note that all the identities in the above Lemma can be proved by induction as well.

Secondly, we present a few basic relations between $\{J_n\}$ and $\{S_n\}$.

LEMMA 12. *The following equalities are true:*

$$\begin{aligned} 2J_n &= S_{n+4} - S_{n+3} - 3S_{n+2} + S_{n+1} + 3S_n, \\ J_n &= -S_{n+2} + S_{n+1} + 2S_n + S_{n-1}, \\ J_n &= S_n - S_{n-2} - 2S_{n-3}, \end{aligned}$$

and

$$\begin{aligned} 11S_n &= 3J_{n+4} - 5J_{n+3} + 4J_{n+2} - 2J_{n+1} + 2J_n, \\ 11S_n &= -2J_{n+3} + 7J_{n+2} + J_{n+1} + 5J_n + 6J_{n-1}, \\ 11S_n &= 5J_{n+2} - J_{n+1} + 3J_n + 4J_{n-1} - 4J_{n-2}, \\ 11S_n &= 4J_{n+1} + 8J_n + 9J_{n-1} + J_{n-2} + 10J_{n-3}, \\ 11S_n &= 12J_n + 13J_{n-1} + 5J_{n-2} + 14J_{n-3} + 8J_{n-4}. \end{aligned}$$

Thirdly, we give a few basic relations between $\{J_n\}$ and $\{R_n\}$.

LEMMA 13. *The following equalities are true:*

$$\begin{aligned} 4805J_n &= 319R_{n+4} + 71R_{n+3} - 425R_{n+2} - 1417R_{n+1} - 518R_n, \\ 4805J_n &= 390R_{n+3} - 106R_{n+2} - 1098R_{n+1} - 199R_n + 638R_{n-1}, \\ 4805J_n &= 284R_{n+2} - 708R_{n+1} + 191R_n + 1028R_{n-1} + 780R_{n-2}, \\ 4805J_n &= -424R_{n+1} + 475R_n + 1312R_{n-1} + 1064R_{n-2} + 568R_{n-3}, \\ 4805J_n &= 51R_n + 888R_{n-1} + 640R_{n-2} + 144R_{n-3} - 848R_{n-4}, \end{aligned}$$

and

$$\begin{aligned} 44R_n &= -97J_{n+4} + 191J_{n+3} - J_{n+2} + 127J_{n+1} + 115J_n, \\ 22R_n &= 47J_{n+3} - 49J_{n+2} + 15J_{n+1} + 9J_n - 97J_{n-1}, \\ 11R_n &= -J_{n+2} + 31J_{n+1} + 28J_n - 25J_{n-1} + 47J_{n-2}, \\ 11R_n &= 30J_{n+1} + 27J_n - 26J_{n-1} + 46J_{n-2} - 2J_{n-3}, \\ 11R_n &= 57J_n + 4J_{n-1} + 76J_{n-2} + 28J_{n-3} + 60J_{n-4}. \end{aligned}$$

Next, we present a few basic relations between $\{j_n\}$ and $\{S_n\}$.

LEMMA 14. *The following equalities are true:*

$$
\begin{aligned}
4j_n &= S_{n+4} + S_{n+3} + S_{n+2} + S_{n+1} - 11S_n, \\
2j_n &= S_{n+3} + S_{n+2} + S_{n+1} - 5S_n + S_{n-1}, \\
j_n &= S_{n+2} + S_{n+1} - 2S_n + S_{n-1} + S_{n-2}, \\
j_n &= 2S_{n+1} - S_n + 2S_{n-1} + 2S_{n-2} + 2S_{n-3}, \\
j_n &= S_n + 4S_{n-1} + 4S_{n-2} + 4S_{n-3} + 4S_{n-4},
\end{aligned}
$$

and

$$
\begin{aligned}
57S_n &= j_{n+4} + j_{n+3} + j_{n+2} + j_{n+1} - 18j_n, \\
57S_n &= 2j_{n+3} + 2j_{n+2} + 2j_{n+1} - 17j_n + 2j_{n-1}, \\
57S_n &= 4j_{n+2} + 4j_{n+1} - 15j_n + 4j_{n-1} + 4j_{n-2}, \\
57S_n &= 8j_{n+1} - 11j_n + 8j_{n-1} + 8j_{n-2} + 8j_{n-3}, \\
57S_n &= -3j_n + 16j_{n-1} + 16j_{n-2} + 16j_{n-3} + 16j_{n-4}.
\end{aligned}
$$

Now, we give a few basic relations between $\{j_n\}$ and $\{R_n\}$.

LEMMA 15. *The following equalities are true:*

$$
\begin{aligned}
4805j_n &= -581R_{n+4} + 907R_{n+3} + 1000R_{n+2} + 1186R_{n+1} + 1558R_n, \\
4805j_n &= 326R_{n+3} + 419R_{n+2} + 605R_{n+1} + 977R_n - 1162R_{n-1}, \\
4805j_n &= 745R_{n+2} + 931R_{n+1} + 1303R_n - 836R_{n-1} + 652R_{n-2}, \\
4805j_n &= 1676R_{n+1} + 2048R_n - 91R_{n-1} + 1397R_{n-2} + 1490R_{n-3}, \\
4805j_n &= 3724R_n + 1585R_{n-1} + 3073R_{n-2} + 3166R_{n-3} + 3352R_{n-4},
\end{aligned}
$$

and

$$
\begin{aligned}
114R_n &= 41j_{n+4} + 3j_{n+3} - 35j_{n+2} - 263j_{n+1} + 79j_n, \\
57R_n &= 22j_{n+3} + 3j_{n+2} - 111j_{n+1} + 60j_n + 41j_{n-1}, \\
57R_n &= 25j_{n+2} - 89j_{n+1} + 82j_n + 63j_{n-1} + 44j_{n-2}, \\
57R_n &= -64j_{n+1} + 107j_n + 88j_{n-1} + 69j_{n-2} + 50j_{n-3}, \\
57R_n &= 43j_n + 24j_{n-1} + 5j_{n-2} - 14j_{n-3} - 128j_{n-4}.
\end{aligned}
$$

Next, we present a few basic relations between $\{S_n\}$ and $\{R_n\}$.

LEMMA 16. *The following equalities are true:*

$$
\begin{aligned}
4805S_n &= 297R_{n+4} - 199R_{n+3} - 230R_{n+2} - 292R_{n+1} - 416R_n, \\
4805S_n &= 98R_{n+3} + 67R_{n+2} + 5R_{n+1} - 119R_n + 594R_{n-1}, \\
4805S_n &= 165R_{n+2} + 103R_{n+1} - 21R_n + 692R_{n-1} + 196R_{n-2}, \\
4805S_n &= 268R_{n+1} + 144R_n + 857R_{n-1} + 361 \times R_{n-2} + 330R_{n-3}, \\
4805S_n &= 412R_n + 1125R_{n-1} + 629R_{n-2} + 598 \times R_{n-3} + 536 \times R_{n-4},
\end{aligned}
$$

and

$$
\begin{aligned}
8R_n &= -7S_{n+4} + S_{n+3} + 9S_{n+2} + 57S_{n+1} - 15S_n, \\
4R_n &= -3S_{n+3} + S_{n+2} + 25S_{n+1} - 11S_n - 7S_{n-1}, \\
2R_n &= -S_{n+2} + 11S_{n+1} - 7S_n - 5S_{n-1} - 3S_{n-2}, \\
R_n &= 5S_{n+1} - 4S_n - 3S_{n-1} - 2S_{n-2} - S_{n-3}, \\
R_n &= S_n + 2S_{n-1} + 3S_{n-2} + 4S_{n-3} + 10S_{n-4}.
\end{aligned}
$$

Note that we also can give some other identities which related to $\{K_n\}$ and $\{Q_n\}$. For example, we have

$$
\begin{aligned}
8K_n &= -13S_{n+4} + 19S_{n+3} + 19S_{n+2} + 19S_{n+1} - 21S_n, \\
8Q_n &= -13S_{n+4} + 19S_{n+3} + 19S_{n+2} + 19S_{n+1} - 29S_n.
\end{aligned}
$$

4. Linear Sums

4.1. Sums of Terms with Positive Subscripts: The following proposition presents some formulas of generalized fifth order Jacobsthal numbers with positive subscripts.

PROPOSITION 17. *For $n \geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} V_k = \frac{1}{5}(V_{n+5} - V_{n+3} - 2V_{n+2} - 3V_{n+1} - V_4 + V_2 + 2V_1 + 3V_0)$.

(b): $\sum_{k=0}^{n} V_{2k} = \frac{1}{15}(-V_{2n+2} + 5V_{2n+1} + 11V_{2n} + 2V_{2n-1} + 8V_{2n-2} + V_4 - 5V_3 + 4V_2 - 2V_1 + 7V_0)$.

(c): $\sum_{k=0}^{n} V_{2k+1} = \frac{1}{15}(4V_{2n+2} + 10V_{2n+1} + V_{2n} + 7V_{2n-1} - 2V_{2n-2} - 4V_4 + 5V_3 - V_2 + 8V_1 + 2V_0)$.

Proof. Take $r = 1, s = 1, t = 1, u = 1, v = 2$, in Theorem 2.1 in [28].

From the last proposition, we have the following corollary which gives sum formulas of fifth order Jacobsthal numbers (take $V_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, J_4 = 1$).

COROLLARY 18. *For $n \geq 0$, we have the following formulas:*

(a): $\sum_{k=0}^{n} J_k = \frac{1}{5}(J_{n+5} - J_{n+3} - 2J_{n+2} - 3J_{n+1} + 2)$.

(b): $\sum_{k=0}^{n} J_{2k} = \frac{1}{15}(-J_{2n+2} + 5J_{2n+1} + 11J_{2n} + 2J_{2n-1} + 8J_{2n-2} - 2)$.

(c): $\sum_{k=0}^{n} J_{2k+1} = \frac{1}{15}(4J_{2n+2} + 10J_{2n+1} + J_{2n} + 7J_{2n-1} - 2J_{2n-2} + 8)$.

Taking $V_n = j_n$ with$j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20$ in the last proposition, we have the following corollary which presents sum formulas of fifth order Jacobsthal-Lucas numbers.

COROLLARY 19. *For $n \geq 0$, we have the following formulas:*

(a): $\sum_{k=0}^{n} j_k = \frac{1}{5}(j_{n+5} - j_{n+3} - 2j_{n+2} - 3j_{n+1} - 7)$.

(b): $\sum_{k=0}^{n} j_{2k} = \frac{1}{15}(-j_{2n+2} + 5j_{2n+1} + 11j_{2n} + 2j_{2n-1} + 8j_{2n-2} + 2)$.

(c): $\sum_{k=0}^{n} j_{2k+1} = \frac{1}{15}(4j_{2n+2} + 10j_{2n+1} + j_{2n} + 7j_{2n-1} - 2j_{2n-2} - 23)$.

From the last proposition, we have the following corollary which gives sum formulas of modified fifth order Jacobsthal numbers (take $V_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10, K_4 = 20$).

COROLLARY 20. *For $n \geq 0$, we have the following formulas:*

(a): $\sum_{k=0}^{n} K_k = \frac{1}{5}(K_{n+5} - K_{n+3} - 2K_{n+2} - 3K_{n+1} - 6)$.

(b): $\sum_{k=0}^{n} K_{2k} = \frac{1}{15}(-K_{2n+2} + 5K_{2n+1} + 11K_{2n} + 2K_{2n-1} + 8K_{2n-2} + 1)$.

(c): $\sum_{k=0}^{n} K_{2k+1} = \frac{1}{15}(4K_{2n+2} + 10K_{2n+1} + K_{2n} + 7K_{2n-1} - 2K_{2n-2} - 19)$.

Taking $V_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8, Q_4 = 16$ in the last proposition, we have the following corollary which presents sum formulas of fifth order Jacobsthal Perrin numbers.

COROLLARY 21. *For $n \geq 0$, we have the following formulas:*

(a): $\sum_{k=0}^{n} Q_k = \frac{1}{5}(Q_{n+5} - Q_{n+3} - 2Q_{n+2} - 3Q_{n+1} - 5)$.

(b): $\sum_{k=0}^{n} Q_{2k} = \frac{1}{15}(-Q_{2n+2} + 5Q_{2n+1} + 11Q_{2n} + 2Q_{2n-1} + 8Q_{2n-2} + 5)$.

(c): $\sum_{k=0}^{n} Q_{2k+1} = \frac{1}{15}(4Q_{2n+2} + 10Q_{2n+1} + Q_{2n} + 7Q_{2n-1} - 2Q_{2n-2} - 20)$.

From the last proposition, we have the following corollary which gives sum formulas of adjusted fifth order Jacobsthal numbers (take $V_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4$).

COROLLARY 22. *For $n \geq 0$, adjusted fifth order Jacobsthal numbers have the following properties:*

(a): $\sum_{k=0}^{n} S_k = \frac{1}{5}(S_{n+5} - S_{n+3} - 2S_{n+2} - 3S_{n+1} - 1)$.

(b): $\sum_{k=0}^{n} S_{2k} = \frac{1}{15}(-S_{2n+2} + 5S_{2n+1} + 11S_{2n} + 2S_{2n-1} + 8S_{2n-2} - 4)$.

(c): $\sum_{k=0}^{n} S_{2k+1} = \frac{1}{15}(4S_{2n+2} + 10S_{2n+1} + S_{2n} + 7S_{2n-1} - 2S_{2n-2} + 1)$.

Taking $V_n = R_n$ with $R_0 = 5, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15$ in the last proposition, we have the following corollary which presents sum formulas of modified fifth order Jacobsthal-Lucas numbers.

COROLLARY 23. *For $n \geq 0$, modified fifth order Jacobsthal-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n} R_k = \frac{1}{5}(R_{n+5} - R_{n+3} - 2R_{n+2} - 3R_{n+1} + 5)$.

(b): $\sum_{k=0}^{n} R_{2k} = \frac{1}{15}(-R_{2n+2} + 5R_{2n+1} + 11R_{2n} + 2R_{2n-1} + 8R_{2n-2} + 25)$.

(c): $\sum_{k=0}^{n} R_{2k+1} = \frac{1}{15}(4R_{2n+2} + 10R_{2n+1} + R_{2n} + 7R_{2n-1} - 2R_{2n-2} - 10)$.

4.2. Sums of Terms with Negative Subscripts. The following proposition presents some formulas of generalized fifth order Jacobsthal numbers with negative subscripts.

PROPOSITION 24. *For $n \geq 1$, we have the following formulas:*

(a): $\sum_{k=1}^{n} V_{-k} = \frac{1}{5}(-V_{-n+4} + V_{-n+2} + 2V_{-n+1} + 3V_{-n} + V_4 - V_2 - 2V_1 - 3V_0)$.

(b): $\sum_{k=1}^{n} V_{-2k} = \frac{1}{15}(-4V_{-2n+3} + 5V_{-2n+2} - V_{-2n+1} + 8V_{-2n} + 2V_{-2n-1} - V_4 + 5V_3 - 4V_2 + 2V_1 - 7V_0)$.

(c): $\sum_{k=1}^{n} V_{-2k+1} = \frac{1}{15}(V_{-2n+3} - 5V_{-2n+2} + 4V_{-2n+1} - 2V_{-2n} - 8V_{-2n-1} + 4V_4 - 5V_3 + V_2 - 8V_1 - 2V_0)$.

Proof. Take $r = 1, s = 1, t = 1, u = 1, v = 2$ in Theorem 3.1 in [28].

From the last proposition, we have the following corollary which gives sum formulas of fifth order Jacobsthal numbers (take $V_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, J_4 = 1$).

COROLLARY 25. *For $n \geq 1$, fifth order Jacobsthal numbers have the following properties.*

(a): $\sum_{k=1}^{n} J_{-k} = \frac{1}{5}(-J_{-n+4} + J_{-n+2} + 2J_{-n+1} + 3J_{-n} - 2)$.

(b): $\sum_{k=1}^{n} J_{-2k} = \frac{1}{15}(-4J_{-2n+3} + 5J_{-2n+2} - J_{-2n+1} + 8J_{-2n} + 2J_{-2n-1} + 2)$.

(c): $\sum_{k=1}^{n} J_{-2k+1} = \frac{1}{15}(J_{-2n+3} - 5J_{-2n+2} + 4J_{-2n+1} - 2J_{-2n} - 8J_{-2n-1} - 8)$.

Taking $V_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20$ in the last proposition, we have the following corollary which presents sum formulas of fifth order Jacobsthal-Lucas numbers.

COROLLARY 26. *For $n \geq 1$, fifth order Jacobsthal-Lucas numbers have the following properties.*

(a): $\sum_{k=1}^{n} j_{-k} = \frac{1}{5}(-j_{-n+4} + j_{-n+2} + 2j_{-n+1} + 3j_{-n} + 7)$.

(b): $\sum_{k=1}^{n} j_{-2k} = \frac{1}{15}(-4j_{-2n+3} + 5j_{-2n+2} - j_{-2n+1} + 8j_{-2n} + 2j_{-2n-1} - 2)$.

(c): $\sum_{k=1}^{n} j_{-2k+1} = \frac{1}{15}(j_{-2n+3} - 5j_{-2n+2} + 4j_{-2n+1} - 2j_{-2n} - 8j_{-2n-1} + 23)$.

From the last proposition, we have the following corollary which gives sum formulas of modified fifth order Jacobsthal numbers (take $V_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10, K_4 = 20$).

COROLLARY 27. *For $n \geq 1$, modified fifth order Jacobsthal numbers have the following properties.*

(a): $\sum_{k=1}^{n} K_{-k} = \frac{1}{5}(-K_{-n+4} + K_{-n+2} + 2K_{-n+1} + 3K_{-n} + 6)$.

(b): $\sum_{k=1}^{n} K_{-2k} = \frac{1}{15}(-4K_{-2n+3} + 5K_{-2n+2} - K_{-2n+1} + 8K_{-2n} + 2K_{-2n-1} - 1)$.

(c): $\sum_{k=1}^{n} K_{-2k+1} = \frac{1}{15}(K_{-2n+3} - 5K_{-2n+2} + 4K_{-2n+1} - 2K_{-2n} - 8K_{-2n-1} + 19)$.

Taking $V_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8, Q_4 = 16$ in the last proposition, we have the following corollary which presents sum formulas of fifth order Jacobsthal Perrin numbers.

COROLLARY 28. *For $n \geq 1$, fifth order Jacobsthal Perrin numbers have the following properties.*

(a): $\sum_{k=1}^{n} Q_{-k} = \frac{1}{5}(-Q_{-n+4} + Q_{-n+2} + 2Q_{-n+1} + 3Q_{-n} + 5)$.

(b): $\sum_{k=1}^{n} Q_{-2k} = \frac{1}{15}(-4Q_{-2n+3} + 5Q_{-2n+2} - Q_{-2n+1} + 8Q_{-2n} + 2Q_{-2n-1} - 5)$.

(c): $\sum_{k=1}^{n} Q_{-2k+1} = \frac{1}{15}(Q_{-2n+3} - 5Q_{-2n+2} + 4Q_{-2n+1} - 2Q_{-2n} - 8Q_{-2n-1} + 20)$.

From the last proposition, we have the following corollary which gives sum formulas of adjusted fifth order Jacobsthal numbers (take $V_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4$).

COROLLARY 29. *For $n \geq 1$, adjusted fifth order Jacobsthal numbers have the following properties:*

(a): $\sum_{k=1}^{n} S_{-k} = \frac{1}{5}(-S_{-n+4} + S_{-n+2} + 2S_{-n+1} + 3S_{-n} + 1)$.

(b): $\sum_{k=1}^{n} S_{-2k} = \frac{1}{15}(-4S_{-2n+3} + 5S_{-2n+2} - S_{-2n+1} + 8S_{-2n} + 2S_{-2n-1} + 4)$.

(c): $\sum_{k=1}^{n} S_{-2k+1} = \frac{1}{15}(S_{-2n+3} - 5S_{-2n+2} + 4S_{-2n+1} - 2S_{-2n} - 8S_{-2n-1} - 1)$.

Taking $V_n = R_n$ with $R_0 = 5, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15$ in the last proposition, we have the following corollary which presents sum formulas of modified fifth order Jacobsthal-Lucas numbers.

COROLLARY 30. *For $n \geq 1$, modified fifth order Jacobsthal-Lucas numbers have the following properties:*

(a): $\sum_{k=1}^{n} R_{-k} = \frac{1}{5}(-R_{-n+4} + R_{-n+2} + 2R_{-n+1} + 3R_{-n} - 5)$.

(b): $\sum_{k=1}^{n} R_{-2k} = \frac{1}{15}(-4R_{-2n+3} + 5R_{-2n+2} - R_{-2n+1} + 8R_{-2n} + 2R_{-2n-1} - 25)$.

(c): $\sum_{k=1}^{n} R_{-2k+1} = \frac{1}{15}(R_{-2n+3} - 5R_{-2n+2} + 4R_{-2n+1} - 2R_{-2n} - 8R_{-2n-1} + 10)$.

4.3. A Sum Formula. The formula in the following proposition can be used to calculate the sum of the generalized fixth order Jacobsthal sequence.

PROPOSITION 31. *For all integers m and j with $S_m^3 - 3S_m^2 - 3S_{2m}S_m + 2S_{3m} + 3S_{2m} + 6S_m - 6 \times 2^m \times (S_{-m} - 1) - 6 \neq 0$, we have*

$$\sum_{k=0}^{n} V_{mk+j} = \frac{\Lambda + \Psi_1}{\Omega} \tag{4.1}$$

where

$\Lambda = 6V_{mn+3m+j} + 6(1 - R_m)V_{mn+2m+j} + 3(R_m^2 - R_{2m} - 2R_m + 2)V_{mn+m+j} + 6V_{mn-m+j} \times 2^m - 6 \times 2^m \times (R_{-m} - 1)\, V_{mn+j}$,

$\Psi_1 = -6V_{3m+j} - 6(1 - R_m)V_{2m+j} - 3(R_m^2 - R_{2m} - 2R_m + 2)V_{m+j} - 6V_{-m+j} \times 2^m + (R_m^3 - 3R_m^2 - 3R_{2m}R_m + 2R_{3m} + 3R_{2m} + 6R_m - 6)V_j$,

$\Omega = R_m^3 - 3R_m^2 - 3R_{2m}R_m + 2R_{3m} + 3R_{2m} + 6R_m - 6 \times 2^m \times (R_{-m} - 1) - 6$.

Proof. Take $r = 1, s = 1, t = 1, u = 1, v = 2$ and $R_n = H_n$ in Soykan [27, Theorem 13.]. □

Note that (4.1) can be written in the following form:

$$\sum_{k=1}^{n} V_{mk+j} = \frac{\Lambda + \Psi_2}{\Omega}$$

where

$\Psi_2 = -6V_{3m+j} - 6(1 - R_m)V_{2m+j} - 3(R_m^2 - R_{2m} - 2R_m + 2)V_{m+j} - 6V_{-m+j} \times 2^m + 6 \times 2^m \times (R_{-m} - 1)\, V_j$.

As special cases of the above proposition, we have the following identities.

COROLLARY 32. *The following identities hold:*

(1) $m = 1$, $j = 0$.

(a): $\sum_{k=0}^{n} J_k = \frac{1}{5}(J_{n+3} - J_{n+1} + 3J_n + 2J_{n-1} + 2)$.

(b): $\sum_{k=0}^{n} j_k = \frac{1}{5}(j_{n+3} - j_{n+1} + 3j_n + 2j_{n-1} - 7)$.

(c): $\sum_{k=0}^{n} K_k = \frac{1}{5}(K_{n+3} - K_{n+1} + 3K_n + 2K_{n-1} - 6)$.

(d): $\sum_{k=0}^{n} Q_k = \frac{1}{5}(Q_{n+3} - Q_{n+1} + 3Q_n + 2Q_{n-1} - 5)$.

(e): $\sum_{k=0}^{n} S_k = \frac{1}{5}(S_{n+3} - S_{n+1} + 3S_n + 2S_{n-1} - 1)$.

(f): $\sum_{k=0}^{n} R_k = \frac{1}{5}(R_{n+3} - R_{n+1} + 3R_n + 2R_{n-1} + 5)$.

(2) $m = -1,\ j = 0$.

(a): $\sum_{k=0}^{n} J_{-k} = \frac{1}{5}\left(-J_{-n+1} - 4J_{-n-1} - 3J_{-n-2} - 2J_{-n-3} - 2\right)$.

(b): $\sum_{k=0}^{n} j_{-k} = \frac{1}{5}\left(-j_{-n+1} - 4j_{-n-1} - 3j_{-n-2} - 2j_{-n-3} + 17\right)$.

(c): $\sum_{k=0}^{n} K_{-k} = \frac{1}{5}\left(-K_{-n+1} - 4K_{-n-1} - 3K_{-n-2} - 2K_{-n-3} + 21\right)$.

(d): $\sum_{k=0}^{n} Q_{-k} = \frac{1}{5}\left(-Q_{-n+1} - 4Q_{-n-1} - 3Q_{-n-2} - 2Q_{-n-3} + 20\right)$.

(e): $\sum_{k=0}^{n} S_{-k} = \frac{1}{5}\left(-S_{-n+1} - 4S_{-n-1} - 3S_{-n-2} - 2S_{-n-3} + 1\right)$.

(f): $\sum_{k=0}^{n} R_{-k} = \frac{1}{5}\left(-R_{-n+1} - 4R_{-n-1} - 3R_{-n-2} - 2R_{-n-3} + 20\right)$.

(3) $m = 3,\ j = 1$.

(a): $\sum_{k=0}^{n} J_{3k+1} = \frac{1}{35}(J_{3n+10} - 6J_{3n+7} - 13J_{3n+4} + 15J_{3n+1} + 8J_{3n-2} + 3)$.

(b): $\sum_{k=0}^{n} j_{3k+1} = \frac{1}{35}(j_{3n+10} - 6j_{3n+7} - 13j_{3n+4} + 15j_{3n+1} + 8j_{3n-2} - 38)$.

(c): $\sum_{k=0}^{n} K_{3k+1} = \frac{1}{35}(K_{3n+10} - 6K_{3n+7} - 13K_{3n+4} + 15K_{3n+1} + 8K_{3n-2} - 44)$.

(d): $\sum_{k=0}^{n} Q_{3k+1} = \frac{1}{35}(Q_{3n+10} - 6Q_{3n+7} - 13Q_{3n+4} + 15Q_{3n+1} + 8Q_{3n-2} - 50)$.

(e): $\sum_{k=0}^{n} S_{3k+1} = \frac{1}{35}(S_{3n+10} - 6S_{3n+7} - 13S_{3n+4} + 15S_{3n+1} + 8S_{3n-2} + 6)$.

(f): $\sum_{k=0}^{n} R_{3k+1} = \frac{1}{35}(R_{3n+10} - 6R_{3n+7} - 13R_{3n+4} + 15R_{3n+1} + 8R_{3n-2} - 45)$.

(4) $m = -3,\ j = -1$.

(a): $\sum_{k=0}^{n} J_{-3k-1} = \frac{1}{35}(-J_{-3n+2} + 6J_{-3n-1} - 22J_{-3n-4} - 15J_{-3n-7} - 8J_{-3n-10} - 13)$.

(b): $\sum_{k=0}^{n} j_{-3k-1} = \frac{1}{35}(-j_{-3n+2} + 6j_{-3n-1} - 22j_{-3n-4} - 15j_{-3n-7} - 8j_{-3n-10} + 13)$.

(c): $\sum_{k=0}^{n} K_{-3k-1} = \frac{1}{35}(-K_{-3n+2} + 6K_{-3n-1} - 22K_{-3n-4} - 15K_{-3n-7} - 8K_{-3n-10} + 39)$.

(d): $\sum_{k=0}^{n} Q_{-3k-1} = \frac{1}{35}(-Q_{-3n+2} + 6Q_{-3n-1} - 22Q_{-3n-4} - 15Q_{-3n-7} - 8Q_{-3n-10} + 30)$.

(e): $\sum_{k=0}^{n} S_{-3k-1} = \frac{1}{35}(-S_{-3n+2} + 6S_{-3n-1} - 22S_{-3n-4} - 15S_{-3n-7} - 8S_{-3n-10} + 9)$.

(f): $\sum_{k=0}^{n} R_{-3k-1} = \frac{1}{35}(-R_{-3n+2} + 6R_{-3n-1} - 22R_{-3n-4} - 15R_{-3n-7} - 8R_{-3n-10} - 15)$.

(5) $m = 4,\ j = -2$.

(a): $\sum_{k=0}^{n} J_{4k-2} = \frac{1}{300}(4J_{4n+10} - 56J_{4n+6} - 116J_{4n+2} + 124J_{4n-2} + 64J_{4n-6} + 160)$.

(b): $\sum_{k=0}^{n} j_{4k-2} = \frac{1}{300}(4j_{4n+10} - 56j_{4n+6} - 116j_{4n+2} + 124j_{4n-2} + 64j_{4n-6} - 70)$.

(c): $\sum_{k=0}^{n} K_{4k-2} = \frac{1}{300}(4K_{4n+10} - 56K_{4n+6} - 116K_{4n+2} + 124K_{4n-2} + 64K_{4n-6} - 275)$.

(d): $\sum_{k=0}^{n} Q_{4k-2} = \frac{1}{300}(4Q_{4n+10} - 56Q_{4n+6} - 116Q_{4n+2} + 124Q_{4n-2} + 64Q_{4n-6} - 295)$.

(e): $\sum_{k=0}^{n} S_{4k-2} = \frac{1}{300}(4S_{4n+10} - 56S_{4n+6} - 116S_{4n+2} + 124S_{4n-2} + 64S_{4n-6} + 20)$.

(f): $\sum_{k=0}^{n} R_{4k-2} = \frac{1}{300}(4R_{4n+10} - 56R_{4n+6} - 116R_{4n+2} + 124R_{4n-2} + 64R_{4n-6} - 305)$.

(6) $m = -4,\ j = -2$.

(a): $\sum_{k=0}^{n} J_{-4k-2} = \frac{1}{75}(14J_{4n-2} - 16J_{4n-14} - J_{-4n+2} - 46J_{-4n-6} - 31J_{-4n-10} - 40)$.

(b): $\sum_{k=0}^{n} j_{-4k-2} = \frac{1}{75}(14j_{4n-2} - 16j_{4n-14} - j_{-4n+2} - 46j_{-4n-6} - 31j_{-4n-10} + 55)$.

(c): $\sum_{k=0}^{n} K_{-4k-2} = \frac{1}{75}(14K_{4n-2} - 16K_{4n-14} - K_{-4n+2} - 46K_{-4n-6} - 31K_{-4n-10} + 125)$.

(d): $\sum_{k=0}^{n} Q_{-4k-2} = \frac{1}{75}(14Q_{4n-2} - 16Q_{4n-14} - Q_{-4n+2} - 46Q_{-4n-6} - 31Q_{-4n-10} + 130)$.

(e): $\sum_{k=0}^{n} S_{-4k-2} = \frac{1}{75}(14S_{4n-2} - 16S_{4n-14} - S_{-4n+2} - 46S_{-4n-6} - 31S_{-4n-10} - 5)$.

(f): $\sum_{k=0}^{n} R_{-4k-2} = \frac{1}{75}(14R_{4n-2} - 16R_{4n-14} - R_{-4n+2} - 46R_{-4n-6} - 31R_{-4n-10} + 20)$.

5. Matrices Related with Generalized Fifth-Order Jacobsthal Numbers

We define the square matrix A of order 5 as:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 2$. We also define

$$B_n = \begin{pmatrix} S_{n+1} & S_n + S_{n-1} + S_{n-2} + 2S_{n-3} & S_n + S_{n-1} + 2S_{n-2} & S_n + 2S_{n-1} & 2S_n \\ S_n & S_{n-1} + S_{n-2} + S_{n-3} + 2S_{n-4} & S_{n-1} + S_{n-2} + 2S_{n-3} & S_{n-1} + 2S_{n-2} & 2S_{n-1} \\ S_{n-1} & S_{n-2} + S_{n-3} + S_{n-4} + 2S_{n-5} & S_{n-2} + S_{n-3} + 2S_{n-4} & S_{n-2} + 2S_{n-3} & 2S_{n-2} \\ S_{n-2} & S_{n-3} + S_{n-4} + S_{n-5} + 2S_{n-6} & S_{n-3} + S_{n-4} + 2S_{n-5} & S_{n-3} + 2S_{n-4} & 2S_{n-3} \\ S_{n-3} & S_{n-4} + S_{n-5} + S_{n-6} + 2S_{n-7} & S_{n-4} + S_{n-5} + 2S_{n-6} & S_{n-4} + 2S_{n-5} & 2S_{n-4} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} V_{n+1} & V_n + V_{n-1} + V_{n-2} + 2V_{n-3} & V_n + V_{n-1} + 2V_{n-2} & V_n + 2V_{n-1} & 2V_n \\ V_n & V_{n-1} + V_{n-2} + V_{n-3} + 2V_{n-4} & V_{n-1} + V_{n-2} + 2V_{n-3} & V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2} + V_{n-3} + V_{n-4} + 2V_{n-5} & V_{n-2} + V_{n-3} + 2V_{n-4} & V_{n-2} + 2V_{n-3} & 2V_{n-2} \\ V_{n-2} & V_{n-3} + V_{n-4} + V_{n-5} + 2V_{n-6} & V_{n-3} + V_{n-4} + 2V_{n-5} & V_{n-3} + 2V_{n-4} & 2V_{n-3} \\ V_{n-3} & V_{n-4} + V_{n-5} + V_{n-6} + 2V_{n-7} & V_{n-4} + V_{n-5} + 2V_{n-6} & V_{n-4} + 2V_{n-5} & 2V_{n-4} \end{pmatrix}.$$

THEOREM 33. *For all integer $m, n \geq 0$, we have*

(a): $B_n = A^n$

(b): $C_1 A^n = A^n C_1$

(c): $C_{n+m} = C_n B_m = B_m C_n$.

Proof. Take $r = 1, s = 1, t = 1, u = 1, v = 2$ and $S_n = G_n$ in Soykan [27, Theorem 16.]. □

THEOREM 34. *For all integers m, n, we have*

$$\begin{aligned} V_{n+m} &= V_n S_{m+1} + V_{n-1}(S_m + S_{m-1} + S_{m-2} + 2S_{m-3}) \\ &\quad + V_{n-2}(S_m + S_{m-1} + 2S_{m-2}) + V_{n-3}(S_m + 2S_{m-1}) + 2V_{n-4}S_m. \end{aligned}$$

Proof. Take $r = 1, s = 1, t = 1, u = 1, v = 2$ and $S_n = G_n$ in Soykan [27, Theorem 17.]. □

COROLLARY 35. *For all integers m, n, we have*

$$\begin{aligned}
J_{n+m} &= J_n S_{m+1} + J_{n-1}(S_m + S_{m-1} + S_{m-2} + 2S_{m-3}) \\
&\quad + J_{n-2}(S_m + S_{m-1} + 2S_{m-2}) + J_{n-3}(S_m + 2S_{m-1}) + 2J_{n-4}S_m, \\
j_{n+m} &= j_n S_{m+1} + j_{n-1}(S_m + S_{m-1} + S_{m-2} + 2S_{m-3}) \\
&\quad + j_{n-2}(S_m + S_{m-1} + 2S_{m-2}) + j_{n-3}(S_m + 2S_{m-1}) + 2j_{n-4}S_m, \\
K_{n+m} &= K_n S_{m+1} + K_{n-1}(S_m + S_{m-1} + S_{m-2} + 2S_{m-3}) \\
&\quad + K_{n-2}(S_m + S_{m-1} + 2S_{m-2}) + K_{n-3}(S_m + 2S_{m-1}) + 2K_{n-4}S_m, \\
Q_{n+m} &= Q_n S_{m+1} + Q_{n-1}(S_m + S_{m-1} + S_{m-2} + 2S_{m-3}) \\
&\quad + Q_{n-2}(S_m + S_{m-1} + 2S_{m-2}) + Q_{n-3}(S_m + 2S_{m-1}) + 2Q_{n-4}S_m, \\
S_{n+m} &= S_n S_{m+1} + S_{n-1}(S_m + S_{m-1} + S_{m-2} + 2S_{m-3}) \\
&\quad + S_{n-2}(S_m + S_{m-1} + 2S_{m-2}) + S_{n-3}(S_m + 2S_{m-1}) + 2S_{n-4}S_m, \\
R_{n+m} &= R_n S_{m+1} + R_{n-1}(S_m + S_{m-1} + S_{m-2} + 2S_{m-3}) \\
&\quad + R_{n-2}(S_m + S_{m-1} + 2S_{m-2}) + R_{n-3}(S_m + 2S_{m-1}) + 2R_{n-4}S_m.
\end{aligned}$$

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