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A New Instability Result for a Certain System of Fourth-Order Non-Linear Delay Differential Equation

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Abstract: *Our goal in this work, is to obtain sufficient conditions for the instability of the zero solution for a certain fourth-order vector delay differential equation on the following form*

$$X^{(4)} + \Psi(\ddot{X})\dddot{X} + G(X, \dot{X}, \ddot{X}, \dddot{X})\ddot{X} + H(\dot{X}(t-\tau)) + F(X(t-\tau)) = 0.$$

An example is given to illustrate our main result.

Keywords and phrases: *Instability, Lyapunov functional, Fourth-order vector delay differential equations.*

I. Introduction

It is worth-mentioning that, in the literature, specific methods have been developed to obtain information on the qualitative behavior of solutions of delay differential equations, when there is no analytical expression for the solutions. One of those methods is known as Lyapunov's second method, which has been successfully used and is still being used to obtain stability, instability, boundedness and existence of periodic solutions for ordinary differential equations, delay differential equations and functional differential equations. Study on the unstable behavior of solutions for various certain scalar and vector, linear and nonlinear delay differential equations of fourth-order is a new topic. Since we have not been able to locate many results on the instability of solutions of certain nonlinear vector delay differential equations of the fourth-order. From those results in the literature see, e.g., Ezeilo [2, 3], Li and Yu [6], Lu and Liao [7], Sadek [8, 9], Tiryaki [10], Tunç [11-16].

Now we summarize some of few papers focused on the instability of nonlinear differential equations of fourth-order with or without delay. First in 2003 Sadek [9] considered the vector differential equation of the fourth-order

$$X^{(4)} + A\dddot{X} + H(X, \dot{X}, \ddot{X}, \dddot{X})\ddot{X} + G(X)\dot{X} + F(X) = 0.$$

Second in 2004 Tunç [11] was concerned with the study of the instability of the zero solution of fourth-order vector differential equation of the form

$$X^{(4)} + \Psi(\ddot{X})\dddot{X} + \Phi(\dot{X})\ddot{X} + H(\dot{X}) + F(X) = 0.$$

Later in 2011 Tunç [14] studied the instability of the solutions for the fourth-order nonlinear vector differential equation

$$X^{(4)} + A\dddot{X} + H(\dot{X})\ddot{X} + G(X)\dot{X} + F(X) = 0.$$

Recently in 2013 Tunç [15] established new sufficient conditions, which guarantee the instability of the zero solution for the fourth-order vector delay differential equation of the form

$$X^{(4)} + A\dddot{X} + H(\dot{X})\ddot{X} + G(X)\dot{X} + F(X(t-\tau)) = 0.$$

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Now our purpose in this paper, is to obtain sufficient conditions for the instability of the zero solution $X = 0$ to the following fourth-order vector delay differential equation

$$X^{(4)} + \Psi(\ddot{X})\dddot{X} + G(X, \dot{X}, \ddot{X}, \dddot{X})\ddot{X} + H(\dot{X}(t-\tau)) + F(X(t-\tau)) = 0, \tag{1.1}$$

in which $X \in R^n$; τ is a positive constant and is a fixed delay. H and F are n-vector continuous functions; Ψ and G are continuous $n \times n$-symmetric matrices.

Equation (1.1) represents a system of real fourth-order differential equations with delay

$$x_i^{(4)} + \psi_i(\ddot{x}_1,...,\ddot{x}_n)\dddot{x}_k + \sum_{k=1}^{n} g_{ik}(x_1,...,x_n, \dot{x}_1,...,\dot{x}_n, \ddot{x}_1,...,\ddot{x}_n, \dddot{x}_1,...,\dddot{x}_n)\ddot{x}_k$$
$$+ h_i(\dot{x}_1(t-\tau),...,\dot{x}_n(t-\tau)) + f_i(x_1(t-\tau),...,x_n(t-\tau)) = 0, \ (i = 1,2,...,n).$$

Throughout this paper we consider instead of equation (1.1), the equivalent differential system

$$\dot{X} = Y, \qquad \dot{Y} = Z, \qquad \dot{Z} = W,$$
$$\dot{W} = -\Psi(Z)W - G(X,Y,Z,W)Z - H(Y) - F(X) + \int_{t-\tau}^{t} J_H(Y(s))Z(s)ds \tag{1.2}$$
$$+ \int_{t-\tau}^{t} J_F(X(s))Y(s)ds.$$

Let $J_H(Y)$ and $J_F(X)$ denote the Jacobian matrices corresponding to the functions $H(Y)$ and $F(X)$ respectively, that is $J_H(Y) = (\frac{\partial h_i}{\partial y_j})$ and $J_F(X) = (\frac{\partial f_i}{\partial x_j}), (i, j = 1,2,\ldots,n)$ exist and are continuous, where $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$, $(h_1, h_2, \ldots, h_n)$ and $(f_1, f_2, \ldots, f_n)$ represent X, Y, H and F respectively.

The symbol $\langle X, Y \rangle$ will be used to denote the usual scalar product in R^n for any X, Y in R^n, that is $\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i$. Thus $\langle X, X \rangle = \|X\|^2$. It is well-known that the real symmetric matrix $A = (a_{ij})$, $(i, j = 1,2,\ldots,n)$ is said to be positive-definite, if and only if, the quadratic form $X^T AX$ is positive-definite, where $X \in R^n$ and X^T denotes the transpose of X.

II. Main Result

Consider the general autonomous differential system with finite delay:

$$\dot{\bar{x}} = \bar{F}(\bar{x}_t), \qquad \bar{x}_t = \bar{x}(t+\theta), \quad -r \le \theta \le 0, \ t \ge 0, \tag{2.1}$$

where $\bar{F} : G \to R^n$ is a continuous mapping and maps closed and bounded sets into bounded sets, since G is an open subset of $C = C([-h,0], R^n)$. It follows from these conditions on F that each initial value problem:

$$\dot{\bar{x}} = \bar{F}(\bar{x}_t), \qquad x_0 = \phi \in G,$$

has a unique solution defined on some interval $[0, A), 0 < A \le \infty$. This solution will be denoted by $\bar{x}(\phi)(.)$ so that $x_0(\phi) = \phi.$

In order to prove our main result, we give some basic information, which plays an essential role throughout the paper.

Definition 2.1 The zero solution $x = 0$ of (2.1) is said to be **stable** for $t \geq t_0$, if for each $\varepsilon > 0$ and $t_0 \in R$ there exists a positive constant $\delta = \delta(\varepsilon, t_0)$, such that for $t \geq t_0, \|\phi\| < \delta$ implies that $|x(\phi)(t)| < \varepsilon$ for all $t \geq 0$. Otherwise the zero solution is said to be **unstable** for $t \geq t_0$.

Here we give a basic idea of the method about the instability of solutions of ordinary differential equations. The following theorem is due to Č etaev, LaSalle and Lefschetz 1961 [5].

Theorem 2.1 Instability Theorem of Č etaev: Let Ω be a neighborhood of the origin. Let there be given a function $V(x)$ and region Ω_1 in Ω with the following properties:

- $V(x)$ has continuous first partial derivatives in Ω_1.
- $V(x)$ and $\dot{V}(x)$ are positive in Ω_1.
- At the boundary points of Ω_1 inside $\Omega, V(x) = 0$.
- The origin is a boundary point of Ω_1.

Under these conditions the origin is unstable.

The following theorem will be our main stability result for (1.1).

Theorem 2.2 In addition to the basic assumptions imposed on the functions Ψ, G, H and F appearing in equation (1.1), we assume that there exist non-zero constants a_1, a_2, a_4, a_4', a_5 and μ such that the following conditions hold:

(i) $F(0) = 0, F(X) \neq 0, X \neq 0, J_F(X)$ *is symmetric and* $a_4' \leq \lambda_i(J_F(X)) \leq a_4$.

(ii) $H(0) = 0, H(Y) \neq 0, Y \neq 0, J_H(Y)$ *is symmetric and* $0 < \lambda_i(J_H(Y)) \leq a_5$.

(iii) $G(X,Y,Z,W)$ *is symmetric and* $\lambda_i(G(X,Y,Z,W)) \leq -a_2$, *for all* $X,Y,Z,W \in R^n, (i = 1,2,\ldots,n)$.

(iv) $\lambda_i(\Psi(Z)) \geq a_1$, *for all* $Z \in R^n$.

Then the zero solution $X = 0$ of the system (1.2) is unstable provided that

$$\tau < \min \left\{ \frac{1}{2}\sqrt{\frac{a_1}{\mu}}, \frac{a_4' - \frac{1}{4}}{a_4\sqrt{n}}, \frac{1 + a_2}{\sqrt{n}(2a_5 + a_4)} \right\}.$$

The following two lemmas are important for proving Theorem 2.2.

Lemma 2.1 Let A be a real symmetric $n \times n$-matrix and

$$\alpha' \geq \lambda_i(A) \geq \alpha > 0 \quad (i = 1,2,\ldots,n),$$

where α', α are constants. Then

$$\alpha'\langle X, X\rangle \geq \langle AX, X\rangle \geq \alpha\langle X, X\rangle,$$

$$\alpha'^2\langle X, X\rangle \geq \langle AX, AX\rangle \geq \alpha^2\langle X, X\rangle.$$

For a proof of the above lemma, see Bellman [1].

Lemma 2.2 Assume that $\dot{X} = Y, \dot{Y} = Z$ and $\dot{Z} = W$. Then

1. $\frac{d}{dt}\int_0^1 \langle F(\sigma X), X\rangle d\sigma = \langle F(X), Y\rangle.$

2. $\frac{d}{dt}\int_0^1 \langle H(\sigma Y), Y\rangle d\sigma = \langle H(Y), Z\rangle.$

3. $\frac{d}{dt}\int_0^1 \langle \sigma\Psi(\sigma Z)Z, Z\rangle d\sigma = \langle \Psi(Z)W, Z\rangle.$

Proof. The proof of (2) is similar to that of (1).

Preliminaries. The proof of Theorem 2.2 is based on the instability criterion created by Krasovskii [4]. According to these criteria it is necessary to show that there exists a continuously differentiable function $V(X_t, Y_t, Z_t, W_t)$, which has the following Krasovskii properties:

1. In every neighborhood of $(0,0,0,0)$ there exists a point (ξ, η, ζ, ρ) such that $V(\xi, \eta, \zeta, \rho) > 0$.
2. The time derivative $\dot{V} = \frac{d}{dt}V(X_t, Y_t, Z_t, W_t)$ along solution paths of the system (1.2) is positive-semi definite.
3. The only solution $(X, Y, Z, W) = (X(t), Y(t), Z(t), W(t))$ of the system (1.2), which satisfies $\dot{V}(X_t, Y_t, Z_t, W_t) = 0$, $(t \geq 0)$ is the trivial solution $(0,0,0,0)$.

Since the zero solution of (1.2) is isolated, the existence of a function V with the properties (K1), (K2) and (K3) is sufficient for the instability of the trivial solution of (1.2).

III. Proof of Theorem 2.2

We define the Lyapunov-Krasovskii functional $V = V(X_t, Y_t, Z_t, W_t)$ as the following:

$$\begin{aligned} V(X_t, Y_t, Z_t, W_t) = \quad & \langle Y, Z\rangle + \langle Z, W\rangle + \langle Y, F(X)\rangle + \int_0^1 \langle H(\sigma Y, Y\rangle d\sigma \\ & + \int_0^1 \langle \sigma\Psi(\sigma Z)Z, Z\rangle d\sigma - \lambda\int_{-\tau}^0 \int_{t+s}^t \|Y(\theta)\|^2 d\theta ds \\ & - \mu\int_{-\tau}^0 \int_{t+s}^t \|Z(\theta)\|^2 d\theta ds, \end{aligned} \tag{3.1}$$

where s is a real variable such that the integrals $\int_{-\tau}^0 \int_{t+s}^t \|Y(\theta)\|^2 d\theta ds$ and $\int_{-\tau}^0 \int_{t+s}^t \|Z(\theta)\|^2 d\theta ds$ are non-negative; λ and μ are some positive constants, which will be determined later in the proof.

It is clear from (3.1) that $V(0,0,0,0) = 0$ and obviously, it follows from assumption (iv) of Theorem 2.2 and (3.1) that

$$V(0,0,\varepsilon,0) \geq \frac{a_1}{2}\langle \varepsilon, \varepsilon\rangle - \mu\|\varepsilon\|^2 \frac{\tau^2}{2} = \frac{1}{2}(a_1 - \mu\tau^2)\|\varepsilon\|^2 > 0,$$

for all arbitrary $\varepsilon \in R^n, \varepsilon \neq 0$. Thus in every neighborhood of $(0,0,0,0)$ there exists a point (ξ, η, ζ, ρ) such that $V(\xi, \eta, \zeta, \rho) > 0$, for all ξ, η, ζ and ρ in R^n. Hence V has the property (K1) of Krasovskii [4].

Next, let $(X,Y,Z,W)=(X(t),Y(t),Z(t),W(t))$ be an arbitrary solution of the system (1.2). Then the total derivative of the function V with respect to t along this solution path is

$$\begin{aligned}\dot{V} &= \frac{d}{dt}V(X_t,Y_t,Z_t,W_t)=\langle Z,Z\rangle+\langle Y,W\rangle+\langle W,W\rangle-\langle G(X,Y,Z,W)Z,Z\rangle\\ &+\langle Z,\int_{t-\tau}^{t}J_H(Y(s))Z(s)ds\rangle+\langle Z,\int_{t-\tau}^{t}J_F(X(s))Y(s)ds\rangle+\langle Y,J_F(X)Y\rangle\\ &-\mu\tau\|Z\|^2+\mu\int_{t-\tau}^{t}\|Z(\theta)\|^2d\theta-\lambda\tau\|Y\|^2+\lambda\int_{t-\tau}^{t}\|Y(\theta)\|^2d\theta. \qquad (3.2)\end{aligned}$$

Under the assumptions (i) and (ii) of the theorem and by using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\langle Z,\int_{t-\tau}^{t}J_F(X(s))Y(s)ds\rangle &\geq -\|Z\|\left\|\int_{t-\tau}^{t}J_F(X(s))Y(s)ds\right\|\\ &\geq -\sqrt{n}a_4\|Z\|\left\|\int_{t-\tau}^{t}Y(s)ds\right\|\\ &\geq -\sqrt{n}a_4\|Z\|\int_{t-\tau}^{t}\|Y(s)\|ds\\ &\geq -\frac{1}{2}\sqrt{n}a_4\tau\|Z\|^2-\frac{1}{2}\sqrt{n}a_4\int_{t-\tau}^{t}\|Y(s)\|^2ds,\end{aligned}$$

also

$$\begin{aligned}\langle Z,\int_{t-\tau}^{t}J_H(Y(s))Z(s)ds\rangle &\geq -\|Z\|\left\|\int_{t-\tau}^{t}J_H(Y(s))Z(s)ds\right\|\\ &\geq -\sqrt{n}a_5\|Z\|\left\|\int_{t-\tau}^{t}Z(s)ds\right\|\\ &\geq -\sqrt{n}a_5\|Z\|\int_{t-\tau}^{t}\|Z(s)\|ds\\ &\geq -\frac{1}{2}\sqrt{n}a_5\tau\|Z\|^2-\frac{1}{2}\sqrt{n}a_5\int_{t-\tau}^{t}\|Z(s)\|^2ds.\end{aligned}$$

Hence it follows that

$$\begin{aligned}\dot{V}(X_t,Y_t,Z_t,W_t)\geq\ &\langle Z,Z\rangle+\langle Y,W\rangle+\langle W,W\rangle-\langle G(X,Y,Z,W)Z,Z\rangle\\ &+\langle Y,J_F(X)Y\rangle-\frac{1}{2}\sqrt{n}a_4\tau\|Z\|^2-\frac{1}{2}\sqrt{n}a_5\tau\|Z\|^2\\ &-\lambda\tau\|Y\|^2+(\lambda-\frac{1}{2}\sqrt{n}a_4)\int_{t-\tau}^{t}\|Y(\theta)\|^2d\theta\\ &-\mu\tau\|Z\|^2+(\mu-\frac{1}{2}\sqrt{n}a_5)\int_{t-\tau}^{t}\|Z(\theta)\|^2d\theta.\end{aligned}$$

If we let $\lambda=\frac{1}{2}\sqrt{n}a_4$ and $\mu=\frac{1}{2}\sqrt{n}a_5$, then we get

$$\begin{aligned}\dot{V}(X_t,Y_t,Z_t,W_t) &\geq\{(a_4'-\frac{1}{4})-a_4\frac{\sqrt{n}}{2}\tau\}\|Y\|^2+\{(1+a_2)-\frac{\sqrt{n}}{2}(2a_5+a_4)\tau\}\|Z\|^2\\ &\quad+\left\|W+\frac{Y}{2}\right\|^2\\ &\geq\{(a_4'-\frac{1}{4})-a_4\frac{\sqrt{n}}{2}\tau\}\|Y\|^2+\{(1+a_2)-\frac{\sqrt{n}}{2}(2a_5+a_4)\tau\}\|Z\|^2\\ &>0, \qquad (3.3)\end{aligned}$$

provided that $\tau <$ min $\left\{ \frac{1}{2}\sqrt{\frac{a_1}{\mu}}, \frac{a_4' - \frac{1}{4}}{a_4\sqrt{n}}, \frac{1+a_2}{\sqrt{n}(2a_5+a_4)} \right\}$, which verifies that V has the property (K2) of Krasovskii [4].

On the other hand, it follows that

$$\frac{d}{dt}V(X_t, Y_t, Z_t, W_t) = 0 \Leftrightarrow Y = \dot{X} = 0, Z = \dot{Y} = 0, W = \dot{Z} = 0, \text{ for all } t \geq 0,$$

which implies that

$$X = \xi, Y = Z = W = 0.$$

Substituting foregoing estimates in the system (1.2), we get that $F(\xi) = 0$, which necessarily implies that $\xi = 0$ since $F(0) = 0, F(X) \neq 0$ for all $X \neq 0$, besides by $H(0) = 0, H(Y) \neq 0$ for all $Y \neq 0$. Thus, we can conclude that

$$\frac{d}{dt}V(X_t, Y_t, Z_t, W_t) = 0, \text{if and only if } X = Y = Z = W = 0.$$

Hence, the property (K3) of Krasovskii [4] holds for the Lyapunov-Krasovskii functional $V(.)$. Thus the proof of Theorem 2.2 is now complete.

IV. Example

In this section, we give an example to illustrate the main result obtained in the previous section.

Example 4.1. As a special case of the system (1.2), let us choose, for the case $n = 2, \Psi, G, H$ and F that appeared in (1.2) as follows:

$$\Psi(Z) = \begin{bmatrix} 1+z_1^2 & 0 \\ 0 & 1+z_2^2 \end{bmatrix}, G(X,Y,Z,W) = \begin{bmatrix} -1-\frac{1}{1+x_1^2+y_1^2+z_1^2+w_1^2} & 0 \\ 0 & -1-\frac{1}{1+x_2^2+y_2^2+z_2^2+w_2^2} \end{bmatrix}$$

$$H(Y(t-\tau)) = \begin{bmatrix} 4\,y_1(t-\tau) \\ 2\,y_2(t-\tau) \end{bmatrix}, \; F(X(t-\tau)) = \begin{bmatrix} x_1(t-\tau) \\ 5\,x_2(t-\tau) \end{bmatrix}.$$

Then, the eigenvalues of the matrices $\Psi(Z)$ and $G(X,Y,Z,W)$ are

$$\lambda_1(\Psi(Z)) = 1+z_1^2, \lambda_2(\Psi(Z)) = 1+z_2^2, \; \lambda_1(G(X,Y,Z,W)) = -1-\frac{1}{1+x_1^2+y_1^2+z_1^2+w_1^2}$$

$$\lambda_2(G(X,Y,Z,W)) = -1-\frac{1}{1+x_2^2+y_2^2+z_2^2+w_2^2}.$$

Next, observe that

$$J_H(Y(t-\tau)) = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}, \; J_F(X(t-\tau)) = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}.$$

So

$$\lambda_1(J_H(Y)) = 4, \lambda_2(J_H(Y)) = 2, \lambda_1(J_F(X)) = 1, \lambda_2(J_F(X)) = 5.$$

Therefore

$$\lambda_i(\Psi(Z)) \geq 1 = a_1, \lambda_i(G(X,Y,Z,W)) \leq -1 = -a_2, 0 < \lambda_i(J_H(Y)) \leq 4 = a_5$$

and $a_4' = 1 \leq \lambda_i(J_F(X)) \leq 5 = a_4.$

Thus, in view of the above estimates, we conclude that all the conditions of Theorem 2.2 hold. Hence, we conclude that the zero solution $X = 0$ of the system (1.2) is unstable provided that $\tau < \dfrac{3}{20\sqrt{2}}$.

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