IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 3 Ver. IV (May. - Jun. 2016), PP 36-41
www.iosrjournals.org

Harmonic Mean Derivative - Based Closed Newton Cotes Quadrature

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Abstract: *A New method of evaluation of Numerical integration by using Harmonic Mean derivative - based closed Newton cotes quadrature rule (HMDCNC) is presented, in which the Harmonic mean value is used for Computing the function derivative. It has shown that the proposed rule gives increase of single order of precision over the existing closed Newton cotes rule. The error terms are also obtained by using the concept of precision and are compared with the existing methods. Finally, the accuracy of the proposed rule is analyzed using Numerical Examples and the results are compared with the existing methods.*

Keyword: *Closed Newton-Cotes formula, Error terms, Harmonic Mean Derivative, Numerical Examples, Numerical Integration .*

I. Introduction

Numerical integration is the study of how the numerical value of an integral can be found. It has several applications in the field of statistics. In Statistics, it is used to evaluate distribution functions and other quantities. Many recent Statistical Methods are dependent especially on multiple integration possibly in very high dimensions[1].In the field of Mathematics, One of the most common method for the evaluation of numerical integration is quadrature rule given by

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i) \tag{1}$$

Where there are (n+1) distinct points $a=x_0 < x_1 < \dots < x_n=b$, $x_i=x_0+ih$, i=0,1,2,..n, $h = \frac{b-a}{n}$ and (n+1) weights $w_0, w_1, \dots, w_n$ .These w_i can be obtained by using the method based on the precision of a quadrature formula. Select the values for w_i,i=0,1,...n. so that the error is zero, that is

$$E_n[f] = \int_a^b f(x)dx - \sum_{i=0}^{n} w_i f(x_i) = 0\ , for\ f(x) = x^j \quad j = 0,1,\dots n \tag{2}$$

Definition 1. An integration method of the form (1) is said to be of order P, if it produces exact results ($E_n[f] = 0$) for all polynomials of degree less than or equal to P [2].

The list of Closed Newton cotes formulas (CNC) that depend on the integer value of n , are given below,

When n=1 : Trapezoidal rule

$$\int_a^b f(x)dx = \frac{b-a}{2}(f(a)+f(b)) - \frac{(b-a)^3}{12} f''(\xi), \qquad where\ \xi \in (a,b) \tag{3}$$

When n =2: Simpson's 1/3rd rule

$$\int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}(\xi), \qquad where\ \xi \in (a,b) \tag{4}$$

When n =3: Simpson's 3/8th rule

$$\int_a^b f(x)dx = \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{(b-a)^5}{6480} f^{(4)}(\xi), \qquad where\ \xi \in (a,b) \tag{5}$$

When n=4 : Boole's rule

$$\int_a^b f(x)dx = \frac{b-a}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b)\right] - \frac{(b-a)^7}{1935360} f^{(6)}(\xi), \qquad where\ \xi \in (a,b) \quad (6)$$

It is known that the degree of precision is n+1 for even value of n and n for odd value of n.

There are so many methods are available to increase the order of accuracy in the closed Newton cotes formula. Dehghan et al., increased the order of accuracy in the closed Newton cotes formula[3] by including the location of boundaries of the interval as two additional parameter, and rescaling the original integral to fit the optimal boundary locations. They have applied this technique to open , semi-open , Gauss legendre and Gauss Chebyshev integration rules[4,5,6,7]. Burg has proposed a derivative based closed ,open and Midpoint quadrature rules[8,9,10]. Weijing Zhao and Hongxing Li took a different approach by introducing a Midpoint technique at the computation of derivative[11]. Recently, we proposed Midpoint derivative based open Newton cotes Quadrature rule[12] and Geometric mean derivative based closed Newton cotes Quadrature rule[13].

In this paper, the use of Harmonic mean derivative at the end points is investigated in closed Newton cotes quadrature formula. This new scheme gives a single order of precision than the existing formula. The error terms are obtained and compared with the existing method. The accuracy of the proposed rule is observed by illustrating the Numerical examples.

II. Harmonic Mean derivative -based Closed Newton Cotes Quadrature rule

To evaluate the definite integral, a new method of evaluation of Harmonic mean derivative - based Closed Newton Cotes formulas are derived. In this method, the Harmonic mean derivative is zero if either a=0 or b=0.That is, this method is not applicable if either a=0 or b=0.

Theorem 2.1. Closed Trapezoidal Rule (n=1) using Harmonic Mean derivative is

$$\int_a^b f(x)dx \approx \frac{b-a}{2}(f(a)+f(b)) - \frac{(b-a)^3}{12} f''\left(\frac{2ab}{a+b}\right), \qquad (7)$$

The precision of this method is 2.

Proof: Since the rule (3) has the degree of precision 1. Now use the rule (7) for $f(x) = x^2$.

$$\text{When } f(x) = x^2\ , \quad \int_a^b x^2\, dx = \frac{1}{3}(b^3 - a^3);$$

$$[n=1] \Rightarrow \frac{b-a}{2}(a^2+b^2) - \frac{2(b-a)^3}{12} = \frac{1}{3}(b^3 - a^3).$$

It shows that the solution is Exact. Therefore, the precision of Closed Trapezoidal Rule with Harmonic Mean derivative is 2.

Theorem 2.2. Closed Simpson's1/3rd Rule with Harmonic Mean derivative (n=2) is

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}\left(\frac{2ab}{a+b}\right), \qquad (8)$$

The precision of this method is 4.

Proof: Since the rule (4) has the degree of precision 3.Now use the rule (8) for $f(x) = x^4$.

$$\text{When } f(x) = x^4\ , \int_a^b x^4\, dx = \frac{1}{5}(b^5 - a^5);$$

$$[n=2] \Rightarrow \left(\frac{b-a}{6}\right)\left[a^4 + 4\left(\frac{a+b}{2}\right)^4 + b^4\right] - \frac{24(b-a)^5}{2880} = \frac{1}{5}(b^5 - a^5).$$

It shows that the solution is Exact. Therefore, the precision of Closed Simpson's1/3rd Rule with Harmonic Mean derivative is 4.

Theorem 2.3. Closed Simpson's3/8rd Rule with Harmonic Mean derivative (n=3) is

$$\int_a^b \mathrm{f(x)dx} \approx \frac{\mathrm{b-a}}{8}\left[\mathrm{f(a)} + 3\mathrm{f}\left(\frac{2\mathrm{a}+\mathrm{b}}{3}\right) + 3\mathrm{f}\left(\frac{\mathrm{a}+2\mathrm{b}}{3}\right) + \mathrm{f(b)}\right] - \frac{(b-a)^5}{6480} f^{(4)}\left(\frac{2ab}{a+b}\right), \quad (9)$$

The precision of this method is 4.

Proof: Since the rule (5) has the degree of precision 3.Now use the rule (9) for f(x) = x^4.

$$\text{When } f(x) = x^4, \int_a^b x^4\,dx = \frac{1}{5}(b^5 - a^5);$$

$$[n=3] \Rightarrow \left(\frac{b-a}{8}\right)\left[a^4 + 3\left(\frac{2a+b}{3}\right)^4 + 3\left(\frac{a+2b}{3}\right)^4 + b^4\right] - \frac{24(b-a)^5}{6480} = \frac{1}{5}(b^5 - a^5).$$

It shows that the solution is Exact. Therefore, the precision of Closed Simpson's3/8rd Rule with Harmonic Mean derivative is 4.

Theorem 2.4. Closed Boole's Rule with Harmonic Mean derivative (n=4) is

$$\int_a^b \mathrm{f(x)dx} \approx \frac{\mathrm{b-a}}{90}\left[7\mathrm{f(a)} + 32\mathrm{f}\left(\frac{3\mathrm{a}+\mathrm{b}}{4}\right) + 12\mathrm{f}\left(\frac{\mathrm{a}+\mathrm{b}}{2}\right) + 32\mathrm{f}\left(\frac{\mathrm{a}+3\mathrm{b}}{4}\right) + 7\mathrm{f(b)}\right] - \frac{(b-a)^7}{1935360} f^{(6)}\left(\frac{2ab}{a+b}\right) \quad (10)$$

The precision of this method is 6.

Proof: Since the rule (6) has the degree of precision 5.Now use the rule (10) for f(x) = x^6.

$$\text{When } f(x) = x^6, \int_a^b x^6\,dx = \frac{1}{7}(b^7 - a^7);$$

$$[n=4] \Rightarrow \left(\frac{b-a}{90}\right)\left[7a^6 + 32\left(\frac{3a+b}{4}\right)^6 + 12\left(\frac{a+b}{2}\right)^6 + 32\left(\frac{a+3b}{4}\right)^6 + 7b^6\right] - \frac{720(b-a)^7}{1935360} = \frac{1}{7}(b^7 - a^7).$$

It shows that the solution is Exact. Therefore, the precision of Closed Boole's Rule with Mean derivative is 6

III. The Error terms of Harmonic Mean derivative -based Closed Newton Cotes Quadrature rule

The error of approximation for the method based on the precision of a quadrature formula is obtained by using the difference between the quadrature formula for the monomial $\frac{x^{p+1}}{(p+1)!}$ and the exact result $\frac{1}{(p+1)!}\int_a^b x^{p+1}\,dx$ where p is the precision of the quadrature formula.

Theorem 3.1. Harmonic Mean derivative-based Closed Trapezoidal Rule (n=1)with the error term is

$$\int_a^b \mathrm{f(x)dx} \approx \frac{\mathrm{b-a}}{2}(\mathrm{f(a)} + \mathrm{f(b)}) - \frac{(b-a)^3}{12} f''\left(\frac{2ab}{a+b}\right) - \frac{(b-a)^5}{24(a+b)} f^{(4)}(\xi), \quad (11)$$

where $\xi \in$ (a, b).This is fifth order accurate with the error term

$$E_1[f] = -\frac{(b-a)^5}{24(a+b)} f^{(4)}(\xi).$$

Proof:

$$\text{Let } f(x) = \frac{x^3}{3!}, \quad \frac{1}{3!}\int_a^b x^3\,dx = \frac{1}{24}(b^4 - a^4);$$

$$\frac{\mathrm{b-a}}{2}(\mathrm{f(a)} + \mathrm{f(b)}) - \frac{(b-a)^3}{12} f''\left(\frac{2ab}{a+b}\right) = \frac{b-a}{12(a+b)}((b^3 + a^3)(a+b) - (b-a)^2(2ab)),$$

Therefore,

$$\frac{1}{24}(b^4 - a^4) - \frac{b-a}{12(a+b)}((b^3 + a^3)(a+b) - 2(b-a)^2 ab) = -\frac{(b-a)^5}{24(a+b)}.$$

Therefore the Error term is,

$$E_1[f] = -\frac{(b-a)^5}{24(a+b)} f^{(4)}(\xi).$$

Theorem 3.2. Harmonic Mean derivative-based Closed Simpson's 1/3rd Rule (n=2)with the error term is

$$\int_a^b \mathrm{f(x)dx} \approx \frac{\mathrm{b-a}}{6}\left[\mathrm{f(a)} + 4\mathrm{f}\left(\frac{\mathrm{a+b}}{2}\right) + \mathrm{f(b)}\right] - \frac{(b-a)^5}{2880} f^{(4)}\left(\frac{2ab}{a+b}\right) - \frac{(b-a)^7}{5760(a+b)} f^{(6)}(\xi) \qquad (12)$$

where $\xi \in$ (a, b).This is seventh order accurate with the error term

$$E_2[f] = -\frac{(b-a)^7}{5760(a+b)} f^{(6)}(\xi).$$

Proof:

$$\text{Let } f(x) = \frac{x^5}{5!}, \qquad \frac{1}{5!}\int_a^b x^5\,dx = \frac{1}{720}(b^6 - a^6);$$

$$\frac{\mathrm{b-a}}{6}\left[\mathrm{f(a)} + 4\mathrm{f}\left(\frac{\mathrm{a+b}}{2}\right) + \mathrm{f(b)}\right] - \frac{(b-a)^5}{2880} f^{(4)}\left(\frac{2ab}{a+b}\right)$$
$$= \frac{b-a}{5760(a+b)}\big((8a^5 + (a+b)^5 + 8b^5)(a+b) - 4(b-a)^4 ab\big),$$

Therefore,

$$\frac{1}{720}(b^6 - a^6) - \frac{b-a}{5760(a+b)}\big((8a^5 + (a+b)^5 + 8b^5)(a+b) - 4(b-a)^4 ab\big) = -\frac{(b-a)^7}{5760(a+b)}.$$

Therefore the Error term is,

$$E_2[f] = -\frac{(b-a)^7}{5760(a+b)} f^{(6)}(\xi).$$

Theorem 3.3. Harmonic Mean derivative-based Closed Simpson's 3/8th Rule (n=3) with the error term is

$$\int_a^b \mathrm{f(x)dx} \approx \frac{\mathrm{b-a}}{8}\left[\mathrm{f(a)} + 3\mathrm{f}\left(\frac{\mathrm{2a+b}}{3}\right) + 3\mathrm{f}\left(\frac{\mathrm{a+2b}}{3}\right) + \mathrm{f(b)}\right] - \frac{(b-a)^5}{6480} f^{(4)}\left(\frac{2ab}{a+b}\right)$$
$$- \frac{(b-a)^7}{12960(a+b)} f^{(6)}(\xi), \qquad (13)$$

Where $\xi \in$ (a, b).This is seventh order accurate with the error term

$$E_3[f] = -\frac{(b-a)^7}{12960(a+b)} f^{(6)}(\xi)$$

Proof:

$$\text{Let } f(x) = \frac{x^5}{5!}, \qquad \frac{1}{5!}\int_a^b x^5\,dx = \frac{1}{720}(b^6 - a^6);$$

$$\frac{\mathrm{b-a}}{8}\left[\mathrm{f(a)} + 3\mathrm{f}\left(\frac{\mathrm{2a+b}}{3}\right) + 3\mathrm{f}\left(\frac{\mathrm{a+2b}}{3}\right) + \mathrm{f(b)}\right] - \frac{(b-a)^5}{6480} f^{(4)}\left(\frac{2ab}{a+b}\right)$$
$$= \frac{b-a}{12960(a+b)}((81a^5 + (2a+b)^5 + (a+2b)^5 + 81b^5)(a+b) - 24(b-a)^4 ab),$$

Therefore,

$$\frac{1}{720}(b^6 - a^6) - \frac{b-a}{12960(a+b)}((81a^5 + (2a+b)^5 + (a+2b)^5 + 81b^5)(a+b) - 24(b-a)^4 ab)$$
$$= -\frac{(b-a)^7}{12960(a+b)}.$$

Therefore the Error term is,

$$E_3[f] = -\frac{(b-a)^7}{12960(a+b)} f^{(6)}(\xi).$$

Theorem 3.4. Harmonic Mean derivative-based Closed Boole's rule (n=4) with the error term is

$$\int_a^b f(x)dx \approx \frac{b-a}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b)\right] - \frac{(b-a)^7}{1935360} f^{(6)}\left(\frac{2ab}{a+b}\right)$$
$$- \frac{(b-a)^9}{3870720(a+b)} f^{(8)}(\xi), \qquad (14)$$

Where $\xi \in (a, b)$.This is ninth order accurate with the error term

$$E_4[f] = -\frac{(b-a)^9}{3870720(a+b)} f^{(8)}(\xi).$$

Proof:

$$\text{Let } f(x) = \frac{x^7}{7!}, \quad \frac{1}{7!}\int_a^b x^7\,dx = \frac{1}{40320}(b^8 - a^8);$$

$$\frac{b-a}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b)\right] - \frac{(b-a)^7}{1935360} f^{(6)}\left(\frac{2ab}{a+b}\right)$$
$$= \frac{b-a}{7!.768}((97a^7 + 91a^6b + 105a^5b^2 + 91a^4b^3 + 91a^3b^4 + 105a^2b^5 + 91ab^6 + 97b^7)(a+b)$$
$$-4(b-a)^6ab),$$

Therefore

$$\frac{1}{40320}(b^8 - a^8) - \frac{b-a}{7!.768}((97a^7 + 91a^6b + 105a^5b^2 + 91a^4b^3 + 91a^3b^4 + 105a^2b^5 + 91ab^6$$
$$+97b^7)(a+b) - 4(b-a)^6ab),$$

$$= -\frac{(b-a)^9}{3870720(a+b)}.$$

Therefore the Error term is,

$$E_4[f] = -\frac{(b-a)^9}{3870720(a+b)} f^{(8)}(\xi).$$

The summary of Precision, the orders and the error terms for Harmonic mean derivative based Closed Newton- Cotes Quadrature are shown in Table 1.

Table 1: Comparison of Error terms

Rules	Precision	Order	Error terms
Trapezoidal rule (n=1)	2	5	$-\frac{(b-a)^5}{24(a+b)} f^{(4)}(\xi)$
Simpson's 1/3rd rule (n=2)	4	7	$-\frac{(b-a)^7}{5760(a+b)} f^{(6)}(\xi)$
Simpson's 3/8th rule (n=3)	4	7	$-\frac{(b-a)^7}{12960(a+b)} f^{(6)}(\xi).$
Boole's rule (n=4)	6	9	$-\frac{(b-a)^9}{3870720(a+b)} f^{(8)}(\xi)$

IV. Numerical Results

In this section, The values of $\int_1^2 e^x\,dx$ and $\int_1^2 \frac{dx}{1+x}$are estimated using the Harmonic Mean derivative - based closed Newton cotes formula and the results are compared with the existing closed Newton-Cotes quadrature formula. The comparisons are shown in Table 2 and 3.
we know that
Error = |Exact value - Approximate value|

Example 4.1:solve $\int_1^2 e^x\,dx$ and compare the solutions with the CNC and HMDCNC rules.

Solution: Exact value of $\int_1^2 e^x dx$=4.67077427

Table 2: Comparison of CNC and HMDCNC rules

value of n	CNC		HMDCNC	
	Approximate value	Error	Approximate value	Error
n=1	5.053668964	0.382894694	4.737529973	0.066755703
n=2	4.672349035	0.001574765	4.671031784	0.000257519
n=3	4.671476470	0.000702200	4.670891027	0.000116757
n=4	4.670776607	0.000002337	4.670774647	0.000000371

Example 2: solve $\int_1^2 \frac{dx}{1+x}$ and compare the solutions with the CNC and HMDCNC rules.

Solution: Exact value of $\int_1^2 \frac{dx}{1+x}$=0.405465108

Table 3 : Comparison of CNC and HMDCNC rules

Value of n	CNC		HMDCNC	
	Approximate value	Error	Approximate value	Error
n=1	0.416666667	0.011201559	0.403547132	0.001917976
n=2	0.405555556	0.000090448	0.405435069	0.000030039
n=3	0.405505952	0.000040844	0.405452402	0.000012706
n=4	0.405465768	0.000000660	0.405464780	0.000000328

V. Conclusion

In this paper, a new scheme of Harmonic mean derivative - based Closed Newton-Cotes quadrature formulas were presented along with their error terms. This Harmonic mean derivative value is included in the existing formula and the error terms were derived by using the difference between the quadrature formula for the monomials and the exact results. This proposed scheme gives a single order of precision than the existing formula. Finally, the accuracy of the proposed scheme is illustrated with a Numerical examples.

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