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A New Version of the Proof Of $\Gamma(n)\Gamma(1-n)$

Amusa I. Sesan[1], Mohammed A. Modiu[2], Baiyeri J. Funso[3] Ekakitie E.T[4]
Adeniji A. Adesola[5]

Department Of Mathematics, University Of Lagos, Nigeria[1&5].
Department Of Mathematics, Yaba College Of Technology [2, 3&4]

Abstract: *In this paper, we obtain a new version of the proof of* $\Gamma(n)\Gamma(1-n)$ *and the Legendre duplicating formulas for positive integer n, by using a simple analytical technique*

Key words*: gamma function, factorial and Legendre duplicating formulas*

I. Introduction

If *n* is such that $n \notin \mathbf{Z}$, then $\Gamma(n)\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$ (1.1)

Several authors have proved this result as follows

$$\Gamma(n)\Gamma(1-n) = \beta(n, 1-n) = \int_0^1 x^{n-1}(1-x)^{-n}\,dx = \int_0^1 \left(\frac{x}{1-x}\right)^n x^{-1}dx$$

By transformation $y = \dfrac{x}{1-x}$, we have that

$$\Gamma(n)\Gamma(1-n) = \beta(n, 1-n) = \int_0^\infty \frac{y^{n-1}}{1+y}\,dx$$

This formed integral cannot be solved easily by the elementary integral calculus, therefore it will be evaluated by the calculus of residue (by using contour integration) method, for the case of a multi-valued function. The function in the integral has a real singular point; the integral is then evaluated along an indented circle. Hence, the solution becomes

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi} \qquad \forall n \notin \square$$

This method of proof is tedious to apply.
Our aim here is to give a simple analytical method of proof of (1.1), which will be easier and faster than the previous way of the proof.
Before we proof this result (1.1), the proofs of the Legendre duplicating formulas are necessary.

II. Legendre Duplicating Formulas

If *m* is a positive integer, then

$$\Gamma\left(-m+\frac{1}{2}\right) = \frac{(-1)^m 2^{2m} m!\sqrt{\pi}}{(2m)!}$$

(2.1)

$$\Gamma\left(m+\frac{1}{2}\right) = \frac{(2m)!\sqrt{\pi}}{2^{2m} m!}$$

(2.2)

Proof of 2.1 and 2.2

$$2.1:\quad \Gamma\left(-m+\frac{1}{2}\right)=\Gamma\left(-\frac{2m-1}{2}\right)$$

$$=\frac{\sqrt{\pi}}{-\left(\frac{2m-1}{2}\right)\times-\left(\frac{2m-1}{2}-1\right)\times-\left(\frac{2m-1}{2}-2\right)\ldots-\frac{5}{2}.-\frac{3}{2}.-\frac{1}{2}}$$

$$=\frac{\sqrt{\pi}}{(-1)^m 2^{-m}(2m-1)(2m-3)(2m-5)\ldots5.3.1}$$

$$=\frac{(-1)^m 2^m\sqrt{\pi}\left[2m(2m-2)(2m-4)\ldots6.4.2\right]}{2m(2m-1)(2m-2)(2m-3)(2m-4)\ldots6.5.4.3.2.1}$$

$$=\frac{(-1)^m 2^m\sqrt{\pi}\,2^m m!}{(2m)!}=\frac{(-1)^m 2^{2m} m!\sqrt{\pi}}{(2m)!}$$

$$2.2:\quad \Gamma\left(m+\frac{1}{2}\right)=\Gamma\left(\frac{2m+1}{2}\right)$$

$$=\left(\frac{2m+1}{2}-1\right)\left(\frac{2m+1}{2}-2\right)\left(\frac{2m+1}{2}-3\right)\ldots\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}$$

$$=\left(\frac{2m-1}{2}\right)\left(\frac{2m-3}{2}\right)\left(\frac{2m-5}{2}\right)\ldots\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}$$

$$=\frac{1}{2^m}(2m-1)(2m-3)(2m-5)\ldots5.3.1\sqrt{\pi}$$

$$=\frac{(2m)(2m-1)(2m-2)(2m-3)\ldots6.5.4.3.2.1\sqrt{\pi}}{2^m(2m)(2m-2)(2m-4)\ldots6.4.2}$$

$$=\frac{(2m)!\sqrt{\pi}}{2^m\left[2^m m!\right]}=\frac{(2m)!\sqrt{\pi}}{2^{2m}m!}\qquad \text{Proved.}$$

We now proceed to prove (1.1) using these two results.

III. The New Version Of The Proof Of (1.1)

Multiplying (2.1) and (2.2) together, we obtain

$$\Gamma\left(-m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)=\frac{(-1)^m 2^{2m} m!\sqrt{\pi}}{(2m)!}\times\frac{(2m)!\sqrt{\pi}}{2^{2m}m!}=(-1)^m\pi=\frac{\pi}{(-1)^m}$$

$$But\ (-1)^m=\cos m\pi \qquad ,\quad \forall m\in\mathbb{Z}$$

$$\therefore \Gamma\left(-m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)=\frac{\pi}{\cos m\pi}\qquad (3.1)$$

Now, let $n=-m+\frac{1}{2} \Rightarrow 1-n=m+\frac{1}{2}$ and $m=\frac{1}{2}-n$, then (3.1) becomes

$$\Gamma(n)\Gamma(1-n)=\frac{\pi}{\cos\left(\frac{1}{2}-n\right)\pi}=\frac{\pi}{\cos\left(\frac{\pi}{2}-n\pi\right)}=\frac{\pi}{\sin n\pi} \qquad \forall n \notin \mathbb{Z}$$

This completes the proof.

IV. Conclusion

Most of the ways several authors proved this result are tedious; by first transforming it to a beta function and later applying the calculus of residue to evaluate the formed integral. We conclude that our new version of the proof is better and easier than the previous ways of proving it, and this proof is entering the literature for the first time.

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