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Collocation Method for Fourth Order Boundary Value Problems using Quartic B-splines

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Abstract: *A finite element method involving collocation method with Quartic B-splines as basis functions has been developed to solve fourth order boundary value problems. The fourth order derivative for the dependent variable is approximated by the finite differences. The basis functions are redefined into a new set of basis functions which in number match with the number of collocated points selected in the space variable domain. The proposed method is tested on two linear and three non-linear boundary value problems. The solution to a nonlinear problem has been obtained as the limit of a sequence of solutions of linear problems generated by the quasilinearization technique. Numerical results obtained by the present method are in good agreement with the exact solutions available in the literature.*

Keywords: *Absolute error, Basis function, Collocation method, Fourth order boundary value problem, Quartic B-spline.*

I. Introduction

In this paper, we consider a general fourth order linear boundary value problem

$$a_0(x)y^{(4)}(x)+a_1(x)y'''(x)+a_2(x)y''(x)+a_3(x)y'(x)+a_4(x)y(x)=b(x),\quad c<x<d \tag{1}$$

subject to boundary conditions $y(c)=A_0, y(d)=C_0, y'(c)=A_1, y'(d)=C_1$ (2)

where A_0, C_0, A_1, C_1 are finite real constants and $a_0(x)$, $a_1(x)$, $a_2(x)$, $a_3(x)$, $a_4(x)$ and $b(x)$ are all continuous functions defined on the interval $[c, d]$.

The fourth order boundary value problems occur in a number of areas of applied mathematics, among which are fluid mechanics, elasticity and quantum mechanics as well as in science and engineering. The existence and uniqueness of the solution for these types of problems have been discussed in Agarwal [1]. Finding the analytical solutions of such type of boundary value problems in general is not possible. Over the years, many researchers have worked on fourth order boundary value problems by using different methods for numerical solutions. Wazawz [2] presented modified decomposition method to solve a special case of fourth order boundary value problems. Waleed and Luis [3] developed decomposition method to solve fourth orderboundary value problems. Erturk and Momani [4] presented a numerical comparison between differential transform method and the Adomian decomposition method for solving fourth order boundary value problems. Momani and Noor [5] presented a numerical comparison between the Differential transform method, Adomian decomposition and Homotopy perturbation method for solving a fourth order boundary value problem. Samuel and Sinkala [6] developed higher order B-spline collocation method to solve fourth order boundary value problems. Syed and Noor [7], Noor and Syed [8] developed Homotopy perturbation method and Variational iteration technique respectively for the solution of fourth order boundary value problems. Ahniyaz et al. [9] developed Sinc-Galerkin method to solve a general linear fourth order boundary value problem. Manoj and Pankaj [10], Ramadan et al. [11], Srivastava et al. [12] and Ghazala and Amin [13] presented the solution of a special case of linear fourth order boundary value problems by spline techniques. Rashidinia and Ghasemi [14], Kasi Viswanadham and Showri Raju [15] have developed B-spline collocation method, cubic B-spline collocation method respectively to solve a general fourth order boundary value problem. Kasi Viswanadham et al. [16], Kasi Viswanadham and Sreenivasulu [17] developed Galerkin methods with quintic B-splines and cubic B-splines respectively to solve a general fourth order boundary value problem. Waleed et al. [18] developed Galerkin method with Jacobian polynomials as basis functions to solve a general fourth order boundary value problem. Lazhar et al.[19] presented extension of Duan-Rach modified Adomian decomposition method to solve a general fourth order boundary value problem. So far, fourth order boundary value problems have not been solved by using Collocation method with quartic B-splines as basis functions. This motivated us to solve a fourth order boundary value problem by Collocation method with quartic B-splines as basis functions.

In this paper, we try to present a simple finite element method which involves collocation approach with quartic B-splines as basis functions to solve the fourth order boundary value problem of the type (1)-(2). This paper is organized as follows. In section II of this paper, the justification for using the collocation method has been mentioned. In section III, the definition of quartic B-splines has been described. In section IV, description of the collocation method with quartic B-splines as basis functions has been presented and in section V, solution procedure to find the nodal parameters is presented. In section VI, numerical examples of both linear and non-linear boundary value problems are presented. The solution to a nonlinear problem has been obtained as the limit of a sequence of solution of linear problems generated by the quasilinearization technique [20]. Finally, the last section is dealt with conclusions of the paper.

II. Justification for using Collocation Method

In finite element method (FEM) the approximate solution can be written as a linear combination of basis functions which constitute a basis for the approximation space under consideration. FEM involves variational methods like Ritzs approach, Galerkins approach, least squares method and collocation method etc. The collocation method seeks an approximate solution by requiring the residual of the differential equation to be identically zero at N selected points in the given space variable domain where N is the number of basis functions in the basis [21]. That means, to get an accurate solution by the collocation method one needs a set of basis functions which in number match with the number of collocation points selected in the given space variable domain. Further, the collocation method is the easiest to implement among the variational methods of FEM. When a differential equation is approximated by m^{th} order B-splines, it yields $(m + 1)^{th}$ order accurate results [22]. Hence this motivated us to solve a fourth order boundary value problem of type (1)-(2) by collocation method with quartic B-splines as basis functions.

III. Definition of quartic B-spline

The quartic B-splines are defined in [22-24]. The existence of quartic spline interpolate s(x) to a function in a closed interval [c, d] for spaced knots (need not be evenly spaced) of a partition $c = x_0 < x_1 < \ldots < x_{n-1} < x_n = d$ is established by constructing it. The construction of $s(x)$ is done with the help of the quartic B-splines. Introduce eight additional knots x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3} and x_{n+4} in such a way that

$x_{-4}<x_{-3}<x_{-2}<x_{-1}<x_0$ and $x_n<x_{n+1}<x_{n+2}<x_{n+3}<x_{n+4}$.

Now the quartic B-splines $B_i(x)'s$ are defined by

$$B_i(x) = \begin{cases} \sum_{r=i-2}^{i+3} \dfrac{(x_r - x)_+^4}{\pi'(x_r)}, & x \in [x_{i-2}, x_{i+3}] \\ 0, & otherwise \end{cases}$$

where $$(x_r - x)_+^4 = \begin{cases} (x_r - x)^4, & if\ x_r \ge x \\ 0, & if\ x_r \le x \end{cases}$$

and $$\pi(x) = \prod_{r=i-2}^{i+3} (x - x_r)$$

where $\{B_{-2}(x), B_{-1}(x), B_0(x), B_1(x), \ldots, B_{n-1}(x), B_n(x), B_{n+1}(x)\}$ forms a basis for the space $s_4(\pi)$ of quartic polynomial splines. Schoenberg [24] has proved that quartic B-splines are the unique nonzero splines of smallest compact support with the knots at

$x_{-4}<x_{-3}<x_{-2}<x_{-1}<x_0<x_1<\ldots<x_{n-1}<x_n<x_{n+1}<x_{n+2}<x_{n+3}<x_{n+4}$.

IV. Description of the method

To solve the boundary value problem (1) subject to boundary conditions (2) by the collocation method with quartic B-splines as basis functions, we define the approximation for $y(x)$ as

$$y(x) = \sum_{j=-2}^{n+1} \alpha_j B_j(x) \tag{3}$$

Where $\alpha_j\,'s$ are the nodal parameters to be determined and $B_j(x)'s$ are quartic B-spline basis functions.

In the present method, the mesh points $x_0, x_1, \ldots, x_{n-2}, x_{n-1}$ are selected as the collocation points. In collocation method, the number of basis functions in the approximation should match with the number of collocation points [24]. Here the number of basis functions in the approximation (3) is n+4, where as the number of selected collocation points is n. So, there is a need to redefine the basis functions into a new set of basis functions which in number match with the number of selected collocation points. The procedure for redefining the basis functions is as follows:

Using the definition of quartic B-splines described in section III and the Dirichlet boundary condition of (2), we get the approximate solution at the boundary points as

$$A_0 = y(c) = y(x_0) = \alpha_{-2}B_{-2}(x_0) + \alpha_{-1}B_{-1}(x_0) + \alpha_0 B_0(x_0) + \alpha_1 B_1(x_0) \tag{4}$$

$$C_0 = y(d) = y(x_n) = \alpha_{n-2}B_{n-2}(x_n) + \alpha_{n-1}B_{n-1}(x_n) + \alpha_n B_n(x_n) + \alpha_{n+1}B_{n+1}(x_n) \tag{5}$$

Eliminating α_{-2} and α_{n+1} from the equations (3), (4) and (5), we get

$$y(x) = w_1(x) + \sum_{j=-1}^{n} \alpha_j P_j(x) \tag{6}$$

where

$$w_1(x) = \frac{A_0}{B_{-2}(x_0)} B_{-2}(x) + \frac{C_0}{B_{n+1}(x_n)} B_{n+1}(x) \tag{7}$$

$$P_j(x) = \begin{cases} B_j(x) - \dfrac{B_j(x_0)}{B_{-2}(x_0)} B_{-2}(x), & j = -1, 0, 1 \\ B_j(x), & j = 2, 3, \ldots, n-3 \\ B_j(x) - \dfrac{B_j(x_n)}{B_{n+1}(x_n)} B_{n+1}(x), & j = n-2, n-1, n \end{cases} \tag{8}$$

Using the Neumann boundary conditions of (2) to the approximation y(x) in (6), we get

$$A_1 = y'(c) = y'(x_0) = w_1'(x_0) + \alpha_{-1}P_{-1}'(x_0) + \alpha_0 P_0'(x_0) + \alpha_1 P_1'(x_0) \tag{9}$$

$$C_1 = y'(d) = y'(x_n) = w_1'(x_n) + \alpha_{n-2}P_{n-2}'(x_n) + \alpha_{n-1}P_{n-1}'(x_n) + \alpha_n P_n'(x_n) \tag{10}$$

Eliminating α_{-1} and α_n from the equations (6), (9) and (10), we get approximation for y(x) as

$$y(x) = w(x) + \sum_{j=0}^{n-1} \alpha_j Q_j(x) \tag{11}$$

$$w(x) = w_1(x) + \frac{A_1 - w_1'(x_0)}{P_{-1}'(x_0)} P_{-1}(x) + \frac{C_1 - w_1'(x_n)}{P_{n+1}'(x_n)} P_{n+1}(x) \tag{12}$$

$$Q_j(x) = \begin{cases} P_j(x) - \dfrac{P_j'(x_0)}{P_{-1}'(x_0)} P_{-1}(x), & j = 0, 1 \\ P_j(x), & j = 2, 3, \ldots, n-3 \\ P_j(x) - \dfrac{P_j'(x_n)}{P_n'(x_n)} P_n(x), & j = n-2, n-1 \end{cases} \tag{13}$$

Now the new basis functions for the approximation y(x) are {$Q_j(x)$, j=0, 1,…, n-1} and they are in number match with the number of selected collocated points. Since the approximation for y(x) in (11) is a quartic approximation, let us approximate $y^{(4)}$ at the selected collocation points with finite differences as

$$y_i^{(4)} = \frac{y'''_{i+1} - y'''_i}{h} \quad \text{for } i = 0 \tag{14}$$

$$y_i^{(4)} = \frac{y'''_{i+1} - y'''_{i-1}}{2h} \quad \text{for } i = 1, 2, ..., n-1 \tag{15}$$

where

$$y_i = y(x_i) = w(x_i) + \sum_{j=0}^{n-1} \alpha_j Q_j(x_i) \tag{16}$$

Now applying the collocation method to (1), we get

$$a_0(x_i) y_i^{(4)} + a_1(x_i) y_i + a_2(x_i) y_i + a_3(x_i) y_i + a_4(x_i) y_i = b(x_i) \quad \text{for } i = 0, 1, \cdots, n-1. \tag{17}$$

Substituting (14), (15) and (16) in (17), we get

$$\begin{aligned} &\frac{a_0(x_i)}{h}\left[w'''(x_{i+1}) + \sum_{j=0}^{n-1} \alpha_j Q_j'''(x_{i+1}) - w'''(x_i) - \sum_{j=0}^{n-1} \alpha_j Q_j'''(x_i) \right] \\ &+ a_1(x_i)\left[w'''(x_i) + \sum_{j=0}^{n-1} \alpha_j Q_j'''(x_i) \right] + a_2(x_i)\left[w''(x_i) + \sum_{j=0}^{n-1} \alpha_j Q_j''(x_i) \right] \\ &+ a_3(x_i)\left[w'(x_i) + \sum_{j=0}^{n-1} \alpha_j Q_j'(x_i) \right] + a_0(x_i)\left[w(x_i) + \sum_{j=0}^{n-1} \alpha_j Q_j(x_i) \right] \end{aligned}$$

for i=0. (18)

$$\begin{aligned} &\frac{a_0(x_i)}{2h}\left[w'''(x_{i+1}) + \sum_{j=0}^{n-1} \alpha_j Q_j'''(x_{i+1}) - w'''(x_{i-1}) - \sum_{j=0}^{n-1} \alpha_j Q_j'''(x_{i-1}) \right] \\ &+ a_1(x_i)\left[w'''(x_i) + \sum_{j=0}^{n-1} \alpha_j Q_j'''(x_i) \right] + a_2(x_i)\left[w''(x_i) + \sum_{j=0}^{n-1} \alpha_j Q_j''(x_i) \right] \\ &+ a_3(x_i)\left[w'(x_i) + \sum_{j=0}^{n-1} \alpha_j Q_j'(x_i) \right] + a_0(x_i)\left[w(x_i) + \sum_{j=0}^{n-1} \alpha_j Q_j(x_i) \right] \end{aligned}$$

for i=1,2,…., n-1. (19)

Rearranging the terms and writing the system of equations (18) and (19) in the matrix form, we get

$$\mathbf{A}\alpha = \mathbf{B} \tag{20}$$

where $\mathbf{A} = [a_{ij}]$;

$$\begin{aligned} a_{ij} = &\frac{a_0(x_i)}{h}\left[Q_j'''(x_{i+1}) - Q_j'''(x_i) \right] + a_1(x_i) Q_j'''(x_i) + a_2(x_i) Q_j''(x_i) \\ &+ a_3(x_i) Q_j'(x_i) + a_4(x_i) Q(x_i) \end{aligned}$$

for i= 0; j=0, 1,…, n-1. (21)

$$a_{ij} = \frac{a_0(x_i)}{2h}\left[Q_j'''(x_{i+1}) - Q_j'''(x_{i-1})\right] + a_1(x_i)Q_j'''(x_i) + a_2(x_i)Q_j''(x_i)$$
$$+a_3(x_i)Q_j'(x_i) + a_4(x_i)Q(x_i)$$

for i= 1, 2, …, n-1; j=0, 1,…, n-1. (22)

$$\mathbf{B} = [b_i];$$
$$b_i = \frac{a_0(x_i)}{h}\left[w'''(x_{i+1}) - w'''(x_i)\right] + a_1(x_i)w'''(x_i) + a_2(x_i)w''(x_i)$$
$$+a_3(x_i)w'(x_i) + a_4(x_i)w(x_i)$$

for i= 0. (23)

$$b_i = \frac{a_0(x_i)}{2h}\left[w'''(x_{i+1}) - w'''(x_{i-1})\right] + a_1(x_i)w'''(x_i) + a_2(x_i)w''(x_i)$$
$$+a_3(x_i)w'(x_i) + a_4(x_i)w(x_i)$$

for i= 1, 2,…, n-1. (24)

V. Procedure to find the solution for nodal parameters

The basis function $Q_i(x)$ is defined only in the interval $[x_{i-2}, x_{i+3}]$ and outside of this interval it is zero. Also at the end points of the interval $[x_{i-2}, x_{i+3}]$ the basis function $Q_j(x)$ vanishes. Therefore, $Q_j(x)$ is having non-vanishing values at the mesh points $x_{i-1}, x_i, x_{i+1}, x_{i+2}$ and zero at the other mesh points. The first three derivatives of $Q_j(x)$ also have the same nature at the mesh points as in the case of $Q_j(x)$. Using these facts, we can say that the Thus the stiff matrix $\mathbf{A}$ is a six diagonal band matrix. Therefore, the system of equations (20) is a six band system in α_i's. The nodal parameters α_i's can be obtained by using band matrix solution package. We have used the FORTRAN-90 programming to solve the boundary value problem (1)-(2) by the proposed method

VI. Numerical results

To demonstrate the applicability of the proposed method for solving the fourth order boundary value problems of the type (1) and (2), we considered two linear and three nonlinear boundary value problems. The obtained numerical results for each problem are presented in tabular forms and compared with the exact solutions available in the literature.

Example 1: Consider the linear boundary value problem

$$y^{(4)} - y'' - y = e^x(x-3), \quad 0 < x < 1 \tag{25}$$

subject to

$$y(0) = 1, \;\; y(1) = 0, \; y'(0) = 0, \; y'(1) = -e.$$

The exact solution for the above problem is $y = (1-x)e^x$.

The proposed method is tested on this problem where the domain [0, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 1. The maximum absolute error obtained by the proposed method is 1.782179 $\times 10^{-5}$.

Example 2: Consider the linear boundary value problem

$$y^{(4)} + x^4 y''' + sinx\, y'' + e^{-x} y = (1 + x^4 + sinx + e^{-x})e^x, \quad 0 < x < 1 \tag{26}$$

subject to

$$y(0) = 1, y(1) = e, \; y'(0) = 1, \; y'(1) = e.$$

The exact solution for the above problem is $y = e^x$.

The proposed method is tested on this problem where the domain [0, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 2. The maximum absolute error obtained by the proposed method is 5.483627x10^{-6}.

Example 3: Consider the linear boundary value problem

$$y^{(4)} = y^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48, \quad 0 < x < 1 \tag{27}$$

subject to

$$y(0) = 0, y(1) = 1, y'(0) = 0, y'(1) = 1.$$

The exact solution for the above problem is $y = x^5 - 2x^4 + 2x^2.$

The nonlinear boundary value problem (27) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [20] as

$$y_{(n+1)}^{(4)} - [2y_{(n)}]y_{(n+1)} = -x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48 - [y_{(n)}]^2, \ n = 0,1,2,... \tag{28}$$

subject to

$$y_{(n+1)}(0) = 0, \ y_{(n+1)}(1) = 1, \ y'_{(n+1)}(0) = 0, \ y'_{(n+1)}(1) = 1.$$

Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for $y(x)$. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (28). The obtained numerical results for this problem are presented in Table 4. The maximum absolute error obtained by the proposed method is 2.272427x10^{-5}.

Example 4: Consider the nonlinear boundary value problem

$$y^{(4)} = \sin x + \sin^2 x - [y'']^2, \quad 0 < x < 1 \tag{29}$$

subject to

$$y(0) = 0, \ y(1) = \sin 1, \ y'(0) = 1, \ y'(1) = cos1.$$

The exact solution for the above problem is $y = sinx.$

The nonlinear boundary value problem (29) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [20] as

$$y_{(n+1)}^{(4)} + [2y''_{(n)}]y''_{(n+1)} = \sin x + \sin^2 x + [y''_{(n)}]^2, \quad n = 0,1,2,... \tag{30}$$

subject to

$$y_{(n+1)}(0) = 0, \ y_{(n+1)}(1) = \sin 1, \ y'_{(n+1)}(0) = 1, \ y'_{(n+1)}(1) = \cos 1.$$

Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for $y(x)$. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (30). The obtained numerical results for this problem are presented in Table 5. The maximum absolute error obtained by the proposed method is 1.430511x10^{-6}.

Example 5: Consider the nonlinear boundary value problem

$$y^{(4)} - 6e^{-4y} = -12(1 + x)^{-4}, \quad 0 < x < 1 \tag{31}$$

subject to

$$y(0) = 1, \ y'(0) = \ln 2, \ y(1) = 1, \ y'(1) = 0.5.$$

The exact solution for the above problem is $y = \ln(1 + x).$

The nonlinear boundary value problem (31) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [20] as

$$y_{(n+1)}^{(4)} + [24e^{-4y_{(n)}}]y_{(n+1)} = -12(1 + x)^{-4} + e^{-4y_{(n)}}[6 + 24y_{(n)}], \quad n = 0,1,2,... \tag{32}$$

subject to

$$y_{(n+1)}(0) = 0, y_{(n+1)}(1) = \ln 2, y'_{(n+1)}(0) = 1, y'_{(n+1)}(1) = 0.5.$$

Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for y(x). The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (32) . The obtained numerical results for this problem are presented in Table 5. The maximum absolute error obtained by the proposed method is 3.281236x10^{-5}.

VII. Tables

x	Absolute error by the proposed method
0.1	1.251698E-06
0.2	5.602837E-06
0.3	9.953976E-06
0.4	1.507998E-05
0.5	1.782179E-05
0.6	1.728535E-05
0.7	1.341105E-05
0.8	8.076429E-06
0.9	2.712011E-06

Table 1: Numerical results for example 1

X	Absolute error by the proposed method
0.1	5.960464E-07
0.2	1.668930E-06
0.3	4.649162E-06
0.4	4.529953E-06
0.5	4.768372E-06
0.6	4.529953E-06
0.7	5.483627E-06
0.8	3.337860E-06
0.9	7.152557E-07

Table 2: Numerical results for example 2

x	Absolute error by the proposed method
0.1	1.688488E-05
0.2	2.272427E-05
0.3	1.987815E-05
0.4	1.198053E-05
0.5	9.536743E-07
0.6	1.019239E-05
0.7	1.943111E-05
0.8	2.229214E-05
0.9	1.639128E-05

Table 3: Numerical results for example 3

x	Absolute error by the proposed method
0.1	2.756715E-07
0.2	6.705523E-07
0.3	6.258488E-07
0.4	1.192093E-06
0.5	1.430511E-06
0.6	1.370907E-06
0.7	5.364418E-07
0.8	2.980232E-07
0.9	2.980232E-07

Table 4: Numerical results for example 4

x	Absolute error by the Proposed method proposed method
0.1	9.082258E-06
0.2	2.081692E-05
0.3	2.917647E-05
0.4	3.281236E-05
0.5	3.072619E-05
0.6	2.402067E-05
0.7	1.424551E-05
0.8	5.424023E-06
0.9	2.384186E-07

Table 5: Numerical results for example 5

VIII. Conclusions

In this paper, we have developed a collocation method with quartic B-splines as basis functions to solve fourth order boundary value problems. Here we have taken mesh points $x_0, x_1, ..., x_{n-2}, x_{n-1}$ as the collocation points. The quartic B-spline basis set has been redefined into a new set of basis functions which in number match with the number of selected collocation points. The proposed method is applied to solve several number of linear and non-linear problems to test the efficiency of the method. The numerical results obtained by the proposed method are in good agreement with the exact solutions available in the literature. The objective of this paper is to present a simple method to solve a fourth order boundary value problem and its easiness for implementation

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