*IOSR Journal of Mathematics (IOSR-JM)*
*e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 3 Ver. IV (May. - Jun. 2016), PP 42-56*
*www.iosrjournals.org*

# Degree Of Approximation Of Fourier Series Of Functions In Besove Space By Riesz Means

Madhusmita Mohanty
*Department of Mathematics, Utkal University, Vani-Vihar, Bhubaneswar-751004, India*

***Abstract:*** *The paper studies the degree of approximation of Fourier series of functions in Besove space by Riesz Means and this generalizes many known results.*
***AMS classification:*** *41A25;42A24.*
***Keywords:*** *Besove space,Holders space,Lipschitz space, Riesz mean, Fourier series, modulus of smoothness.*

## I. Introduction

Let $f$ be a $2\pi$ periodic function and let $f \in L_p[0,2\pi], p \geq 1.$ The fourier series of $f$ at $x$ is given by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \qquad (1.1)$$

Let $s_n(x)$ denote the nth partial sums of (1.1). We know ([6],p.50)that

$$s_n(x) - f(x) = \frac{1}{\pi}\int_0^{\pi}\phi_x(u)D_n(u)du \qquad (1.2)$$

$$where \;\; \phi_x(u) = f(x+u) + f(x-u) - 2f(x) \quad (1.3)$$

$$D_n(u) = \frac{1}{2} + \sum_{k=0}^{n} cosku = \frac{sin(k+\frac{1}{2})u}{2sin\frac{u}{2}} \quad (1.4)$$

**Riesz Means:**

Let $\sum a_n$ be an infinite series of partial sum $\{s_n\}$. Let $b_n$ be a sequence of numbers such that

$$B_n = b_0 + b_1 + \ldots + b_n, B_n \neq 0 (n \geq 0)$$

Then the transformation

$$t_n = \frac{1}{B_n}\sum_{k=0}^{n} b_k s_k \; (1.5)$$

is called the Riesz means or $(R, b_n)$ means of the series $\sum a_n$ or the sequence $s_n$.

**Modulus of Continuity:**

Let $A = R, R + [a,b] \subset R$ or $T$ (which usually taken to be $R$ with identification of points modulo $2\pi$ ).

The modulus of continuity $w(f,t) = w(t)$ of a function $f$ on $A$ can be defined as

$$w(t) = w(f,t) = \sup_{\substack{|x-y|\leq t, \\ x,y\in A}} |f(x) - f(y)|, t \geq 0.$$

**Modulus of Smoothness:**

The $k^{th}$ order modulus of smoothness [2] of a function $f : A \to R$ is defined by

$$w_k(f,t) = \sup_{0<h\leq t}\{\sup|\Delta_h^k(f,x)| : x, x+kh \in A\}, t \geq 0 \quad (1.6)$$

where

$$\Delta_h^k(f,x)=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}f(x+ih),k\in N. \qquad (1.7)$$

For $k=1, w_1(f,t)$ is called the modulus of continuity of $f$. The function $w$ is continuous at $t=0$ if and only if f is uniformly continuous on A, that is $f\in\tilde{c}(A)$. The $k^{th}$ order modulus of smoothness of $f\in L_p(A), 0<p<\infty$ or of $f\in\tilde{c}(A)$, if $p=\infty$ is defined by

$$w_k(f,t)_p=\sup_{0<h\le t}\|\Delta_h^k(f,\cdot)\|_p, t\ge 0 \qquad (1.8)$$

if $p\ge 1, k=1$, then $w_1(f,t)_p=w(f,t)_p$ is a modulus of continuity (or integral modulus of continuity). If $p=\infty, k=1$ and $f$ is continuous then $w_k(f,t)_p$ reduces to modulus of continuity $w_1(f,t)$ or $w(f,t)$.

**Lipschitz Space:**
If $f\in\tilde{c}(A)$ and

$$w(f,t)=O(t^{\alpha}), 0<\alpha\le 1 \qquad (1.9)$$

then we write $f\in Lip\alpha$. If $w(f,t)=O(t)$ as $t\to 0+$ (in particular (1.9) holds for $\alpha>1$) then $f$ reduces to a constant.

If $f\in L_p(A), 0<p<\infty$ and

$$w(f,t)_p=O(t^{\alpha}), 0<\alpha\le 1 \qquad (1.10)$$

then we write $f\in Lip(\alpha,p), 0<p<\infty, 0<\alpha\le 1$.
The case $\alpha>1$ is of no interest as the function reduces to a constant, whenever

$$w(f,t)_p=O(t) \text{ as } t\to 0+ \qquad (1.11)$$

We note that if $p=\infty$ and $f\in c(A)$, then $Lip(\alpha,p)$ class reduces to $Lip\,\alpha$ class.

**Generalized Lipschitz Space:**
Let $\alpha>0$ and suppose that $k=[\alpha]+1$. For $f\in L_p(A), 0<p<\infty$, if

$$w_k(f,t)=O(t^{\alpha}), t>0 \qquad (1.12)$$

then we write

$$f\in Lip^*(\alpha,p), \alpha>0, 0<p\le\infty \qquad (1.13)$$

and say that $f$ belongs to generalized Lipschitz space. The seminorm is then

$$|f|_{Lip^*(\alpha,L_p)}=\sup_{t>0}(t^{-\alpha}w_k(f,t)_p).$$

It is known ([2], p-52) that the space $Lip^*(\alpha,L_p)$ contains $Lip(\alpha,L_p)$. For $0<\alpha<1$ the spaces coincide, (for $p=\infty$, it is necessary to replace $L_\infty$ by $\tilde{c}$ of uniformly continuous function on A). For $0<\alpha<1$ and $p=1$ the space $Lip^*(\alpha,L_p)$ coincide with $Lip\alpha$.
For $\alpha=1, p=\infty$, we have

$$Lip(1,\tilde{c})=Lip1 \qquad (1.14)$$

but

$$Lip^*(1,\tilde{c})=z \qquad (1.15)$$

is the Zygmund space [5] which is characterized by (1.12) with $k=2$.

**Holder ($H_\alpha$) Space:**

For $0 < \alpha \le 1$, let

$$H_\alpha = \{ f \in C_{2\pi} : w(f,t) = O(t^\alpha) \}. \qquad (1.16)$$

It is known [3] that $H_\alpha$ is a Banach Space with the norm $\|\cdot\|_\alpha$ defined by

$$\| f \|_\alpha = \| f \|_c + \sup_{t>0} t^{-\alpha} w(t), 0 < \alpha \le 1 \quad (1.17)$$

$$\| f \|_0 = \| f \|_c$$

and

$$H_\alpha \subseteq H_\beta \subseteq C_{2\pi}, 0 < \beta \le \alpha \le 1 \qquad (1.18)$$

**$H_{(\alpha,p)}$ Space:**

For $0 < \alpha \le 1$, let

$$H_{(\alpha,p)} = \{ f \in L_p[0,2\pi] : 0 < p \le \infty, w(f,t)_p = O(t^\alpha) \} \qquad (1.19)$$

and introduce the norm $\|\cdot\|_{(\alpha,p)}$ as follows

$$\| f \|_{(\alpha,p)} = \| f \|_p + \sup_{t>0} t^{-\alpha} w(f,t)_p, 0 < \alpha \le 1. \quad (1.20)$$

$$\| f \|_{(0,p)} = \| f \|_p .$$

It is known [1] that $H_{(\alpha,p)}$ is a Banach space for $p \ge 1$ and a complete $p$-normed space for $0 < p < 1$. Also

$$H_{(\alpha,p)} \subseteq H_{(\beta,p)} \subseteq L_p, 0 < \beta \le \alpha \le 1. \quad (1.21)$$

Note that $H_{(\alpha,\infty)}$ is the space $H_\alpha$ defined above.

For study of degree of approximation problems the natural way to proceed to consider with some restrictions on some modulus of smoothness as prescribed in $H_\alpha$ and $H_{(\alpha,p)}$ spaces. As we have seen above only a constant function satisfies Lipschitz condition for $\alpha > 1$. However for generalized Lipschitz class there is no such restriction on $\alpha$. We required a finer scale of smoothness than is provided by Lipschitz class. For each $\alpha > 0$ Besove developed a remarkable technique for restricting modulus of smoothness by introducing a third parameter $q$ (in addition to $p$ on $\alpha$) and applying $\alpha \cdot q$ norms (rather than $\alpha, \infty$ norms) to the modulus of smoothness $w_k(f,\cdot)_p$ of $f$.

**Besove space:**

Let $\alpha > 0$ be given and let $k = [\alpha] + 1$. For $0 < p, q \le \infty$, the Besove space ([2], p-54) $B_q^\alpha(L_p)$ is defined as follows:

$$B_q^\alpha(L_p) = \{ f \in L_p : | f |_{B_q^\alpha(L_p)} = \| w_k(f,\cdot) \|_{(\alpha,q)} \ is\ finite \}$$

where

$$\| w_k(f,\cdot) \|_{(\alpha,q)} = \begin{cases} \left( \int_0^\infty (t^{-\alpha} w_k(f,t)_p)^q \frac{dt}{t} \right)^{\frac{1}{q}}, & 0 < q < \infty \\ \sup_{t>0} t^{-\alpha} w_k(f,t)_p, & q = \infty. \end{cases}$$

It is known ([2], p-55) that $\| w_k(f,\cdot) \|_{(\alpha,q)}$ is a seminorm if $1 \le p, q \le \infty$ and a quasi-seminorm in other cases.

The Besove norm for $B_q^\alpha(L_p)$ is

$$\| f \|_{B_q^\alpha(L_p)} = \| f \|_p + \| w_k(f,\cdot) \|_{(\alpha,q)} \quad (1.22)$$

It is known ([4], p-237) that for $2\pi$ -periodic function $f$, the integral $\left(\int_0^\infty (t^{-\alpha} w_k(f,t)_p)^q \frac{dt}{t}\right)^{\frac{1}{q}}$ is replaced by $\left(\int_0^\pi (t^{-\alpha} w_k(f,t)_p)^q \frac{dt}{t}\right)^{\frac{1}{q}}$.

We know ([2], p-56, [4], p-236) the following inclusion relations.

- For fixed $\alpha$ and $p$
- $B_q^\alpha(L_p) \subset B_{q_1}^\alpha(L_p), q < q_1.$
- For fixed $p$ and $q$
- $B_q^\alpha(L_p) \subset B_q^\beta(L_p), \beta < \alpha.$
- For fixed $\alpha$ and $q$
- $B_q^\alpha(L_p) \subset B_q^\alpha(L_{p_1}), p_1 < p.$

**Special Cases Of Besove Space:**

For $q = \infty, B_\infty^\alpha(L_p), \alpha > 0, p \geq 1$ is same as $Lip^*(\alpha, L_p)$ the generalized Lipschitz space and the corresponding norm $\| \cdot \|_{B_\infty^\alpha(L_p)}$ is given by

$$\| f \|_{B_\infty^\alpha(L_p)} = \| f \|_p + \sup_{t>0} t^{-\alpha} w_k(f,t)_p \quad (1.23)$$

for every $\alpha > 0$ with $k = [\alpha]+1$.

For the special case when $0 < \alpha < 1, B_\infty^\alpha(L_p)$ space reduces to $H_{(\alpha,p)}$ space due to Das et al. [1] and the corresponding norm is given by

$$\| f \|_{B_\infty^\alpha(L_p)} = \| f \|_{(\alpha,p)} = \| f \|_p + \sup_{t>0} t^{-\alpha} w(f,t)_p, 0 < \alpha < 1. \quad (1.24)$$

For $\alpha = 1$, the norm is given by

$$\| f \|_{B_\infty^1(L_p)} = \| f \|_p + \sup_{t>0} t^{-\alpha} w_2(f,t)_p. \quad (1.25)$$

Note that $\| f \|_{B_\infty^1(L_p)}$ is not same as $\| f \|_{(1,p)}$ and the space $B_\infty^1(L_p)$ includes the space $H(1,p), p \geq 1$.

If we further specialize by taking $p = \infty, B_\infty^\alpha, 0 < \alpha < 1$, coincides with $H_\alpha$ space due to Prossodorf [3] and the norm is given by

$$\| f \|_{B_\infty^\alpha(L_\infty)} = \| f \|_\alpha = \| f \|_c + \sup_{t>0} t^\alpha w(f,t), 0 < \alpha < 1. \quad (1.26)$$

For $\alpha = 1, p = \infty$, the norm is given by

$$\| f \|_{B_\infty^1(L_\infty)} = \| f \|_c + \sup_{t>0} t^{-1} w_2(f,t), \alpha = 1 \quad (1.27)$$

which is different from $\| f \|_1$ and $B_\infty^1(L_\infty)$ includes the $H_1$ space.

## II. Main Result

**Theorem:** *Let* $0 < \alpha < 2$ *and* $0 \le \beta < \alpha$. *If* $f \in B_q^{\alpha}(L_p), p \ge 1$ *and* $1 < q \le \infty$. *Let* $T_n(x)$ *be the* $(R, b_n)$ *of Fourier series when* $b_n \ge 0$ *and non-incresaing. Then*

*Case 1: (1<q<∞)*

$$\| T_n(\cdot) \|_{B_q^{\beta}(L_p)} = O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}\left(\frac{B_{k+1}}{k^{\alpha-\beta-\frac{2}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}}$$

*Case 2: (q=∞)*

$$\| T_n(\cdot) \|_{B_\infty^{\beta}(L_p)} = O\left(\frac{1}{B_n}\right)\sum_{k=1}^{n}\frac{B_{k+1}}{k^{\alpha-\beta}}$$

For the proof of the theorem we need the following Lemmas.

**Lemma 2.1** *Let* $1 \le p \le \infty$ *and* $0 < \alpha < 2$. *If* $f \in L_p[0, 2\pi]$, *then for* $0 < t, u \le \pi$

i. $\| \phi(\cdot, t, u) \|_p \le 4 w_k(f, t)_p$

ii. $\| \phi(\cdot, t, u) \|_p \le 4 w_k(f, u)_p$

iii. $\| \phi_{\cdot}(u) \|_p \le 2 w_k(f, u)_p$

**Lemma 2.2** *Let* $0 < \alpha < 2$. *Suppose that* $0 \le \beta < \alpha$. *If* $f \in B_q^{\alpha}(L_p), p \ge 1, 1 < q < \infty$, *then*

i. $$\int_0^{\pi} | K_n(u) | \left(\int_0^{u} \frac{\| \phi(\cdot, t, u) \|_p^q}{t^{\beta q}} \frac{dt}{t}\right)^{\frac{1}{q}} du = O(1)\left\{\int_0^{\pi}\left(u^{\alpha-\beta} | K_n(u) |\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

ii. $$\int_0^{\pi} | K_n(u) | \left(\int_u^{\pi} \frac{\| \phi(\cdot, t, u) \|_p^q}{t^{\beta q}} \frac{dt}{t}\right)^{\frac{1}{q}} du = O(1)\left\{\int_0^{\pi}\left(u^{\alpha-\beta+\frac{1}{q}} | K_n(u) |\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

**Lemma 2.3** *Let* $0 < \alpha < 2$. *Suppose that* $0 \le \beta < \alpha$. *If* $f \in B_q^{\alpha}(L_p), p \ge 1$ *and* $q = \infty$, *then*

$$\sup_{0 < t, u \le \pi} t^{-\beta} \| \phi(\cdot, t, u) \|_p = O(u^{\alpha-\beta})$$

**Lemma 2.4** *Let* $b_n \ge 0$ *and non-increasing and* $K_n(u) = \frac{1}{B_n}\sum_{k=0}^{n} b_k D_k(u)$, *then for* $0 < u \le \pi$ *and* $m = [\frac{\pi}{u}]$

$$K_n(u) = \begin{cases} O(n) \\ O\left(\frac{B_m}{u^2 B_n}\right), & u \ge \frac{\pi}{n}. \end{cases}$$

*Proof.* We know

$$D_k(u) = \frac{1}{2} + \sum_{v=0}^{k} \cos vu = \frac{\sin(k+\frac{1}{2})u}{2\sin\frac{u}{2}} \quad (2.1)$$

$$\Rightarrow |D_k(u)| = \left|\frac{1}{2} + \sum_{v=0}^{k} \cos vu\right|$$

$$\le \frac{1}{2} + \sum_{v=0}^{k} |\cos vu|$$

$$\le k+1 \quad (2.2)$$

Now

$$|K_n(u)| \le \frac{1}{B_n}\sum_{k=0}^{n} b_k |D_k(u)|$$

$$\le \frac{1}{B_n}\sum_{k=0}^{n} b_k (k+1) \; (by\ using\ (2.2)$$

$$= O\left(\frac{n+1}{B_n}\sum_{k=0}^{n} b_k\right)$$

$$= O(n)$$

Again

$$K_n(u) = \frac{1}{B_n}\sum_{k=0}^{n} b_k \frac{\sin(k+\frac{1}{2})u}{2\sin\frac{u}{2}} \quad (by\ using\ (2.1)$$

$$= \frac{1}{B_n 2\sin\frac{u}{2}}\left(\sum_{k=0}^{m} + \sum_{k=m+1}^{n}\right) b_k \sin(k+\frac{1}{2})u$$

$$\Rightarrow |K_n(u)| \le \left|\frac{1}{B_n 2\sin\frac{u}{2}}\right| + \left[\left|\sum_{k=0}^{m}\right| + \left|\sum_{k=m+1}^{n}\right|\right] b_k \sin(k+\frac{1}{2})u$$

$$= O\left(\frac{1}{uB_n}\right) \quad (2.3)$$

$$\left|\sum_{k=0}^{m} b_k \sin(k+\frac{1}{2})u\right| \le \sum_{k=0}^{m} b_k \left|\sin(k+\frac{1}{2})u\right|$$

$$\le \sum_{k=0}^{m} b_k .$$

Again,

$$\left|\sum_{k=m+1}^{n} b_k \sin(k+\frac{1}{2})u\right| \le b_m \max_{m\le n} \sum_{k=m'}^{m} \sin(k+\frac{1}{2})u \quad (\ as\ b_k\ is\ monotonically\ decreasing\ )$$

$$= b_m \cdot \frac{1}{u}$$

From equation (2.3), we have

$$|K_n(u)| \le O\left(\frac{1}{uB_n}\right)\sum_{k=0}^{m} b_k + O\left(\frac{1}{uB_n}\right) b_m \cdot \frac{1}{u}$$

$$= O\left(\frac{B_m}{uB_n}\right) + O\left(\frac{b_m}{u^2 B_n}\right)$$

$$= O\left(\frac{B_m}{u^2 B_n}\right), m \le n$$

$$[\because B_m \ge (u+1)b_m \ge \frac{\pi}{u} b_m]$$

**Proof of the Theorem:**

Case 1: Let $T_n(x)$ be the Riesz transformation of the Fourier series $s_n(x)$, that is by (1.5)

$$T_n(x) = \frac{1}{B_n}\sum_{k=0}^{n} b_k s_k(x) \quad (2.4)$$

$$= \frac{1}{B_n}\sum_{k=0}^{n} b_k \left(\frac{1}{\pi}\int_0^{\pi} \phi_x(u) D_k(u) du + f(x)\right)$$

$$= \frac{1}{\pi B_n}\int_0^{\pi} \phi_x(u) \sum_{k=0}^{n} b_k D_k(u) du + f(x)\frac{1}{B_n}\sum_{k=0}^{n} b_k$$

$$= \frac{1}{\pi}\int_0^{\pi} \phi_x(u) \left\{\frac{1}{B_n}\sum_{k=0}^{n} b_k D_k(u)\right\} du + f(x)$$

$$= \frac{1}{\pi}\int_0^{\pi} \phi_x(u) K_n(u) du \qquad (2.5)$$

$$where\ K_n(u) = \frac{1}{B_n}\sum_{k=0}^{n} b_k D_k(u) \qquad (2.6)$$

We first consider the case $1 < q < \infty$.

We have for $p \ge 1$ and $0 \le \beta < \alpha < 2$, by use of Besove norm defined in (1.24)

$$\| T_n(\cdot) \|_{B_q^{\beta}(L_p)} = \| T_n(\cdot) \|_p + \| w_k(T_n, \cdot) \|_{\beta, q} \qquad (2.7)$$

Applying Lemma 2.1(iii) in equation (2.7), we have

$$\| T_n(\cdot) \|_p \le \frac{1}{\pi}\int_0^{\pi} \| \phi_\cdot(u) \|_p |K_n(u)| du$$

$$\le \frac{1}{\pi}\int_0^{\pi} 2 w_k(f, u)_p |K_n(u)| du$$

$$= \frac{2}{\pi}\int_0^{\pi} |K_n(u)| w_k(f, u)_p du$$

Applying H $o$ lder's inequality, we have

$$\| T_n(\cdot) \|_p \le \frac{2}{\pi}\left\{\int_0^{\pi}\left(|K_n(u)| u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \left\{\int_0^{\pi}\left(\frac{w_k(f, u)_p}{u^{\alpha+\frac{1}{q}}}\right)^{q} du\right\}^{\frac{1}{q}}$$

By definition of Besove Space, we have

$$\| T_n(\cdot) \|_p \leq O(1)\left\{\int_0^{\pi}\left( | K_n(u) | u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$= O(1)\left[\left\{\int_0^{\frac{\pi}{n}}\left( | K_n(u) | u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} + \left\{\int_{\frac{\pi}{n}}^{\pi}\left( | K_n(u) | u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}\right]$$

$$= O(1)[I + J], \quad (say) \qquad (2.8)$$

By using Lemma 2.4 in I of (2.8), we have

$$I = \left\{\int_0^{\frac{\pi}{n}}( | K_n(u) | u^{\alpha+\frac{1}{q}})^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$= O(n)\left\{\int_0^{\frac{\pi}{n}} u^{(\alpha+\frac{1}{q})\cdot\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$= O(n)\left\{\int_0^{\frac{\pi}{n}} u^{\frac{q}{q-1}(\alpha+1)-1} du\right\}^{1-\frac{1}{q}}$$

$$= O\left(\frac{1}{n^{\alpha}}\right) \qquad (2.9)$$

Applying Lemma 2.4 in J of (2.8), we have

$$J = \left\{\int_{\frac{\pi}{n}}^{\pi}\left( | K_n(u) | u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$= O\left(\frac{1}{B_n}\right)\left\{\int_{\frac{\pi}{n}}^{\pi}\left(\frac{B_m}{u^2} u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$= O\left(\frac{1}{B_n}\right)\left\{\int_{\frac{\pi}{n}}^{\pi}\left(B_m u^{\alpha+\frac{1}{q}-2}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$= O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n-1}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}} B_m^{\frac{q}{q-1}} u^{(\alpha+\frac{1}{q}-2)\cdot\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$= O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}} B_m^{\frac{q}{q-1}} u^{\frac{q}{q-1}(\alpha-1)-1} du\right\}^{1-\frac{1}{q}}$$

$$= O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n} B_{k+1}^{\frac{q}{q-1}} \int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}} u^{\frac{q}{q-1}(\alpha-1)-1} du\right\}^{1-\frac{1}{q}}$$

$$= O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n} B_{k+1}^{\frac{q}{q-1}} \frac{1}{k^{\frac{q}{q-1}(\alpha-1)+1}}\right\}^{1-\frac{1}{q}}$$

$$= O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n} \left(\frac{B_{k+1}}{k^{\alpha-\frac{1}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \qquad (2.10)$$

Substitute the value of (2.9) and (2.10) in (2.8), we have

$$\| T_n(\cdot) \|_p = O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n} \left(\frac{B_{k+1}}{k^{\alpha-\frac{1}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \qquad (2.11)$$

By using Besove space, we have

$$\| w_k(T_n,\cdot) \|_{\beta,q} = \left\{\int_0^{\pi} \left(t^{-\beta} w_k(T_n,t)_p\right)^q \frac{dt}{t}\right\}^{\frac{1}{q}}$$

$$= \int_0^{\pi} \left\{\left(\frac{w_k(T_n,t)_p}{t^{\beta}}\right)^q \frac{dt}{t}\right\}^{\frac{1}{q}}$$

From definition of $w_k(T_n,t)_p$, we have

$$w_k(T_n,t)_p = \| T_n(\cdot,t) \|_p$$

$$\le \int_0^{\pi} \left\{\left(\frac{\| T_n(\cdot,t) \|_p}{t^{\beta}}\right)^q \frac{dt}{t}\right\}^{\frac{1}{q}}$$

$$= \left[\int_0^{\pi} \left\{\int_0^{\pi} | T_n(x,t) |^p \, dx\right\}^{\frac{q}{p}} \frac{dt}{t^{\beta q+1}}\right]^{\frac{1}{q}}$$

$$= \left[\int_0^{\pi} \left\{\int_0^{\pi} \left|\int_0^{\pi} \phi(x,t,u) K_n(u) du\right|^p dx\right\}^{\frac{q}{p}} \frac{dt}{t^{\beta q+1}}\right]^{\frac{1}{q}}$$

By repeated application of generalized Minkowski's inequality, we have

$$\| w_k(T_n,\cdot) \|_{\beta,q} \le \frac{1}{\pi}\left[\int_0^{\pi} \left\{\int_0^{\pi} \left(\int_0^{\pi} | \phi(x,t,u) |^p | K_n(u) |^p \, dx\right)^{\frac{1}{p}} du\right\}^q \frac{dt}{t^{\beta q+1}}\right]^{\frac{1}{q}}$$

$$= \frac{1}{\pi}\left[\int_0^\pi \left\{\int_0^\pi |K_n(u)| \| \phi(\cdot,t,u)\|_p \, du\right\}^q \frac{dt}{t^{\beta q+1}}\right]^{\frac{1}{q}}$$

$$\le \frac{1}{\pi}\int_0^\pi |K_n(u)|\, du \left(\int_0^\pi \frac{\|\phi(\cdot,t,u)\|_p^q}{t^{\beta q}}\frac{dt}{t}\right)^{\frac{1}{q}}$$

$$= \frac{1}{\pi}\int_0^\pi |K_n(u)|\, du \left\{\left(\int_0^u + \int_u^\pi\right)\frac{\|\phi(\cdot,t,u)\|_p^q}{t^{\beta q}}\frac{dt}{t}\right\}^{\frac{1}{q}}$$

$$\le \frac{1}{\pi}\int_0^\pi |K_n(u)|\, du \left\{\int_0^u \frac{\|\phi(\cdot,t,u)\|_p^q}{t^{\beta q}}\frac{dt}{t}\right\}^{\frac{1}{q}}$$

$$+ \frac{1}{\pi}\int_0^\pi |K_n(u)|\, du \left\{\int_u^\pi \frac{\|\phi(\cdot,t,u)\|_p^q}{t^{\beta q}}\frac{dt}{t}\right\}^{\frac{1}{q}}$$

$$= O(1)\left[\left\{\int_0^\pi \left(|K_n(u)| u^{\alpha-\beta}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}\right.$$

$$\left. + \left\{\int_0^\pi \left(|K_n(u)| u^{\alpha-\beta+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}\right] (using\ Lemma\ 2.2)$$

$$= O(1)[I' + J'], \quad (say) \qquad (2.12)$$

$$I' = \left\{\int_0^\pi \left(|K_n(u)| u^{\alpha-\beta}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$= \left\{\left(\int_0^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^\pi\right)\left(|K_n(u)| u^{\alpha-\beta}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$\le \left\{\int_0^{\frac{\pi}{n}} \left(|K_n(u)| u^{\alpha-\beta}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} + \left\{\int_{\frac{\pi}{n}}^\pi \left(|K_n(u)| u^{\alpha-\beta}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$= I_1' + I_2', \quad (say) \qquad (2.13)$$

Applying Lemma 2.4 in $I_1'$ and $I_2'$, we have

$$I_1' = \left\{\int_0^{\frac{\pi}{n}} \left(|K_n(u)| u^{\alpha-\beta}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$= O(n)\left\{\int_0^{\frac{\pi}{n}} u^{\alpha-\beta(\frac{q}{q-1})} du\right\}^{1-\frac{1}{q}}$$

$$= O(n)\left\{\int_0^{\frac{\pi}{n}} u^{\frac{q}{q-1}(\alpha-\beta+1-\frac{1}{q})-1} du\right\}^{1-\frac{1}{q}}$$

$$= O(n)\left(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}}\right) \qquad (2.14)$$

$$I_2' = \left\{\int_{\frac{\pi}{n}}^{\pi}\left(|K_n(u)|\, u^{\alpha-\beta}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$= O(\frac{1}{B_n})\left\{\int_{\frac{\pi}{n}}^{\pi}\left(\frac{B_m}{u^2} u^{\alpha-\beta}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$= O(\frac{1}{B_n})\left\{\int_{\frac{\pi}{n}}^{\pi}\left(B_m u^{\alpha-\beta-2}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$= O(\frac{1}{B_n})\left\{\sum_{k=1}^{n}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}\left(B_m u^{\alpha-\beta-2}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

Let $h(u) = (B_m u^{\alpha-\beta-2})^{\frac{q}{q-1}}$ and $H(u)$ is a primitive of $h(u)$, then

$$\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}\left(B_m u^{\alpha-\beta-2}\right)^{\frac{q}{q-1}} du = \int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}} h(u)du$$

$$= H(\frac{\pi}{k}) - H(\frac{\pi}{k+1})$$

$$= \frac{\pi}{k} - \frac{\pi}{k+1} h(c), \quad for\ some\ \frac{\pi}{k+1} < c < \frac{\pi}{k}$$

$$= O(1)\frac{1}{k^2}\left(\frac{B_{k+1}}{k^{\alpha-\beta-2}}\right)^{\frac{q}{q-1}}$$

$$= O\left(\frac{B_{k+1}}{k^{\alpha-\beta-\frac{2}{q}}}\right)^{\frac{q}{q-1}}$$

$$I_2' = O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}\left(\frac{B_{k+1}}{k^{\alpha-\beta-\frac{2}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \qquad (2.15)$$

From (2.13), (2.14) and (2.15), we have

$$I' = O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}\left(\frac{B_{k+1}}{k^{\alpha-\beta-\frac{2}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \qquad (2.16)$$

$$J' = \left\{\int_0^{\pi}\left(|K_n(u)|\,u^{\alpha-\beta+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$= \left\{\left(\int_0^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^{\pi}\right)\left(|K_n(u)|\,u^{\alpha-\beta+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$\le \left\{\int_0^{\frac{\pi}{n}}\left(|K_n(u)|\,u^{\alpha-\beta+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} + \left\{\int_{\frac{\pi}{n}}^{\pi}\left(|K_n(u)|\,u^{\alpha-\beta+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$= \left(J_1^1 + J_2^1\right), \quad (say) \ (2.17)$$

Applying Lemma 2.4 in $J_1^1$ and $J_2^1$ , we have

$$J_1^1 = \left\{\int_0^{\frac{\pi}{n}}\left(|K_n(u)|\,u^{\alpha-\beta+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$= O(n)\left\{\int_0^{\frac{\pi}{n}} u^{\frac{q}{q-1}(\alpha-\beta+\frac{1}{q})} du\right\}^{1-\frac{1}{q}}$$

$$= O(n)\left\{\int_0^{\frac{\pi}{n}} u^{\frac{q}{q-1}(\alpha-\beta+1)-1} du\right\}^{1-\frac{1}{q}}$$

$$= O\left(\frac{1}{n^{\alpha-\beta}}\right) \ (2.18)$$

$$J_2^1 = \left\{\int_{\frac{\pi}{n}}^{\pi}\left(|K_n(u)|\,u^{\alpha-\beta+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$= O\left(\frac{1}{B_n}\right)\left\{\int_{\frac{\pi}{n}}^{\pi}\left(B_m u^{\alpha-\beta-2+\frac{1}{q}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}}$$

$$= O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n-1}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}\left(B_m u^{\alpha-\beta-2+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$= O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}\left(B_m u^{\alpha-\beta-2+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

Proceeding as in $I_2^1$, we have

$$J_2^1 = O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}\left(\frac{B_{k+1}}{k^{\alpha-\beta-\frac{2}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \qquad (2.19)$$

From (2.17), (2.18), (2.19), we have

$$J^1 = O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}\left(\frac{B_{k+1}}{k^{\alpha-\beta-\frac{2}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \qquad (2.20)$$

From (2.12), (2.16) and (2.20), we have

$$\| w_k(T_{n,\cdot})\|_{\beta,q} = O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}\left(\frac{B_{k+1}}{k^{\alpha-\beta-\frac{2}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \qquad (2.21)$$

From (2.21),(2.11) and (2.7), for $p \geq 1,\ 1 < q < \infty$ and $0 \leqq \beta < \alpha < 2$, we have

$$\| T_n(\cdot)\|_{B_q^\beta(L_p)} = \| T_n(\cdot)\|_p + \| w_k(T_n,\cdot)\|_{\beta,q}$$

$$= O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}\left(\frac{B_{k+1}}{k^{\alpha-\beta-\frac{2}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \qquad (2.22)$$

Case 2: Now, we consider the case $q = \infty$

$$\| T_n(\cdot)\|_{B_\infty^\beta(L_p)} = \| T_n(\cdot)\|_p + \| w_k(T_n,\cdot)\|_{\beta,\infty} \qquad (2.23)$$

We know $T_n(x) = \frac{1}{\pi}\int_0^\pi \phi_x(u) K_n(u)du$

Applying Lemma 2.1(iii), we have

$$\| T_n(\cdot)\|_p \leq \frac{1}{\pi}\int_0^\pi \|\phi.(u)\|_p\ K_n(u)du$$

$$\leq \frac{2}{\pi}\int_0^\pi |K_n(u)|\, w_k(f,u)_p\, du$$

$$= O(1)\int_0^\pi |K_n(u)|\, u^\alpha du \quad (\textit{by the hypothesis})$$

$$= O(1)\left[\int_0^{\frac{\pi}{n}} |K_n(u)|\,u^{\alpha}du + \int_{\frac{\pi}{n}}^{\pi} |K_n(u)|\,u^{\alpha}du\right]$$
$$= O(1)[I^{II} + J^{II}], \quad (say) \qquad (2.24)$$

$$I^{II} = \int_0^{\frac{\pi}{n}} |K_n(u)|\,u^{\alpha}du$$
$$= O(n)\int_0^{\frac{\pi}{n}} u^{\alpha}du$$
$$= O\left(\frac{1}{n^{\alpha}}\right) \quad (by\ lemma\ 2.4) \qquad (2.25)$$

$$J^{II} = \int_{\frac{\pi}{n}}^{\pi} |K_n(u)|\,u^{\alpha}du$$
$$= O\left(\frac{1}{B_n}\right)\int_{\frac{\pi}{n}}^{\pi} B_m u^{\alpha-2}du$$
$$= O\left(\frac{1}{B_n}\right)\sum_{k=1}^{n-1}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}} B_m u^{\alpha-2}du$$
$$= O\left(\frac{1}{B_n}\right)\sum_{k=1}^{n}\frac{B_{k+1}}{k^{\alpha}} \qquad (2.26)$$

From (2.24), (2.25) and (2.26), we have

$$\| T_n(\cdot) \|_p = O\left(\frac{1}{B_n}\right)\sum_{k=1}^{n}\frac{B_{k+1}}{k^{\alpha}} \qquad (2.27)$$

Again,

$$\| w_k(T_n,\cdot) \|_{\beta,\infty} = O(1)\left[\int_0^{\frac{\pi}{n}} u^{\alpha-\beta}K_n(u)du + \int_{\frac{\pi}{n}}^{\pi} u^{\alpha-\beta}K_n(u)du\right]$$
$$= O(1)[I^{III} + J^{III}], \quad (say) \qquad (2.28)$$

Using Lemma 2.4 in $I^{III}$ and $J^{III}$ , we have

$$I^{III} = \int_0^{\frac{\pi}{n}} |K_n(u)|\,u^{\alpha-\beta}du$$
$$= O\left(\frac{1}{n^{\alpha-\beta}}\right) \qquad (2.29)$$

$$J^{III} = \int_{\frac{\pi}{n}}^{\pi} u^{\alpha-\beta}|K_n(u)|\,du$$
$$= O\left(\frac{1}{B_n}\right)\int_{\frac{\pi}{n}}^{\pi} B_m u^{\alpha-\beta-2}du$$
$$= O\left(\frac{1}{B_n}\right)\sum_{k=1}^{n}\frac{B_{k+1}}{k^{\alpha-\beta}} \qquad (2.30)$$

From (2.28), (2.29) and (2.30), we have

$$\| w_k(T_n,\cdot) \|_{\beta,\infty} = O\left(\frac{1}{B_n}\right)\sum_{k=1}^{n}\frac{B_{k+1}}{k^{\alpha-\beta}} \qquad (2.31)$$

From (2.23),(2.27) and (2.31), we have

$$\| T_n(\cdot) \|_{B_\infty^\beta (L_p)} = O\left(\frac{1}{B_n}\right) \sum_{k=1}^{n} \frac{B_{k+1}}{k^{\alpha-\beta}} \qquad (2.32)$$

Hence the Theorem follows from (2.22) and (2.32).

## Acknowledgement

I thank Dr. G. Das and Dr. B.K. Ray for their supervision and valuable suggestions.

## References

[1]. Das, G., Ghosh, T. and Ray, B.K: Degree of Approximation of function by their Fourier series in the generalized Holder metric, proc. Indian Acad. Sci. (Math.Sci) 106(2)(1996) 139-153.
[2]. Devore Ronald A. Lorentz, G.: Constructive approximation, Springler- Verlay, Berlin Heidelberg New York, 1993.
[3]. Prossdorf, S.: Zur Konvergenz der Fourier richen Holder stetiger Funktionen math.Nachar, 69(1975),7-14.
[4]. Wojtaszczyk, P.: A Mathematical Introduction to Wevlets, London Mathematical Society students texts 37, Cambridge University Press, New York, 1997.
[5]. Zygmund, A .: Smooth fuctions, Duke math. Jounal 12(1945), 47-56.
[6]. Zygmund,A.: Trigonometric series vols I & II combined, Cambridge Univ. Press, New York, 1993.