IOSR Journal Of Humanities And Social Science (IOSR-JHSS)
Volume 19, Issue 8, Ver. I (Aug. 2014), PP 120-129
e-ISSN: 2279-0837, p-ISSN: 2279-0845.
www.iosrjournals.org

Some Properties of the Annihilator Graph of a Commutative Ring

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Abstract
Let R be a commutative ring with unity. A. Badawi defined and studied annihilator graph AG(R) of R. In this chapter we introduce a new annihilator graph. of the commutative ring R by taking the new definition and labeling it newannihilator chart by ANNG(R). We examine the relationship between graphs ANNG(R), AG(R) and (R), where (R) is the zero divisor graph of R defined by D. F. Anderson and P.S. Livingston. We study some properties of the commutative ring R ANNG(R) related to connectivity, diameter and circumference. We will create a result set which describe certain situations where ANNG(R) is identical to AG(R) and (R). For reduced commutative ring R, we study some characteristics of the annihilator graph ANNG(R) associated with the minimal primes of R. For a reduced commutative ring R, we are establish some equivalent conditions that describe certain situations where ANNG(R) is a complete bipartite graph or a star graph. In addition, we examine some properties ANNG(R) when R is an irreducible commutative ring. In this chapter, R is a commutative ring with unity, Z(R) is a set all zero divisors of R, N(R) is the set of all nilpotent elements of R, U(R) is the group units, T(R) is the total quotient ring of R, and (R) is the set of all minimal primes ideals of R. For each X R we denote X {0} X*. For any two graphs G and H, if G is identical to H, then we write G = H; otherwise we write G H. Distance between two distinct vertices and the graph of the zero divisor (R) will be denoted (R) (,).
Keywords:- properties of the annihilator graph , commutative ring , new annihilator graph ANNG etc .

Definitions and preliminary questions :-

Here we introduce a new annihilator graph of the commutative ring R and define new annihilator chart like this:

Definition :- Let R be a commutative ring and Z(R) be the set of all zero-divisors of R. For ∈ Z(R), let ( ) = { ∈ R | = 0}. We define the new annihilator graph of R, denoted by ANNG(R), as the undirected graph whose set of vertices is Z(R)* = Z(R) − {0}, and two distinct vertices and are adjacent if and only if

( ) ≠ ( ) ∩ ( ).

A. Badawi defined the annihilator graph of a commutative ring R as follows:

Definition Let R be a commutative ring and Z(R) be the set of all zero-divisors of R. For ∈ Z(R), let ( ) = { ∈ R | = 0}. The annihilator graph of R, denoted by AG(R), is the undirected graph whose vertex set is Z(R)* = Z(R) − {0}, and two distinct vertices and are adjacent if and only if ( ) ≠

( ) ∪ ( ).

D. F. Anderson and P. S. Livingston defined the zero-divisor graph of a commutative ring R as follows:

Definition Let R be a commutative ring. The zero-divisor graph of R, denoted by Γ(R), is the undirected graph whose vertices are the nonzero zero-divisors of R and two distinct vertices and are adjacent if and only if = 0.

Theorem Let R be a commutative ring. Then ANNG(R) is an empty graph if and only if R is an integral domain.

Proof. Suppose that ANNG(R) is an empty graph. Then Z(R)* = ∅ by definition. Hence R is an integral domain. Conversely, suppose that R is an integral domain. Then Z(R)* = ∅, and hence ANNG(R) is an empty graph.

We are now going to present the following results without proof.

Lemma Let R be a commutative ring.

(1) Let and be distinct elements of Z(R)*. Then — is not an edge of AG(R) if and only if ( ) = ( ) or ( ) = ( ).

(2) If — is an edge of Γ(R) for some distinct , ∈ Z(R)*, then — is an edge of AG(R). In particular, if P is a path in Γ(R), then P is a path in AG(R).

(3) If Γ(R) (,) = 3 for some distinct , ∈ Z(R)*, then — is an edge of AG(R).

(4) If — is not an edge of AG(R) for some distinct , ∈ Z(R)*, then there is a ∈ Z(R)* – { , } such that — — is a path in Γ(R) and AG(R), and hence — — is also a path in AG(R).

Lemma Let R be a reduced commutative ring that is not an integral domain and let ∈ Z(R)*. Then

(1) ( ) = ( ) for each positive integer ≥ 2;

(2) If + ∈ Z(R) for some ∈ ( ) – {0}, then (+) is properly contained in ( ) (i.e., (+) ⊂ ( )). In particular, if Z(R) is

an ideal of R and c ∈ ( ) − {0} , then (+) is properly contained in ( ).

Lemma Let R be a non-reduced commutative ring with |N(R)*| ≥ 2, and let $\Gamma_{NG}(R)$ be the induced subgraph of Γ(R) with vertices N(R)*. Then $\Gamma_{NG}(R)$ is complete if and only if $N(R)^2 = \{0\}$.

Lemma Let R be a non-reduced commutative ring. If Z(R) is not an ideal of R then (Γ(R)) = 3.

Theorem Let R be commutative ring that is not an integral domain. Then Γ(R) is connected and (Γ(R)) ≤ 3.

Theorem Let R be commutative ring. If (R) contains a cycle, then ((Γ(R)) ≤ 4.

Theorem Let R be a commutative ring. Then (Γ(R)) = 2 if and only if either (1) R is reduced with exactly two minimal primes and at least three nonzero zero-divisors, or (2) Z(R) is an ideal whose square is not {0} and each pair of distinct zero-divisors has a nonzero annihilator.

Theorem Let R be commutative ring with |Z(R)*|≥ 2. Then AG(R) is connected and (AG(R)) ≤ 2.

Theorem Let R be a reduced commutative ring that is not an integral domain. Then AG(R) = Γ(R) if and only if | (R)| = 2.

Theorem Let R be a reduced commutative ring. Then the following statements are equivalent:

(1) (AG(R)) = 4;

(2) (Γ (R)) = 4;

(3) T(R) is ring-isomorphic to $K_1 \times K_2$, where each K_i is a field with $|K_i| \geq 3$;

(4) | (R) | = 2 and each minimal prime ideal of R has at least three distinct elements;

(5) Γ(R) = , with , ≥ 2;

(6) AG(R) = ' with , ≥ 2.

Theorem Let R be a reduced commutative ring that is not an integral domain. Then the following statements are equivalent:

(1) (AG(R)) = ∞;

(2) (Γ (R)) = ∞;

(3) T(R) is ring-isomorphic to ℤ × K, where K is a field;

(4) | (R) | = 2 and at least one minimal prime ideal of R has exactly two distinct elements;

(5) Γ(R) = ' for some ≥ 1;

(6) AG(R) = ' for some ≥ 1.

Theorem Let R be a non-reduced commutative ring. Then (AG(R)) = 4 if and only AG(R) ≠ Γ(R) and (AG(R)) = 4.

Theorem Let R be a non-reduced commutative ring with |Z(R)*| ≥ 2. Then the following statements are equivalent:

(1) (AG(R)) = ∞;

(2) N(R) is a prime ideal of R and either Z(R) = N(R) = {0, – , }(– ≠) for some nonzero ∈ R or Z(R) ≠ N(R) and N(R) = {0, } for some nonzero ∈ R (and hence Z(R) = {0});

(3) Either AG(R) = ' or AG(R) = ' ;

(4) Either Γ(R) = ' or Γ(R) = ' .

Some basic properties of ANN$_G$(R)

In this section we study the some basic properties of the new annihilator graph ANN$_G$(R). We show that ANN$_G$(R) is connected with diameter at most two. If ANN$_G$(R) contains a cycle, we show that girth of ANN$_G$(R) is at most four. If | Z(R)*| = 1 for a commutative ring R, then assume Z(R)* = { } and hence = 0 . In this case R is ring-isomorphic to either ℤ or ℤ [X] /⟨ X ^{2} ⟩. Thus all the graphs ANN$_G$(R), AG(R) and Γ(R) are trivial with vertex and hence ANNG(R) = AG(R) = Γ(R). In this case (ANNG(R)) = 0. Hence throughout this article, we consider commutative rings with more than one nonzero zero-divisors.

Theorem Let R be a commutative ring.

(1) Let and be distinct elements of Z(R)*. Then — is not an edge of ANNG(R) if and only if ( ) = ( ) = ( ).

(2) If — is an edge of Γ(R) for some distinct , ∈ Z(R)*, then — is an edge of ANNG(R). In particular, if P is a path in Γ(R), then P is a path in ANNG(R).

(3) If — is an edge of AG(R) for some distinct , ∈ Z(R)*, then — is an edge of ANNG(R). In particular, if P is a path in AG(R), then P is a path in ANNG(R).

(4) If Γ(R) (,) = 3 for some distinct , ∈ Z(R)*, then — is an edge of ANNG(R).

(5) If — is not an edge of ANNG(R) for some distinct , ∈ Z(R)*, then there is a ∈ Z(R)* − { , } such that — — is a path in Γ(R) and AG(R), and hence — — is also a path in ANNG(R).

(6) If ANNG(R) = Γ(R), then ANNG(R) = AG(R).

Proof. (1) Suppose that — is not an edge of ANNG(R). Then ( ) = ( ) ∩ ( ) by definition. Thus ( ) ⊆ ( ) and ( ) ⊆ ( ). But ( ) ⊆ ( ) and ( ) ⊆ ( ). Hence ( ) = ( ) = ( ). Conversely, suppose that ( ) = ( ) = ( ). Then ( ) = ( ) ∩ ( ). Hence — is not an edge of ANNG(R) by definition.

(2) Suppose that — is an edge of Γ(R) for some distinct , ∈ Z(R)*. Then = 0 and ( ) = (0) = R. Since ≠ 0, ≠ 0, we have ( ) ≠ R and ( ) ≠ R. Therefore ( ) ≠ ( ) and ( ) ≠

( ). Hence — is an edge of ANNG(R) by (1). In particular, suppose that P : — — — …— is a path of length in Γ(R). Then — is an edge of Γ(R) for all (0 ≤ < − 1). This implies — is an edge of ANNG(R) for all (0 ≤ < − 1). Hence P : — — — …— is a path of length in ANNG(R).

(3) Suppose that — is an edge of AG(R) for some distinct , ∈ Z(R)*. Then ( ) ≠ ( ) and ( ) ≠ ( ) by Lemma (1). Hence — is an edge of ANNG(R) by (1). In particular, suppose that P : — — — …— is a path of length in AG(R). Then — is an edge of AG(R) for all (0 ≤ < − 1). This implies — is an edge of ANNG(R) for all (0 ≤ < − 1). Hence P : — — — …— is a path of length in ANNG(R).

(4) Suppose that Γ(R) (,) = 3 for some distinct , ∈ Z(R)*. So assume — — — is a shortest path connecting and in Γ(R), where , ∈ Z(R)* and ≠ . This implies = 0, = 0, = 0, ≠ 0 and ≠ 0. This implies ∈ ( ) and ∈ ( ). Thus { , } ⊆ ( ) such that ∉ ( ) and ∉ ( ). Therefore ( ) ≠ ( ) and ( ) ≠ ( ). Hence — is an edge of ANNG(R) by (1).

Alternative proof of (4) . Suppose that Γ(R) (,) = 3 for some distinct , ∈ Z(R)*. Then — is an edge of AG(R) by Lemma (3). Hence — is an edge of ANNG(R) by (3).

(5) Suppose that — is not an edge of ANNG(R) for some distinct , ∈ Z(R)*. Then ( ) = ( ) = ( ) by (1). Also — is not an edge of Γ(R) by (2) and hence ≠ 0. Therefore there is a ∈ ( ) = ( ) such that ≠ 0. If ∈ { , }, then = 0, a contradiction. Thus ∈ Z(R)* − { , } such that — — is a path in Γ(R) and also a path in AG(R) by Lemma (2). Hence — — is a path in ANNG(R) by (2) or (3).

Alternative proof of (5)**.** Suppose that — is not an edge of ANNG(R) for some distinct , $\in$ Z(R)*. Then — is not an edge of AG(R) by (3). Thus $\in$ Z(R)* $-$ { , } such that — — is a path in Γ(R) and AG(R) by Lemma (4). Hence — — is a path in ANNG(R) by (2) or (3).

(6) Let ANNG(R) = Γ(R). If possible, suppose that ANNG(R) $\neq$ AG(R). Then there are some distinct , $\in$ Z(R)* such that — is an edge of ANNG(R) that is not an edge of AG(R). So — is not an edge of Γ(R) by Lemma (2) , and hence ANNG(R) $\neq$ Γ(R), a contradiction. Thus ANNG(R) = AG(R).

Remark. (1) The converse of the Theorem (2) is not true in general. In ℤ ,
2 — 6 is an edge of ANNG(ℤ), but 2 — 6 is not an edge of Γ(ℤ).

(2) The converse of the Theorem (3) is not true in general. In ℤ ,
2 — 4 is an edge of ANNG(ℤ), but 2 — 4 is not an edge of AG (ℤ).

(3) Every edge of Γ(R) is an edge of ANNG(R) by Theorem (2) and V(ANNG(R)) = V (Γ(R)). So Γ(R) is a spanning subgraph of ANNG(R). Again every edge of AG(R) is an edge of ANNG(R) by Theorem (3) and V(ANNG(R)) = V(AG(R)). So AG(R) is also a spanning subgraph of ANNG(R).

Theorem. Let R be a commutative ring with |Z(R)*| $\geq$ 2. Then ANNG(R) is connected and (ANNG(R)) $\in$ {1, 2}.

Proof. Let and be two distinct elements of Z(R)*. If — is an edge of ANNG(R), then (,) = 1. Suppose that — is not an edge of ANNG(R). Then there is a $\in$ Z(R)* $-$ { , } such that — — is a path in Γ(R) and AG(R), and hence — — is also a path in ANNG(R) by Theorem (5). Thus (,) = 2. Hence ANNG(R) is connected and (ANNG(R)) $\in$ {1, 2}.

Example. (1) Consider the non-reduced commutative ring R = ℤ . Then ANNG(R) = and hence (ANNG(R)) = 1.

(2) Consider the non-reduced commutative ring R = ℤ $\times$ ℤ . Then (0, 1) — (0, 3) is

not an edge of ANNG(R). Let be the edge (0, 1) — (0, 3). Then ANNG(R) = −

and hence (ANNG(R)) = 2.

(3) Consider the reduced commutative ring R = ℤ × ℤ . Then ANNG(R) = 1,1 and

hence (ANNG(R)) = 1.

(4) Consider the reduced commutative ring R = ℤ . Then ANNG(R) = 1,2 and hence

(ANNG(R)) = 2.

Theorem. Let R be a commutative ring. Suppose that — is an edge of ANNG(R) that is not an edge of Γ(R) for some distinct , ∈ Z(R)*. If Γ(R) (,) = 3, then ANNG(R) contains a cycle of length 3 and (ANNG(R)) = 3.

Proof. Suppose that — is an edge of ANNG(R) that is not an edge of Γ(R) for some distinct , ∈ Z(R)*. Suppose that Γ(R) (,) = 3. So assume — — — is a shortest path connecting and in Γ(R), where , ∈ Z(R)* and ≠ . This implies = 0, = 0, = 0, ≠ 0 and ≠ 0. This implies ∈ ( ). Since ∉ ( ), we have ( ) ≠ ( ). Thus — is an edge of ANNG(R) by Theorem(1). We have — — is a path in ANNG(R) by Theorem (2). Thus — — — is a cycle of length 3 in ANNG(R), and hence (ANNG(R)) = 3.

Theorem Let R be a commutative ring and suppose that ANNG(R) ≠ Γ(R). Then (ANNG(R)) = 3.

Proof. Since ANNG(R) ≠ Γ(R), there are some distinct , ∈ Z(R)* such that — is an edge of ANNG(R) that is not an edge of Γ(R). Since Γ(R) is connected, we have |Z(R)*| ≥ 3. Again, since (Γ(R)) ∈ {0, 1, 2, 3}, we have Γ(R) (,) ∈ {2, 3}.

Case 1. Let Γ(R) (,) = 2. So assume — — is a shortest path connecting and in Γ(R). Then — — is a path of length 2 from to in ANNG(R) by Theorem (2). Since — is an edge of ANNG(R), we have ANNG(R) contains a cycle of length 3. Hence (ANNG(R)) = 3.

Case 2. Let $\Gamma(R)$ (,) = 3. Then (ANNG(R)) = 3 by Theorem.

Thus combining both the cases, we have (ANNG(R)) = 3.

Example (1) Consider the reduced commutative ring R $= \mathbb{Z} \times \mathbb{Z}$. Then (2, 3) — (0, 3) is an edge of ANNG(R) that is not an edge of $\Gamma(R)$. Thus ANNG(R) $\neq$ $\Gamma(R)$ and (2, 3) — (0, 2) — (0,3) — (2, 3) is a cycle of length 3. Hence (ANNG(R)) = 3.

(2) Consider the non-reduced commutative ring R $= \mathbb{Z}$. Then ANNG(R) = 3 and $\Gamma(R)$ = 1,2 . Thus ANNG(R) $\neq \Gamma(R)$ and (ANNG(R)) = 3.

Conclusion :-

Let R be a commutative ring with unity. In this chapter, we defined a new annihilator graph ANNG(R) of R. We proved that the zero divisor graph (R) defined by D. F. Anderson and P. S. Livingston and the annihilator graph AG(R) defined by A. Badawim are spanning subgraphs of ANNG(R). We find that ANNG(R) is always associated with a mean of at most two. If ANNG(R) contains a cycle, we have shown that the circuit of ANNG(R) is at most four. We also investigated certain situations where ANNG(R) is identical to □ (R) and AG(R) for both reduced and irreducible commutative ring R.

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