*IOSR Journal of Mathematics (IOSR-JM)*
*e-ISSN: 2278-5728, p-ISSN:2319-765X. Volume 10, Issue 2 Ver. V (Mar-Apr. 2014), PP 08-13*
*www.iosrjournals.org*

# Intuitionistic Fuzzy R-Ideals of BCI-Algebras with Interval Valued Membership& Non Membership Functions

C.Ragavan[1], A.Solairaju[2] and M.Balamurugan[3]
*[1, 3]Asst Prof. Department of Mathematics, Sri Vidya Mandir Arts & Science College, Uthangarai, T.N. India.*
*[2]Associate Professor of Mathematics Jamal Mohamed College, Trichy, T.N, India.*

**Abstract:** *The purpose of this paper is to define the notion of an interval valued Intuitionistic Fuzzy R-ideal (briefly, an i-v IF R-ideal) of a BCI – algebra. Necessary and sufficient conditions for an i-v Intuitionistic Fuzzy R-ideal are stated. Cartesian product of i-v Fuzzy ideals are discussed.*

## I. Introduction:

The notion of BCK-algebras was proposed by Imai and Iseki in 1996. In the same year, Iseki [6] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK/BCI-algebras and their relationship with other universal structures including lattices and Boolean algebras. Fuzzy sets were initiated by Zadeh [10]. In [9], Zadeh made an extension of the concept of a Fuzzy set by an interval-valued fuzzy set. This interval-valued fuzzy set is referred to as an i-v fuzzy set. InZadeh also constructed a method of approximate inference using his i-v fuzzy sets. In Birwa's defined interval valued fuzzy subgroups of Rosenfeld`s nature, and investigated some elementary properties. The idea of "intuitionistic fuzzy set" was first published by Atanassov as a generalization of notion of fuzzy sets. After that many researchers considers the Fuzzifications of ideal and sub algebras in BCK/BCI-algebras. In this paper, using the notion of interval valued fuzzy set, we introduce the concept of an interval-valued intuitionistic fuzzy BCI-algebra of a BCI-algebra, and study some of their properties. Using an i-v level set of i-v intuitionistic fuzzy set, we state a characterization of an intuitionistic fuzzy R-ideal of BCI-algebra. We prove that every intuitionistic fuzzy R-ideal of a BCI-algebra X can be realized as an i-v level R-ideal of an i-v intuitionistic fuzzyR-ideal of X. in connection with the notion of homomorphism, we study how the images and inverse images of i-v intuitionistic fuzzy R-ideal become i-v intuitionistic fuzzy R-ideal.

## II. Preliminaries:

Let us recall that an algebra (X,*,0) of type (2,0) is called a BCI-algebra if it satisfies the following conditions:1.((x*y)*(x*z))*(z*y)=0,2.(x*(x*y))*y=0,3.x*x=0,4.x*y=0 and y*x=0 imply x=y,for all x,y,z $\epsilon$ X. In a BCI-algebra, we can define a partial ordering"≤" by x≤y if and only if x*y=0.in a BCI-algebra X, the set M={x$\epsilon$X/0*x=0} is a sub algebra and is called the BCK-part of X. A BCI-algebra X is called proper if X-M≠ϕ. otherwise it is improper. Moreover, in a BCI-algebra the following conditions hold:
1. (x*y)*z=(x*z)*y, 2.x*0=0, 3. x ≤y imply x*z ≤y*z and z*y ≤z*x, 4. 0*(x*y) = (0*x)*(0*y),
5. 0*(x*y) = (0*x)*(0*y), 6. 0*(0*(x*y)) =0*(y*x), 7. (x *z)*(y*z) ≤x*y
An intuitionistic fuzzy set A in a non-empty set X is an object having the form A= {<x,$\mu_A$(x),$\upsilon_A$(x)>/x$\epsilon$X},Where the functions $\mu_A$ : X→[0,1] and $\upsilon_A$: X→[0,1] denote the degree of the membership and the degree of non membershipof each element x$\epsilon$ X to the set A respectively, and 0≤ $\mu_A$(x) +$\upsilon_A$(x) ≤ 1 for all x$\epsilon$ X.Such defined objects are studied by many authors and have many interesting applications not only in the mathematics. For the sake of simplicity, we shall use the symbol A=[$\mu_A$, $\upsilon_A$] for the intuitionistic fuzzy set A={[ $\mu_A$(x),$\upsilon_A$(x)]/ x$\epsilon$X}.

**Definition 2.1**:A non empty subset I of X is called an ideal of X if it satisfies:1. 0$\epsilon$I,2.x*y$\epsilon$I and y$\epsilon$I ⇒ x$\epsilon$I.

**Definition 2.2:**A fuzzy subset μ of a BCI-algebra X is called afuzzy ideal of X if it satisfies:
1.μ(0)≥μ(x), 2. μ(x)≥min {μ(x*y),μ(y)},for all x,y$\epsilon$X.

**Definition 2.3:** A non empty subset I of X is called anR- ideal of X if it satisfies:
1. 0$\epsilon$I. 2. (x*z)*(z*y)$\epsilon$I and y$\epsilon$I imply x*z$\epsilon$I.Putting z=0 in(2) then we see that every R- ideal is an ideal.

**Definition 2.4:**A fuzzy set μ in a BCI-algebra X is called an fuzzyR- ideal of X if
1.μ(0)≥μ(x), 2. μ(x)≥min {μ ((x*z)*(z*y)),μ(y)}.

**Definition 2.5:** Let A and B be two fuzzy ideal of BCI algebra X. The fuzzy set $A \cap B$ membership function $\mu_{A\cap B}$ is defined by $\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}, x \in X$.

**Definition 2.6:** Let A and B be two fuzzy ideal of BCI algebra X. The fuzzy set $A \cup B$ with membership function $\mu_{A\cup B}$ is defined by $\mu_{A\cup B}(\mathrm{x}) = \max\{\mu_A(\mathrm{x}), \mu_B(\mathrm{x})\}, \forall x \in X.$

**Definition 2.7:** Let A and B be two fuzzy ideal of BCI algebra X with membership functionand respectively. A is contained in B if $\mu_A(\mathrm{x}) \le \mu_B(\mathrm{x}),\ \forall\ x \in X$

**Definition 2.8:** Let A be a fuzzy ideal of BCI algebra X.The fuzzy set $A^m$ with membership function $\mu_{A^m}$ is defined by $\mu_{A^m}(\mathrm{x}) \le (\mu_A(\mathrm{x}))^m, \forall\, x \in X$

**Definition 2.9:** An IFS A=< X, $\mu_A$,$\upsilon_A$> in a BCI-algebra X is called an intuitionisticfuzzy ideal of X if it satisfies:(F1) $\mu_A(0) \ge \mu_A(x)$ &$\upsilon_A(0) \ge \upsilon_A(x)$,(F2) $\mu_A(x) \ge \min\{\mu_A(x*y), \mu_A(y)\}$,
(F3) $\upsilon_A(x) \le \max\{\upsilon_A(x*y), \upsilon_A(y)\}$, for all x,y$\epsilon$X.

**Definition 2.10:** An intuitionistic fuzzy set A=< $\mu_A$, $\upsilon_A$> of a BCI-algebra X is called an intuitionistic fuzzy R-ideal if it satisfies (F1) and(F4) $\mu_A(x) \ge \min\{\mu_A((x*z)*(z*y)), \mu_A(y)\}$,
(F5) $\upsilon_A(y*x) \le \max\{\upsilon_A((x*z)*(z*y)), \upsilon_A(y)\}$, for all x,y,z $\epsilon$ X.

An interval-valued intuitionistic fuzzy set A defined on X is given byA={(x,[$\mu_A^L(x)\mu_A^U(x)$],[$\upsilon_A^L(x)\upsilon_A^U(x)$])},$\forall x\epsilon X$ where $\mu_A^L$,$\mu_A^U$ are two membership functions and $\upsilon_A^L$,$\upsilon_A^U$ aretwo non-membership functions X such that $\mu_A^L \le \mu_A^U$&$\upsilon_A^L \ge \upsilon_A^U$,$\forall$ x$\epsilon$X. Let $\overline{\mu}_A(x)=[\mu_A^L,\mu_A^U]$&$\overline{\upsilon_A}(x)=[\upsilon_A^L,\upsilon_A^U]$,$\forall$ x$\epsilon$X and let D[0,1]denote the family of all closed subintervals of [0,1].If $\mu_A^L(x)=\mu_A^U(x)=c, 0\le c\le 1$ and if $\upsilon_A^L(x)=\upsilon_A^U(x)=k$, $0\le k\le 1$,then we have $\overline{\mu}_A(x)=[c,c]$&$\overline{\upsilon}_A(x)=[k,k]$ which we also assume, for the sake of convenience, to belong to D[0,1]. thus $\overline{\mu}_A(x)$&$\overline{\upsilon}_A(x)\epsilon[0,1]$,$\forall$ x$\epsilon$X,and therefore the i-v IFS a is given by A=[(x,$\overline{\mu}_A(x)$,$\overline{\upsilon}_A(x)$)},$\forall$ x$\epsilon$X,where $\overline{\mu}_A(x):X\to D[0,1]$. Now let us define what is known as refined minimum, refined maximum of two elements in D[0,1].we also define the symbols" ≤","≥" and "=" in the case of two elements in D[0,1]. Consider two elements $D_1$:[$a_1$,$b_1$]and $D_2$:[$a_2$,$b_2$]$\epsilon$D[0,1]. Then rmin($D_1$,$D_2$)=[min{$a_1$,$a_2$},min{$b_1$,$b_2$}], rmax($D_{1}$,$D_{2}$)=[max{$a_1$,$a_2$},max{$b_1$,$b_2$}]
$D_1 \ge D_2 \Leftrightarrow a_1 \ge a_2, b_1 \ge b_2$; $D_1 \le D_2 \Leftrightarrow a_1 \le a_2, b_1 \le b_2$ and $D_1 = D_2$.

## III. Interval-valued Intuitionistic fuzzy R-ideals of BCI-algebras

**Definition 3.1:**An interval-valued intuitionistic fuzzy set A in BCI-algebra X is called an interval-valued intuitionistic fuzzy a-ideal of X if it satisfies ($FI_1$)$\overline{\mu}_A(0) \ge \overline{\mu}_A(x)$,$\overline{\upsilon}_A(0) \le \overline{\upsilon}_A(x)$,($FI_2$)$\overline{\mu}_A(x)$ $\ge$r min {$\overline{\mu}_A((x*z)*(z*y))$,$\overline{\mu}_A(y)$}, ( $FI_3$)$\overline{\upsilon}_A(x) \le$ r max {$\overline{\upsilon}_A((x*z)*(z*y))$,$\overline{\upsilon}_A(y)$}.

**Theorem 3.2**Let A be an i-v intuitionistic fuzzyR-ideal of X. if there exists a sequence {$x_n$} in X such that $\lim_{n\to\infty} \overline{\mu}_A(x_n)$ =[1,1], $\lim_{n\to\infty} \overline{\nu}_A(x_n)$ =[0,0] then $\overline{\mu}_A(0)$=[1,1] and$\overline{\upsilon}_A(0)$=[0,0].

**Proof:**Since $\overline{\mu}_A(0) \ge \overline{\mu}_A(x)$and $\overline{\upsilon}_A(0) \le \overline{\upsilon}_A(x)$ for all x$\epsilon$X, we have $\overline{\mu}_A(0) \ge \overline{\mu}_A(x_n)$ and $\overline{\upsilon}_A(0) \le \overline{\upsilon}_A(x_n)$, for every positive integer n. note that $[\mu_A^L, \mu_A^U] \ge \overline{\mu}_A(0)$ .[1,1] $\ge \overline{\mu}_A(x) \ge \overline{\mu}_A(0) \ge \lim_{n\to\infty} \overline{\mu}_A(x_n)$ =[1,1]. $[\lambda_A^L, \lambda_A^U] \le \overline{\lambda}_A(0)$ .[0,0] $\le \overline{\upsilon}_A(x) \le \overline{\upsilon}_A(0) \le \lim_{n\to\infty} \overline{\nu}_A(x_n)$ =[0,0].Hence $\overline{\mu}_A(0)$=[1,1] and$\overline{\upsilon}_A(0)$=[0,0].

**Lemma3.3:**An i-v intuitionistic fuzzy set A= [$\langle\mu_A^L, \mu_A^U\rangle$,$\langle\upsilon_A^L, \upsilon_A^U\rangle$] in X is an i-v intuitionistic fuzzy R-ideal of X if and only if $\langle\mu_A^L, \mu_A^U\rangle$and $\langle\upsilon_A^L, \upsilon_A^U\rangle$ are intuitionistic fuzzy ideals of X.

**Proof:**Since $\mu_A^L(0) \ge \mu_A^L(x)$; $\mu_A^U(0) \ge \mu_A^U(x)$;$\upsilon_A^L(0) \le \upsilon_A^L(x)$and $\upsilon_A^U(0) \le \upsilon_A^U(x)$, therefore $\overline{\mu}_A(0) \ge \overline{\mu}_A(x)$, $\overline{\upsilon}_A(0) \le \overline{\upsilon}_A(x)$. Suppose that $\langle\mu_A^L, \mu_A^U\rangle$ and $\langle\upsilon_A^L, \upsilon_A^U\rangle$are intuitionistic fuzzy ideal of X. let x,y$\epsilon$X, then

$\overline{\mu}_A(x)=[\mu_A^L(x),\mu_A^U(x)] \ge [\min\{\mu_A^L(x*y),\mu_A^L(y)\},\min\{\mu_A^U(x*y),\mu_A^U(y)\}]$
=r min {[$\mu_A^L(x*y)$, $\mu_A^U(x*y)$],[$\mu_A^L(y)$, $\mu_A^U(y)$]}
= r min {$\overline{\mu}_A(x*y)$,$\overline{\mu}_A(y)$} and

$\overline{\upsilon}_A(x)$= [$\upsilon_A^L(x)$,$\upsilon_A^U(x)$]$\le$ [max{$\upsilon_A^L(x*y)$,$\upsilon_A^L(y)$},max{$\upsilon_A^U(x*y)$,$\upsilon_A^U(y)$}]
=r max {[$\upsilon_A^L(x*y)$, $\upsilon(x*y)$],[$\upsilon_A^L(y)$, $\upsilon(y)$]}
= r max {$\overline{\upsilon}_A(x*y)$,$\overline{\upsilon}_A(y)$}.Hence A is an i-v intuitionistic fuzzy ideal of X.

Conversely,
Assume that A is an i-v intuitionistic fuzzy ideal of X. for any x,y$\epsilon$X,we have
[$\mu_A^L(x)$,$\mu_A^U(x)$]= $\overline{\mu}$A(x) $\ge$ r min{[$\overline{\mu}_A(x*y)$,$\overline{\mu}_A(y)$]}
=r min {[$\mu_A^L(x*y)$, $\mu_A^U(x*y)$],[$\mu_A^L(y)$, $\mu_A^U(y)$]}
= [min {$\mu_A^L(x*y)$,$\mu_A^L(y)$},min{$\mu_A^U(x*y)$,$\mu_A^U(y)$}]

And [$\upsilon_A^L(x)$,$\upsilon_A^U(x)$] =$\overline{\upsilon}_A(x) \le$ r max{$\overline{\upsilon}_A(x*y)$,$\overline{\upsilon}_A(y)$}
=r max {[$\upsilon_A^L(x*y)$, $\upsilon_A^U(x*y)$],[$\upsilon_A^L(y)$, $\upsilon_A^U(y)$]}
= [max {$\upsilon_A^L(x*y)$,$\upsilon_A^L(y)$},min{$\upsilon_A^U(x*y)$,$\upsilon_A^U(y)$}]

It follows that $\mu_A^L(x) \geq \min\{\mu_A^L(x*y),\mu_A^L(y)\}$, $\upsilon_A^L(x) \leq \max\{\upsilon_A^L(x*y),\upsilon_A^L(y)\}$
And $\mu_A^U(x) \geq \min\{\mu_A^U(x*y),\mu_A^U(y)\}$, $\upsilon_A^U(x) \leq \max\{\upsilon_A^U(x*y),\upsilon_A^U(y)\}$
Hence $\langle\mu_A^L,\mu_A^U\rangle$ and $\langle\upsilon_A^L,\upsilon_A^U\rangle$ are intuitionistic fuzzy ideals of X.

**Theorem 3.4.** Every i-v intuitionistic fuzzy R-ideal of a BCI-algebraX is an i-v intuitionistic fuzzy ideal.

**Proof:** Let A=[$\langle\mu_A^L,\mu_A^U\rangle$, $\langle\upsilon_A^L,\upsilon_A^U\rangle$] be an i-v intuitionistic fuzzy R-ideal of X, where $\langle\mu_A^L,\mu_A^U\rangle$ and $\langle\upsilon_A^L,\upsilon_A^U\rangle$ are intuitionistic fuzzy R-ideal of X. thus $\langle\mu_A^L,\mu_A^U\rangle$ and $\langle\upsilon_A^L,\upsilon_A^U\rangle$ are intuitionistic fuzzy R-ideals of X. hence by lemma 3.3, A is i-v intuitionistic fuzzy ideal of X.

**Definition 3.5:** An i-v intuitionistic fussy set A in X is called an interval-valued intuitionistic fuzzy BCI-sub algebraof Xif $\overline{\mu}_A(x*y) \geq r\min\{\overline{\mu}_A(x),\overline{\mu}_A(y)\}$ and $\overline{\upsilon}_A(x*y) \leq \{\overline{\upsilon}_A(x),\overline{\upsilon}_B(y)\}$, for all x,y$\epsilon$X.

**Theorem 3.6:** Every i-v intuitionisticfuzzy R-ideal of a BCI-algebra X is i-v intuitionistic fuzzy sub algebra of X.

**Proof:** Let A=[ $\langle\mu_A^L,\mu_A^U\rangle$, $\langle\upsilon_A^L,\upsilon_A^U\rangle$]be an i-v intuitionistic fuzzy R-ideal of X, where $\langle\mu_A^L,\mu_A^U\rangle$, and $\langle\upsilon_A^L,\upsilon_A^U\rangle$ are intuitionistic fuzzy R-ideal of BCI-algebra X. thus $\langle\mu_A^L,\mu_A^U\rangle$, and $\langle\upsilon_A^L,\upsilon_A^U\rangle$ are intuitionistic fuzzy subalgebra of X. Hence, A is i-v intuitionistic fuzzysub algebra of X.

**4. Cartesian product of i-v intuitionistic fuzzy R-ideals**

**Definition 4.1** An intuitionistic fuzzy relation A on any set a is a intuitionistic fuzzy subset A with a membership function $\Omega_A$: X×X→ [0, 1] and non membership function $\Psi_A$: X×X→ [0, 1].

**Lemma 4.2** Let $\overline{\mu}_A$ and $\overline{\mu}_B$ be two membership functions and $\overline{\upsilon}_A$ and $\overline{\upsilon}_B$ be two non membership functions of each x $\epsilon$X to the i-v subsets A and B, respectively. Then $\mu_A\times\mu_B$ is membership function and $\upsilon_A\times\upsilon_B$ is non membership function of each element(x,y)$\epsilon$X×X to the set A×B and defined by $(\overline{\mu}_A\times\overline{\mu}_B)(x,y)=r\min\{\overline{\mu}_A(x),\overline{\mu}_B(y)\}$ and
$(\overline{\upsilon}_A\times\overline{\upsilon}_B)(x,y)=r\max\{\overline{\upsilon}_A(x),\overline{\upsilon}_B(y)\}$.

**Definition 4.3** Let A= [$\langle\mu_A^L,\mu_A^U\rangle$, $\langle\upsilon_A^L,\upsilon_A^U\rangle$]and B=[ $\langle\mu_B^L,\mu_B^U\rangle$, $\langle\upsilon_{\square},\upsilon_{\square}\rangle$] be two i-v intuitionistic fuzzy subsets in a set X. the Cartesian product of A×B is defined byA×B= {((x,y), $\overline{\mu}_A\times\overline{\mu}_B$, $\overline{\upsilon}_A\times\overline{\upsilon}_B$);∀x,y□X×X}Where
A×B: X×X→D[0,1].

**Theorem 4.4.** Let A=[$\langle\mu_{\square},\mu_{\square}\rangle$, $\langle\upsilon_{\square},\upsilon_{\square}\rangle$]and B=[ $\langle\mu_{\square},\mu_{\square}\rangle$, $\langle\upsilon_{\square},\upsilon_{\square}\rangle$] be two i-v intuitionistic fuzzy subsets in a set X,then A×B is an i-v intuitionistic fuzzy a-ideal of X×X.

**Proof:** Let(x,y) □X×X, then by definition
$(\overline{\mu}_A\times\overline{\mu}_B)(0,0)=r\min\{\overline{\mu}_A(0),\overline{\mu}_B(o) = r\min\{[\mu_{\square}(0),\mu_{\square}(0)],[\mu_{\square}(0),\mu_{\square}(0)]\}$
$=[\min\{\mu_{\square}(0),\mu_{\square}(0)\},\min\{\mu_{\square}(0),\mu_{\square}(0)\}]$
$\geq[\min\{\mu_{\square}(x),\mu_{\square}(y)\},\min\{\mu_{\square}(x),\mu_{\square}(y)\}]$
$=r\min\{[\mu_{\square}(x),\mu_{\square}(x)],[\mu_{\square}(y),\mu_{\square}(y)]\}$
$= r\min\{\overline{\mu}_A(x),\overline{\mu}_B(y)\}=(\overline{\mu}_A\times\overline{\mu}_B)(x,y)$
And $(\overline{\upsilon}_A\times\overline{\upsilon}_B)(0,0)=r\max\{\overline{\upsilon}_A(0),\overline{\upsilon}_B(o)$
$= r\max\{[\upsilon_{\square}(0),\upsilon_{\square}(0)],[\upsilon_{\square}(0),\upsilon_{\square}(0)]\}$
$=[\max\{\upsilon_{\square}(0),\upsilon_{\square}(0)\},\max\{\upsilon_{\square}(0),\upsilon_{\square}(0)\}]$
$\leq[\max\{\upsilon_{\square}(x),\upsilon_{\square}(y)\},\max\{\upsilon_{\square}(x),\upsilon_{\square}(y)\}]$
$=r\max\{[\upsilon_{\square}(x),\upsilon_{\square}(x)],[\upsilon_{\square}(y),\upsilon_{\square}(y)]\}$
$= r\max\{\overline{\upsilon}_A(x),\overline{\upsilon}_B(y) =(\overline{\upsilon}_A\times\overline{\upsilon}_B)(x,y)$
Therefore ($FI_2$) holds.Now, for all x,y,z□X, we have
$(\overline{\mu}_A\times\overline{\mu}_B)((x,x'))=r\min\{\mu_A(x),\mu B(x')\}$

$$\geq r\min\{r\min\{\overline{\mu}_A((x*z)*(z*y)),\overline{\mu}_A(y)\},r\min\{\overline{\mu}_A((x^1*z^1)*(z^1*y^1)),\overline{\mu}_A(y^1)\}\}$$

$$=r\min\{\{\min\{\mu^L{}_A((x*z)*(z*y)),\mu^L{}_A(y)\},\ \min\{\mu^U{}_A((x*z)*(z*y)),\mu^U{}_A(y)\}\},$$

$$\{\min\{\mu^L{}_B((x^1*z^1)*(z^1*y^1)),\mu^L{}_B(y^1)\},\min\{\mu^U{}_B((x^1*z^1)*(z^1*y^1)),\mu^U{}_B(y^1)\}\}$$

$$=\{\min\{\min\{\mu^L{}_A((x*z)*(z*y)),\mu^L{}_B((x^1*z^1)*(z^1*y^1))\},\min\{\mu^L{}_A(y),\ \mu^L{}_B(y^1)\}\},$$

$$\min\{\min\{\mu^U{}_A((x*z)*(z*y)),\mu^U{}_B((x^1*z^1)*(z^1*y^1))\},\min\{\mu^U{}_A(y),\mu^U{}_B(y^1)\}\}\}$$

$=r\min\{(\overline{\mu}_A\times\overline{\mu}_B)(((x*z)*(z*y)),\ ((x^1*z^1)*(z^1*y^1))),(\overline{\mu}_A\times\overline{\mu}_B)(y,y')\}$
Also, $(\overline{\upsilon}_A\times\overline{\upsilon}_B)((x,x')) =r\max\{\upsilon_A(x),\upsilon_B(x')\}$

$$\leq r\max\{r\max\{\overline{\nu}_A((x*z)*(z*y)),\overline{\nu}_A(y)\},r\max\{\overline{\nu}_A((x^1*z^1)*(z^1*y^1)),\overline{\nu}_A(y^1)\}\}$$

$$=r\max\{\{\max\{\nu^L{}_A((x*z)*(z*y)),\nu^L{}_A(y)\},\ \max\{\nu^U{}_A((x*z)*(z*y)),\nu^U{}_A(y)\}\},$$

$$\{\max\{\nu^L{}_B((x^1*z^1)*(z^1*y^1)),\nu^L{}_B(y^1)\},\max\{\nu^U{}_B((x^1*z^1)*(z^1*y^1)),\nu^U{}_B(y^1)\}\}$$

$$=\{\max\{\max\{\nu^L{}_A((x*z)*(z*y)),\nu^L{}_B((x^1*z^1)*(z^1*y^1))\},\max\{\nu^L{}_A(y),\nu^L{}_B(y^1)\}\},$$
$$\max\{\max\{\nu^U{}_A((x*z)*(z*y)),\nu^U{}_B((x^1*z^1)*(z^1*y^1))\},\max\{\nu^U{}_A(y),\nu^U{}_B(y^1)\}\}\}$$
$$= r\ max\ \{(\bar{\nu}_A\times\bar{\nu}_B)(((x*z)*(z*y)),((x^1*z^1)*(z^1*y^1)),(\bar{\nu}_A\times\bar{\nu}_B)(y,y^1)\}$$

Hence A×B is an i-v intuitionistic fuzzy R-ideal of X× X

**Definition 4.5**:Let$\bar{\mu}_B$,$\bar{\upsilon}_B$ respectively, be an i-v membership and non membership function of each element x□X to the set B. then strongest i-v intuitionistic fuzzy set relationon X ,that is a membership function relation $\bar{\mu}_A$on$\bar{\mu}_B$ and non membership function relation $\bar{\upsilon}_A$ on$\bar{\upsilon}_B$ and $\mu_{A_B}$ , $\nu_{A_B}$□ whose i-v membership and non membership function, of each element (x,y) □X×X and defined by $\bar{\mu}_{A_B}$ (x,y)=r min{$\bar{\mu}_B$(x),$\bar{\mu}_B$(y)}& $\bar{\nu}_{A_B}$□ (x,y)=r max{$\bar{\upsilon}_B$(x),$\bar{\upsilon}_B$(y)}

**Definition 4.6**Let B=[ ⟨μ□, μ□⟩, ⟨υ□, υ□⟩] be an i-v subset in a set X, then the strongest i-v intuitionistic fuzzy relation on X that is a i-v A on B is $A_B$ and defined by, $A_B=[\langle\mu^L_{A_B},\mu^U_{A_B}\rangle,\langle\nu^L_{A_B},\nu^U_{A_B}\rangle]$

**Theorem 4.7**Let $B=[\langle\mu^L_{A_B},\mu^U_{A_B}\rangle,\langle\nu^L_{A_B},\nu^U_{A_B}\rangle]$ be an i-v subset in a set X and $A_B=[\langle\mu^L_{A_B},\mu^U_{A_B}\rangle,\langle\nu^L_{A_B},\nu^U_{A_B}\rangle]$ be the strongest i-v intuitionistic fuzzy relation on X. then B is an i-v intuitionistic R-ideal of X if and only if $A_B$ is an i-v intuitionistic fuzzy R-ideal of X×X.

Proof: Let B be an i-v intuitionistic fuzzy a-ideal of X. then$\bar{\mu}_{AB}$(0,0)=r min{$\bar{\mu}_B$(0),$\bar{\mu}_B$(0)}
≥r min{$\bar{\mu}_B$(x),$\bar{\mu}_B$(y)}=$\bar{\mu}_{AB}$(x,y) and $\bar{\upsilon}_{AB}$(0,0)=r max{$\bar{\upsilon}_B$(0),$\bar{\upsilon}_B$(0)}≤r max{$\bar{\upsilon}_B$(x),$\bar{\upsilon}_B$(y)}=$\bar{\upsilon}_{AB}$(x,y) ∀(x,y) □X×X.
On the other hand $\bar{\mu}_{A_B}$ (($x_1$,$x_2$))=$\bar{\mu}_{AB}$($x_1$, $x_2$)
` =r min {$\bar{\mu}_B$($x_1$),$\bar{\mu}_B$($x_2$)}
≥r min{r min{$\bar{\mu}_B$(($x_1$*$z_1$)*($z_1$*$y_1$)),$\bar{\mu}_B$($y_1$)},r min{$\bar{\mu}_B$(($x_2$*$z_2$)*($z_2$*$y_2$)),$\bar{\mu}_B$($y_2$)}}
=r min{r min{$\bar{\mu}_B$(($x_1$*$z_1$)*($z_1$ *$y_1$)),$\bar{\mu}_B$(($x_2$*$z_2$)*($z_2$ *$y_2$))},r min {$\bar{\mu}_B$($y_1$),$\bar{\mu}_B$($y_2$)}}
=r min {$\bar{\mu}_{AB}$(($x_1$*$z_1$)*($z_1$ *$y_1$), ($x_2$*$z_2$)*($z_2$ *$y_2$)),$\bar{\mu}_{AB}$($y_1$, $y_2$)}
=r min {$\bar{\mu}_{AB}$ ((($x_1$,$x_2$)*($z_1$,$z_2$))*(( $z_1$,$z_2$)*($y_1$, $y_2$))),$\bar{\mu}_{AB}$($y_1$, $y_2$)}

Also,$\bar{\nu}_{A_B}$ (($x_1$, $x_2$)) =r max {$\bar{\upsilon}_B$ ($x_1$),$\bar{\upsilon}_B$($x_2$)}
≤ r max{r max{$\bar{\upsilon}_B$(($x_1$*$z_1$)*($z_1$ *$y_1$)),$\bar{\upsilon}_B$($y_1$)},r max{$\bar{\upsilon}_B$(($x_2$*$z_2$)*($z_2$ *$y_2$)),$\bar{\upsilon}_B$($y_2$)}}
= r max{r max{$\bar{\upsilon}_B$(($x_1$*$z_1$)*($z_1$ *$y_1$)),$\bar{\upsilon}_B$(($x_2$*$z_2$)*($z_2$ *$y_2$))},r max {$\bar{\upsilon}_B$($z_1$),$\bar{\upsilon}_B$($z_2$)}}
=r max{$\bar{\upsilon}_{AB}$(($x_1$*$z_1$)*($z_1$ *$y_1$), ($x_2$*$z_2$)*($z_2$ *$y_2$)),$\bar{\upsilon}_{AB}$($y_1$, $y_2$)}
=r max{$\bar{\upsilon}_{AB}$((($x_1$,$x_2$)*( $z_1$,$z_2$))*(( $z_1$,$z_2$)*($y_1$, $y_2$))),$\bar{\upsilon}_{AB}$($z_1$, $z_2$)}

For all ($x_1$,$x_2$),($y_1$,$y_2$),($z_1$,$z_2$) in X×X. hence $A_B$ is an i-v intuitionistic fuzzy R-ideal of X×X.
Conversely, let $A_B$ be an i-v intuitionistic fuzzy R-ideal of X×X. then for all (x,x)□X×X.we have
r min {$\bar{\mu}_B$(0),$\bar{\mu}_B$(0)}=$\bar{\mu}_{AB}$(0,0)≥$\bar{\mu}_{AB}$(x,x)= r min{$\bar{\mu}_B$(x),$\bar{\mu}_B$(x)}(or)$\bar{\mu}_B$(0) ≥$\bar{\mu}_B$(x) and
r max {$\bar{\upsilon}_B$(0),$\bar{\upsilon}_B$(0)}=$\bar{\upsilon}_{AB}$(0,0)≤$\bar{\upsilon}_{AB}$(x, x)=rmin{$\bar{\upsilon}_B$(x),$\bar{\mu}_B$(x)}(or)$\bar{\upsilon}_B$(0)≤$\bar{\upsilon}_B$(x)∀x□X. Now,
let ($x_1$,$x_2$),($y_1$,$y_2$),($z_1$,$z_2$) □X×X, then
rmin {$\bar{\mu}_B$($x_1$,$x_2$)}=$\bar{\mu}_{AB}$($x_1$, $x_2$)≥r min {$\bar{\mu}_{AB}$((($x_1$,$x_2$)*(($z_1$, $z_2$))*(($z_1$, $z_2$)*($y_1$, $y_2$))),$\bar{\mu}_{AB}$($y_1$, $y_2$)}
=r min {$\bar{\mu}_{AB}$ (($x_1$*$z_1$)*($z_1$*$y_1$), ($x_2$*$z_2$)*($z_2$*$y_2$)),$\bar{\mu}_{AB}$($y_1$,$y_2$)}
=r min {r min {$\bar{\mu}_B$(($x_1$*$z_1$)*($z_1$*$y_1$)),$\bar{\mu}_B$($z_1$)},r min {$\bar{\mu}_{AB}$(($x_2$*$z_2$)*(0*$y_2$)),$\bar{\mu}_B$($z_2$)}}
Also, rmax {$\bar{\upsilon}_B$($x_1$,$x_2$)}=$\bar{\upsilon}_{AB}$($x_1$, $x_2$)
≤r max {$\bar{\upsilon}_{AB}$((($x_1$,$x_2$)*(($z_1$, $z_2$))*(($z_1$, $z_2$)*($y_1$, $y_2$))),$\bar{\upsilon}_{AB}$($y_1$,$y_2$)}
=r max {$\bar{\upsilon}_{AB}$ (($x_1$*$z_1$)*($z_1$*$y_1$), ($x_2$*$z_2$)*($z_2$*$y_2$)), $\bar{\upsilon}_{AB}$ ($y_1$,$y_2$)}
=r max {r max {$\bar{\upsilon}_B$(($x_1$*$z_1$)*($z_1$*$y_1$)),$\bar{\mu}_B$($y_1$)},r max {$\bar{\upsilon}_{AB}$(($x_2$*$z_2$)*($z_2$*$y_2$)),$\bar{\upsilon}_B$($y_2$)}}
If $x_2$=$y_2$=$z_2$=0, then r min {$\bar{\mu}_B$($x_1$),$\bar{\mu}_B$(0)}≥ r min {r min {$\bar{\mu}_B$(($x_1$*$z_1$)*($z_1$*$y_1$),$\bar{\mu}_B$($y_1$)},$\bar{\mu}_B$(0)} and
r max {$\bar{\upsilon}_B$($x_1$),$\bar{\upsilon}_B$(0)}≥ r max {r max {$\bar{\upsilon}_B$(($x_1$*$z_1$)*($z_1$*$y_1$), $\bar{\upsilon}_B$($y_1$)},$\bar{\upsilon}_B$(0)}
$\bar{\mu}_B$($x_1$)≥r min {$\bar{\mu}_B$(($x_1$*$z_1$)*($z_1$*$y_1$), $\bar{\mu}_B$($y_1$)} and
$\bar{\upsilon}_B$($x_1$) ≥r max {$\bar{\upsilon}_B$(($x_1$*$z_1$)*($z_1$*$y_1$), $\bar{\upsilon}_B$($y_1$)}.
Therefore B is i-v intuitionistic fuzzy R-ideal of X.

**Theorem 4.8:** If$\bar{\mu}_A$ is a i-v intuitionistic fuzzy a-ideal of BCI-algebra X, then $\bar{\mu}_{A^m}$ is also i-v intuitionistic fuzzy R-ideal of BCI-algebra X

Proof: For all x, y, z $\in$ X

$$1.\ \bar{\mu}_A(0)\geq\bar{\mu}_A(x),\ \bar{\nu}_A(0)\leq\bar{\nu}_A(x).\quad [\bar{\mu}_A(0)]^m\geq[\bar{\mu}_A(x)],\ [\bar{\nu}_A(0)]^m\leq[\bar{\nu}_A(x)]$$
$$\bar{\mu}_A(0)^m\geq\bar{\mu}_A(x)^m,\ \bar{\nu}_A(0)^m\leq\bar{\nu}_A(x)^m.\quad \bar{\mu}_{A^m}(0)\geq\bar{\mu}_{A^m}(x),\ \bar{\nu}_{A^m}(0)\leq\bar{\nu}_{A^m}(x)\ \ \forall x\in X$$

$2. \bar{\mu}_A(x) \geq r \min\{\bar{\mu}_A((x*z)*(z*y)), \bar{\mu}_A(y)\}$. $[\bar{\mu}_A(x)]^m \geq [r \min\{\bar{\mu}_A((x*z)*(z*y)), \bar{\mu}_A(y)\}]^m$

$\bar{\mu}_A(x)^m \geq r \min\{\bar{\mu}_A((x*z)*(z*y)), \bar{\mu}_A(y)\}^m$. $\bar{\mu}_{A^m}(x) \geq r \min\{\bar{\mu}_A((x*z)*(z*y))^m, \bar{\mu}_A(y)^m\}$

$\bar{\mu}_{A^m}(x) \geq r \min\{\bar{\mu}_{A^m}((x*z)*(z*y)) \bar{\mu}_{A^m}(y)\}$

$3. \bar{\nu}_A(x) \leq r \max\{\bar{\mu}_A((x*z)*(z*y)), \bar{\mu}_A(y)\}$. $[\bar{\nu}_A(x)]^m \leq [r \max\{\bar{\nu}_A((x*z)*(z*y)) \bar{\nu}_A(y)\}]^m$

$\bar{\nu}_A(x)^m \leq r \max\{\bar{\nu}_A((x*z)*(z*y)), \bar{\nu}_A(y)\}^m$. $\bar{\nu}_{A^m}(x) \leq r \max\{\bar{\nu}_A((x*z)*(z*y))^m, \bar{\nu}_A(y)^m\}$

$\bar{\nu}_{A^m}(x) \leq r \max\{\bar{\nu}_{A^m}((x*z)*(z*y)) \bar{\nu}_{A^m}(y)\}$

**Theorem 4.9:**If $\bar{\mu}_A$is a i-v intuitionistic fuzzy R-ideal of BCI-algebra X, then $\bar{\mu}_{A\cap B}$ isalso a i-v intuitionistic fuzzy R-ideal of BCI-algebra X

Proof: For all x, y, z $\in$ X

$1. \bar{\mu}_A(0) \geq \bar{\mu}_A(x),\ \bar{\nu}_A(0) \leq \bar{\nu}_A(x)$ *and* $\bar{\mu}_B(0) \geq \bar{\mu}_B(x),\ \bar{\nu}_B(0) \leq \bar{\nu}_B(x)$

$\min\{\bar{\mu}_A(0), \bar{\mu}_B(0)\} \geq \min\{\bar{\mu}_A(x), \bar{\mu}_B(x)\}, \min\{\bar{\nu}_A(0), \bar{\nu}_B(0)\} \leq \min\{\bar{\nu}_A(x), \bar{\nu}_B(x)\}$

$\bar{\mu}_{A\cap B}(0) \geq \bar{\mu}_{A\cap B}(x), \bar{\nu}_{A\cap B}(0) \leq \bar{\nu}_{A\cap B}(x)$

$2. \bar{\mu}_A(x) \geq r \min\{\bar{\mu}_A((x*z)*(z*y)), \bar{\mu}_A(y)\}, \bar{\mu}_B(y*x) \geq r \min\{\bar{\mu}_B((x*z)*(z*y)), \bar{\mu}_A(y)\}$

$\{\bar{\mu}_A(x), \bar{\mu}_B(x)\} \geq \{r \min\{\bar{\mu}_A((x*z)*(z*y)), \bar{\mu}_A(y)\}, r \min\{\bar{\mu}_B((x*z)*(z*y)), \bar{\mu}_B(y)\}\}$

$\min\{\bar{\mu}_A(x), \bar{\mu}_B(x)\} \geq \min\{r \min\{\bar{\mu}_A((x*z)*(z*y)), \bar{\mu}_A(y)\}, r \min\{\bar{\mu}_B((x*z)*(z*y)), \bar{\mu}_B(y)\}\}$

$\geq \min\{r \min\{\bar{\mu}_A((x*z)*(z*y)), \bar{\mu}_B((x*z)*(z*y))\}, r \min\{\bar{\mu}_A(y), \bar{\mu}_B(y)\}\}$

$\bar{\mu}_{A\cap B}(x) \geq r \min\{\bar{\mu}_{A\cap B}((x*z)*(z*y)), \bar{\mu}_{A\cap B}(y)\}$

$3. \bar{\nu}_A(x) \leq r \max\{\bar{\nu}_A((x*z)*(z*y)), \bar{\nu}_A(y)\}, \bar{\nu}_B(x) \leq r \max\{\bar{\nu}_B((x*z)*(z*y)), \bar{\nu}_A(y)\}$

$\{\bar{\nu}_A(x), \bar{\nu}_B(x)\} \leq \{r \max\{\bar{\nu}_A((x*z)*(z*y)), \bar{\nu}_A(y)\}, r \max\{\bar{\nu}_B((x*z)*(z*y)), \bar{\nu}_B(y)\}\}$

If one is contained in the other

$\min\{\bar{\nu}_A(x), \bar{\nu}_B(x)\} \leq \min\{r \max\{\bar{\nu}_A((x*z)*(z*y)), \bar{\nu}_A(z)\}, r \max\{\bar{\nu}_B((x*z)*(z*y)), \bar{\nu}_B(y)\}\}$

$\bar{\nu}_{A\cap B}(x) \leq r \max\{\min\{\bar{\nu}_A((x*z)*(z*y)), \bar{\nu}_B((x*z)*(z*y))\}, \min\{\bar{\nu}_A(z), \bar{\nu}_B(y)\}\}$

$\bar{\nu}_{A\cap B}(x) \leq r \max\{\bar{\nu}_{A\cap B}((x*z)*(z*y)), \bar{\nu}_{A\cap B}(y)\}$

**Theorem 4.10:** If $\bar{\mu}_A$is a i-v intuitionistic fuzzy R-ideal of BCI-algebra X, then $\bar{\mu}_{A\cup B}$ isalso a i-v intuitionistic fuzzy R-ideal of BCI-algebra X.

Proof: For all x, y, z $\in$ X

$1. \bar{\mu}_A(0) \geq \bar{\mu}_A(x),\ \bar{\nu}_A(0) \leq \bar{\nu}_A(x)$ *and* $\bar{\mu}_B(0) \geq \bar{\mu}_B(x),\ \bar{\nu}_B(0) \leq \bar{\nu}_B(x)$

$\min\{\bar{\mu}_A(0), \bar{\mu}_B(0)\} \geq \min\{\bar{\mu}_A(x), \bar{\mu}_B(x)\},\ \min\{\bar{\nu}_A(0), \bar{\nu}_B(0)\} \leq \min\{\bar{\nu}_A(x), \bar{\nu}_B(x)\}$

$\bar{\mu}_{A\cup B}(0) \geq \bar{\mu}_{A\cup B}(x), \bar{\nu}_{A\cup B}(0) \leq \bar{\nu}_{A\cup B}(x)$

$2. \bar{\mu}_A(x) \geq r \min\{\bar{\mu}_A((x*z)*(z*y)), \bar{\mu}_A(y)\}, \quad \bar{\mu}_B(x) \geq r \min\{\bar{\mu}_B((x*z)*(z*y)), \bar{\mu}_A(y)\}$

$\{\bar{\mu}_A(x), \bar{\mu}_B(x)\} \geq \{r \min\{\bar{\mu}_A((x*z)*(z*y)), \bar{\mu}_A(y)\},\ r \min\{\bar{\mu}_B((x*z)*(z*y)), \bar{\mu}_B(y)\}\}$

$\max\{\bar{\mu}_A(x), \bar{\mu}_B(x)\} \geq \max\{r \min\{\bar{\mu}_A((x*z)*(z*y)), \bar{\mu}_A(y)\},\ r \min\{\bar{\mu}_B((x*z)*(z*y)), \bar{\mu}_B(y)\}\}$

$\geq \max\{r \min\{\bar{\mu}_A((x*z)*(z*y)), \bar{\mu}_B((x*z)*(z*y))\}, \quad r \max\{\bar{\mu}_A(y), \bar{\mu}_B(y)\}\}$

If one is contained in the other

$\mathrm{r} \min\{\max\{\bar{\mu}_A((x*z)*(z*y)), \bar{\mu}_B((x*z)*(z*y))\}, \quad \max\{\bar{\mu}_A(y), \bar{\mu}_B(y)\}\}$

$\bar{\mu}_{A\cup B}(x) \geq r \min\{\bar{\mu}_{A\cup B}((x*z)*(z*y)), \bar{\mu}_{A\cup B}(y)\}$

$3. \bar{\nu}_A(x) \leq r \max\{\bar{\nu}_A((x*z)*(z*y)), \bar{\nu}_A(y)\}, \quad \bar{\nu}_B(x) \leq r \max\{\bar{\nu}_B((x*z)*(z*y)), \bar{\mu}_A(y)\}$

$\{\bar{\nu}_A(x), \bar{\nu}_B(x)\} \leq \{r \max\{\bar{\nu}_A((x*z)*(z*y)), \bar{\nu}_A(y)\},\ r \max\{\bar{\nu}_B((x*z)*(z*y)), \bar{\nu}_B(z)\}\}$

$\max\{\bar{\nu}_A(x), \bar{\nu}_B(x)\} \leq \max\{r \max\{\bar{\nu}_A((x*z)*(z*y)), \bar{\nu}_A(y)\},\ r \max\{\bar{\nu}_B((x*z)*(z*y)), \bar{\nu}_B(y)\}\}$

$\bar{\nu}_{A\cup B}(x) \leq r \max\{\max\{\bar{\nu}_A((x*z)*(z*y)), \bar{\nu}_B((x*z)*(z*y))\},\ \max\{\bar{\nu}_A(y), \bar{\nu}_B(y)\}\}$

$\bar{\nu}_{A\cup B}(x) \leq r \max\{\bar{\nu}_{A\cup B}((x*z)*(z*y)), \bar{\nu}_{A\cup B}(y)\}$

## References:

[1]. K.T Atanassov, intuitionisticfuzzy sets and systems, 20(1986), 87-96
[2]. K.T Atanassov, intuitionisticfuzzy sets. Theory and applications, studies in fuzziness and soft computing, 35.Heidelberg; physica-verlag
[3]. R.Biswas, Rosenfeld's fuzzy subgroups with interval-valued membership functions, fuzzy sets and systems 63(1994), no.1,87-90
[4]. S.M. Hong, Y.B.Kim and G.I.Kim, fuzzy BCI-sub algebras with interval-valued membership functions, math japonica, 40(2)(1993)199-202
[5]. K.Iseki, an algebra related with a propositional calculus, proc, Japan Acad.42 (1966),26-29
[6]. H.M.Khalid, B.Ahmad, fuzzy H-ideals in BCI-algebras, fuzzy sets and systems 101(1999)153-158.
[7]. L.A.zadeh, the concept of a linguistic variable and its application to approximate reasoning. I, information sci,8(1975),199-249.