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Stability Analysis Of Richardson's Arms Race Model With R&D Expenditure Parameter In Two Competitive Firms

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Abstract:
Richardson's Arms Race Model was originally proposed by mathematician Lewis Fry Richardson on the assumption that the existence of weapons – large military arsenals – increase the likelihood of violent conflict. The Richardson's Arms Race Model is an important tool in the study of arms races and international conflict. In this paper, we demonstrate that its potential applications are not only limited to study the political and military science, but also in economics for modelling competition in competitive firms.
Keywords: *Richardson's Arm Race Model, Economics, Stability Analysis, Mathematical Modelling*

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I. Introduction

The Richardson Arms Race Model was originally proposed and developed by the British Mathematician Lewis Fry Richardson in the mid of 1930s. His scientific training led him to believe that wars were a phenomenon that could be studied and mathematically modelled. In his model, he developed a system of differential equations which is used to describe how two countries (or rivals) can adjust their armament and military expenditures over time in response to each other. This developed model is widely used in political science, conflict studies and international relations. The core idea of this model is built on three forces. First, reaction to the rival's armament, second, integral fatigue or cost and third, grievances or fear. Richardson build a model to examine certain conditions in order to predict whether an arms race was 'stable' or 'unstable'.

Arms races can be described as a situation where two or more countries increase their fighting resources and military capabilities to gain a position of superiority. In an attempt to model these complex events, Richardson developed his arms race model consisting of a pair of first order ordinary differential equations which capture the action-reaction nature of these situations.

The basic idea of this model is based on the competitive exclusion principle, in ecology sometimes referred to as Gause's law (1934) [7] is his book 'The Struggle of Existence'. The 'Competitive exclusion principle' (CEP) states that two species with identical niches cannot coexist indefinitely. Gause's CEP has been one of the central themes of research in ecology: trying to understand the mechanisms of species coexistence and patterns of biodiversity.

The idea of Richardson's arms race model can be explored to other forms of conflict such as amongst competing firms. Research and Development (R&D) expenditure is the money a company spends on research and development to innovate its products, services and processes, which can be recorded as an operating expense or capitalized if it has future economic benefits. Research indicates that R&D spending is positively correlated with a firm's performance, including its market value and can influence firm growth, although many factors are involved.

In 2025, the top companies by projected R&D expenditure were Amazon, Alphabet (Google) and Meta with investments in areas like AI, cloud, and autonomous systems. Hence, for illustration, in this paper we consider the Research and Development (R&D) of Microsoft and Alphabet, since both are competitive and working in the same niche area.

II. Literature Review

Dunne, J. P., et al. (2003)[5] examined Richardson's action-reaction model of an arms race, which has prompted significant research attempting to empirically approximate such models. In general, these attempts were not successful. They used latest developments in time-series econometrics to illustrate problems with estimates for Turkey and Greece, as well as India and Pakistan. They found little proofs for a Richardson-type arms race between Greece and Turkey, whereas India and Pakistan exhibited a stable interaction with a clear equilibrium.

Dunne, J. P. et al. (2005)[6] examined Richardson's action-reaction model of arms races, which has inspired many empirical studies, though most have achieved limited success. His research revisited the estimation challenges associated with such models and, using recent advances in time-series econometrics, analysed military expenditure data for Greece and Turkey. The findings revealed evidence of cointegration between the two countries' military spending, indicating a long-term relationship, but not one consistent with the classical Richardson-type arms race model.

Tishler and Milstein (2009)[17] proposed a two-stage oligopoly model examining how competition influences firms' R&D decisions. They found a U-shaped relationship between competition and R&D effort - innovation declines at moderate competition but rises sharply under intense rivalry as firms enter "R&D wars." The model suggests that high competition can boost total R&D spending even as market output falls, affecting firm survival and welfare.

Lehmann, B., et al. (2009)[13] modified the Richardson Arms Race Model by introducing a carrying capacity term to each equation, similar to the carrying capacity term in a logistic growth model. They found that introducing these terms allowed for the prediction of the level of armament for each country at the onset of war.

Miltiadis Chalikias et al. (2014)[4] represented Lewis Richardson's arms race model in Advertising Expenditure of Two Competitive Firms. Richardson's arms race model was applied to secondary data from the mobile phone industry in Greece. They concluded that the theoretical models are almost identical with reality, which means that they can be applied to firms under the appropriate preconditions.

Kevin Zhang et al. (2021)[11] demonstrated another potential application of Richardson's Arms Race model beyond its original focus on defence and international conflicts. They applied the model to illustrate the competitive behaviour of two oligopolistic companies using R&D as a parameter.

V. S. Izhutkin and H. K. Aung (2021)[8] developed a computer simulation model to predict armed conflicts between organizations based on Richardson's approach. Their study demonstrated how computational techniques modernize the classical arms race model, enabling real-time visualization, sensitivity analysis, and experiments across geopolitical scenarios. The model bridges mathematical theory with practical applications in conflict analysis and defense decision-making.

Shatyrko, D. K. A. (2023)[16] introduces a refined perspective by addressing these limitations. The study initially applied the ODE-based Richardson model, using statistical data of R&D and GDP. This analysis enabled the observation of arms race behaviours through simulations and phase portraits, which provide visual insights into the stability and nature of equilibrium points within the model.

III. Richardson's Arms Race Model

Consider two neighbouring countries A and B, which are involved in arms race. Let $x(t)$ and $y(t)$ be the expenditures on arms respectively by these two countries in some standardized unit. Richardson added assumption that the cause of the rate of increase of a country's armament, not only depend on mutual stimulation but also on the permanent underlying grievances of each country against the other.[2]

The system of differential equations becomes:

$$\frac{dx}{dt} = ay - bx + c$$
$$\frac{dy}{dt} = dx - ey + f$$

where a, d : positive 'defense' coefficients
b, e : negative 'fatigue' coefficients
c, f : 'grievance' constants

A primary weakness with the Richardson's original arms race model is the challenge of finding the parameters through rigorous, quantitative methods.
Hence, the model is converted to a discrete-time model for easier parametrization.[10]

Let X_n be the level of armament of country A at $'n'$time period, and Y_n be the level of armament of country B at $'n'$time period.
The discrete-time version of Richardson's arms race model becomes:

$$X_n = aY_{n-1} + bX_{n-1} + c$$
$$Y_n = dX_{n-1} + eY_{n-1} + f$$

IV. Implementation Of Richardson's Arms Race Model

Richardson's arms race model was originally developed to study the science behind the military rivalry between two competitive countries. Its core principles of interdependence and action–reaction dynamics can be effectively adapted and then reinterpreted through its applications in economic modelling.

The potential application of Richardson Arms race model with one of its important economical parameters as Research and Development (R&D) used to study the behaviour amongst competing firms in the markets. [17]

In terms of our proposed model, the coefficients of parameters are defined as:
a, d : 'Defense' coefficient of R&D – extent to which a firm feels motivated to increase R&D in response to heightened R&D of opposing firms.
b, e : 'Fatigue' coefficient of R&D – extent to which one firm feels less incentivized to increase R&D given pre-existing expenditure levels.
c, f : 'Grievance' constants of R&D – continuing to be accounting for other unconsidered factors.

For illustration purpose, we consider the application of Richardson's arms race model for R&D competition between two firms where the discrete-time model version is applied to the R&D expenditures between two giant software companies Alphabet Inc. and Microsoft Inc., close competitors in markets working in the same niche.

Data Collection

The R&D expenditures were found from annual reports and financial statements publicly made available by Alphabet Inc. and Microsoft Inc. for every year from 2009 to 2025.[1]

Table 1: R&D expenditure per year from 2009-2025, in billions of USD [1]

Year	Microsoft Inc.	Alphabet Inc.
2009	9.01	2.84
2010	8.71	3.76
2011	9.04	5.16
2012	9.81	6.08
2013	10.41	7.14
2014	11.38	9.83
2015	12.05	12.28
2016	11.98	13.95
2017	13.04	16.63
2018	14.73	21.42
2019	16.88	26.02
2020	19.27	27.57
2021	20.72	31.56
2022	26.63	37.64
2023	27.19	45.43
2024	29.51	49.33
2025	32.48	55.63

V. Stability Analysis

Using the data available, multiple linear regression are performed to find the best fit discrete-time model where $'n'$ is the number of years since 2008. The output with all the parameters value is as follows:

$$X_n = 0.31Y_{n-1} + 0.46X_{n-1} + 3.79$$
$$Y_n = 0.36X_{n-1} + 0.94Y_{n-1} - 1.27$$

To check the stability and nature of the dynamical system, we demonstrate two approaches; first, by solving for a theoretical stable state [12]; second, by determining the eigenvalues of the transformation matrix of the system to assess the long-term behaviour of the dynamical system and determine whether it will reach the stability or not.[12]

Stability Analysis - First Approach

Consider the discrete time model, let $A_n = \begin{pmatrix} X(n) \\ Y(n) \end{pmatrix}$ represents the state of the system at time period n. Let $B = \begin{pmatrix} 0.46 & 0.31 \\ 0.36 & 0.94 \end{pmatrix}$ be the transformation matrix of the system, representing the transformation of the system by 'defense' and 'fatigue' coefficients. Let $C = \begin{pmatrix} 3.97 \\ -1.27 \end{pmatrix}$ be the constant vector for the grievance constants.
The system can be rewritten as

$$A_n = BA_{n-1} + C$$

By definition of a stable state, we know that

$$A_s = BA_s + C$$

which gives

$$A_s = (I - B)^{-1} \cdot C$$

where I represents the 2×2 identity matrix.
Applying the values, we have

$$A_s = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.46 & 0.31 \\ 0.36 & 0.94 \end{pmatrix} \right)^{-1} \begin{pmatrix} 3.97 \\ -1.27 \end{pmatrix} = \begin{pmatrix} -7.85 \\ 25.91 \end{pmatrix}$$

We note that the proposed steady state has a value -7.85 for $X(s)$, which is not sensible, as there is no way to achieve negative expenditure.
Therefore, we conclude without further calculation that this system cannot achieve a steady state.

Stability Analysis - Second Approach

Since, $A_n = BA_{n-1} + C$ is a non-homogeneous system, we cannot directly reduce into the form of eigenvalues and eigenvectors.

Instead, we first consider the homogeneous part of the system which we denote $A'_n = BA'_{n-1} = B^n A_0$ where A_0 is the initial state of the system at time zero.

Let x_1 and x_2 be the eigenvectors of B, and μ_1 and μ_2 be their eigenvalues respectively. Then, we know that $A_0 = c_1 x_1 + c_2 x_2$ for some constants c_1 and c_2.
By definition of eigenvalues and eigenvectors, we have
$A'_n = \mu_1^n c_1 x_1 + \mu_2^n c_2 x_2$ for the homogeneous part of the system.
By adding the constant vector D such that
$A_n = A'_n + D = B^n A_0 + D$ which further gives the calculation as,

$$A_n = 1.111^n c_1 \begin{pmatrix} 0.432 \\ 0.902 \end{pmatrix} + 0.288^n c_2 \begin{pmatrix} -0875 \\ 0.484 \end{pmatrix} + \begin{pmatrix} 3.79 \\ -1.27 \end{pmatrix} - (1)$$

Clearly, the eigenvalue with the largest absolute value, in this case 1.111, will dominate as n increases. This dominant eigenvalue determines the long-term dynamics of the system, since $1.111 > 1$, the corresponding component grows without bound as $n \to \infty$. Therefore, the system does not attain the stable state for A_n.

Phase Plane Diagram

The equation (1) is the general solution of the given system of differential equation with the eigen values 1.111 and 0.288 along with two independent eigen vectors. By using the result of linear algebra, if the dynamical systems associated matrix with positive eigen values, the nature of the system is unstable. We plot the phase-plane diagram which represents the dynamical systems of the equation (1). We implement the dynamical systems in Python using the library NumPy and matplotlib, the output is shown below.

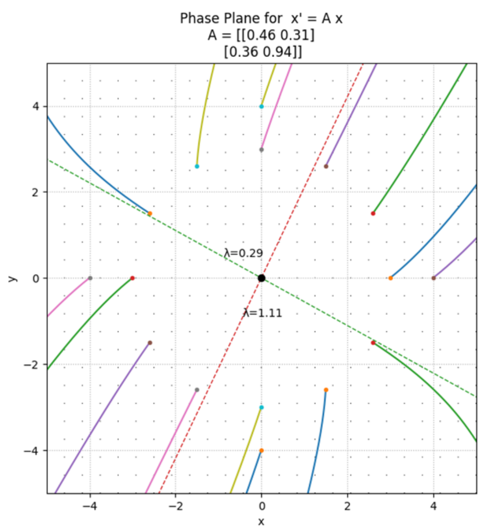

Figure 1: Phase Plane Diagram

The above graph represents the Phase Plane diagram of the dynamical systems with eigen values and clearly the vector fields indicates that the system is unstable.

VI. Conclusion

Both methods conclude that this system cannot reach a stable state and increase in R&D expenditure for both the competitive firms will likely continue without bound.

Based on the R&D expenditure data of both the companies as they are work on same ecological niche, both will continue to increase their R&D expenditure to remain competitive.

The result obtained by this proposed model demonstrates the potential for Richardson's Arms Race Model to be applied in competition situations within economic markets.

For future research, one can use another economical parameter to study the potential applications of Richardson Arms Race model. This model is not only limited to competitive firms but also applicable to study the nature of competition behaviour of social dynamics working in the identical niche.

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