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Connection between the Inversion Formula and the Functional Power Series

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This study derives the connection between the functional power series and the inversion formula, both given in series form. The link is established by differentiation of the inversion formula showing an expression of a functional power series for the reciprocal of the derivative in terms of the function itself. 1

0.1 Mathematical Classification
Mathematics Subject Classification: 11A25, 30E10, 30K05, 30B10, 30C10, 30D10, 40H05
0.2 Keywords
Functional power series, inversion of functions, Taylor series, reversion of series

0.3 Introduction

The purpose of this work is to find a connection between the functional power series [2] and the inversion formula [1]. Their equations resemble each other and also their derivations have similarities. Therefore it is reasonable to expect that some sort of link does exist. In the following both equations are expressed and the derivations are found in the original articles.

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I. The Inversion Formula

The inverse function of an analytic function u(z) with

$z, u \in C$

$$u = u(z) \qquad (1)$$

is defined as

$$z = z(u) \qquad (2)$$

with

$$u = u(z) \qquad (3)$$

$z, u \in C$

and u(z) be analytic over the interior of a circle

$$r = |z - z_0| \qquad (4)$$

Let the inverse function $z(u)$ be analytic over the interior of a circle R_0 at u_0 with $u_0 = u(z_0)$

$$R_0 = |u - u_0| \qquad (5)$$

Then one can find a representation of the unknown inverse function according to [1] with $z_0 = z(u_0)$

$$z(u) = z_0 + \sum_{n=1}^{\infty} \frac{(u-u_0)^n}{n!} [[\frac{1}{\frac{du(z)}{dz}} \cdot \frac{d}{dz}]^{n-1} \frac{1}{(\frac{du(z)}{dz})}]_{z_0} \qquad (6)$$

The necessary, but not sufficient, condition for the inversion formula to converge is that the first derivative of $u(z)$ must be nonzero at z_0 which is the point of focus. This point is a parameter free of choice allowing a huge benefit while attempting to avoid singularities. This series expression allows one to develop an unknown inverse function in a straightforward way as a power series. Complexity of the derivative terms determines the degree of ease.

II. The Functional Power Series

The functional power series is a method for expanding any differentiable function as a power series in terms of powers of practically any differentiable function. The equation for an analytic function $u(z)$ with the variable

$$s = s(z) \in C$$

after we mark $s_0 = s(z_0)$, will be

$$u(z) = \sum_{n=0}^{\infty} \frac{(s(z) - s(z_0))^n}{n!} [(\frac{1}{s'(z)} \frac{d}{dz})^n u(z)]_{z_0} \qquad (7)$$

according to [2]. The necessary but not sufficient conditions for this power series to converge, are that $s(z)$ is differentiable and

$$s'(z_0) \neq 0 \qquad (8)$$

$$|s(z_0)| < \infty \qquad (9)$$

The point of focus z_0 is a free parameter as long as the basic conditions (8) and (9) hold and the resulting series converges. Adjusting the point of focus makes it possible to circumvent singularities and optimize the range of validity. Again, the complexity of the derivative terms will dictate how easy this is to generate.

III. Creating the Connection

As is obvious from the above results there seems to be a number of similarities between the results. Neither is a Taylor's series but of a more advanced form. The equations themselves are similar but have some crucial differences in terms. The methods for their derivations are similar as is obvious from the original articles [2] and [1]. They consist, instead of expanding into fully unmanageable forms of progressively increasing complexity, of compacting the derivative term's into short forms. These allow a significantly easier way of generating the terms and also tend to show the trends in coefficients helping in recognizing a possible closed solution. Also symbolic computer generation of terms is greatly enhanced with the short forms.

To find out how these two formulas are connected, one can start the process by differentiating equation (6) in terms of u getting

$$\frac{1}{\frac{du}{dz}} = \sum_{n=1}^{\infty} \frac{(u-u_0)^{n-1}}{(n-1)!}\left[\left[\frac{1}{\frac{du(z)}{dz}} \cdot \frac{d}{dz}\right]^{n-1} \frac{1}{\left(\frac{du(z)}{dz}\right)}\right]_{z_0} \tag{10}$$

Changing the index leads to

$$\frac{1}{\frac{du}{dz}} = \sum_{n=0}^{\infty} \frac{(u-u_0)^{n}}{n!}\left[\left[\frac{1}{\frac{du(z)}{dz}} \cdot \frac{d}{dz}\right]^{n} \frac{1}{\left(\frac{du(z)}{dz}\right)}\right]_{z_0} \tag{11}$$

Surprisingly, this can be recognized as the functional power series of function

$$\frac{1}{\frac{du}{dz}}$$

expanded in terms of function s(z) = u(z) as a variable. One can also differentiate equation (6) in terms of z getting the same result (11). This equation is the link connecting the equations (6) and (7). The necessary, but not sufficient, condition for the sum to converge is that the first derivative of u(z) must be nonzero at z_0 which is the point of focus. Also u(z) must be differentiable and

$$|u(z_0)| < \infty \tag{12}$$

In the complex variable case, analyticity ensures differentiability.

IV. Conclusions

The inversion formula is a power series for generating the inverse function of a known function. The functional power series is based on a completely arbitrary function s(z) in terms of which the expansion is made. The functional power series is therefore far more general than the inversion formula. From that follows that it has also significantly wider applications. The inversion formula has as ingredients the function and its derivatives only.

The connection between the inversion formula (6) and the functional power series equation (7) is obtained by differentiating the inversion formula in terms of u or z and changing the summation index. The result is a functional power series of function

$$\frac{1}{\frac{du}{dz}}$$

in terms of function $s(z) = u(z)$. It is obvious that by starting from this result seen as a functional power series, one can go backwards by integration to the inversion formula. The treatment is valid both for real and complex valued functions as long as the function $u(z)$ is differentiable.

Equation (11) can be understood in various cases as follows. By default $z(u)$ is unknown and $u(z)$ is known. Then this equation is an expansion of

$$\frac{1}{\frac{du}{dz}}$$

expanded in terms of u as a variable. If $z(u)$ is known and $u(z)$ is unknown, the equation can be changed to

$$\frac{dz}{du} = \sum_{n=0}^{\infty} \frac{(u-u_0)^n}{n!} \left[\frac{d^n}{du^n} \frac{dz}{du}\right]_{z_0} \tag{13}$$

This is actually a Taylor series for

$$\frac{1}{\frac{du}{dz}}$$

in terms of u. If both $z(u)$ and $u(z)$ are known, the equation (11) becomes

$$\frac{dz}{du} = \sum_{n=0}^{\infty} \frac{(u-u_0)^n}{n!} \left[\left[\frac{1}{\frac{du(z)}{dz}} \cdot \frac{d}{dz}\right]^n \frac{dz}{du}\right]_{z_0} \tag{14}$$

This is more or less an identity but may lead in some cases to an interesting expansion boiling down to an identity.

References

[1]. Stenlund, H.: Inversion Formula**, arXiv:1008.0183v3** [math.GM] July 27 (2010)

[2]. Stenlund, H.: Functional Power Series, **arXiv:1204.5992v1** [math.GM] April 24 (2012)

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