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Examining Single step and Multistep Quantum Binomial Option Pricing Model

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Abstract**:** *A well-known challenge in finance is to price options efficiently and precisely. Quantum approach is being examined in the hopes of making more effective computers to price options more accurately than the classical computers. In this paper, the Quantization of efficiently used financial model called Binomial Option Pricing Model by solving rigid equations with the use of quantum theories is explored. It also generates much more effective circuit to price the options called Quantum Binomial Option Pricing Model. The proposed model has time complexity* $O((N-T)\, log_2(N-T))$ *in the quantum computational complexity class. Its risk-neutral system has an interesting structure as a disk in the unit ball of* R^3*. This eliminates arbitrage opportunities from the quantum field along with two thresholds.*

Key Word**:** *Quantum Field, Binomial Option Pricing Model, Quantum Binomial Option Pricing Model, Time Complexity, Quantum Computational Complexity Class*

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I. Introduction

A well-known challenge in finance is to price options efficiently and precisely. Quantum approach is being examined in the hopes of making more effective computers to price options more accurately than the classical computers. Qualitative research explains financial markets as price motion. It can be interpreted in terms of moving averages, momentums, and acceleration. In the financial world, whenever the price of any financial commodity is measured, it can be considered as a quantum particle. Moreover, financial markets appear with both particle and wave properties simultaneously. Richard Feynman [7] proposed that if the computers can be modelled on the basis of the theories of quantum mechanics, it would allow to resolve the difficulties that classical computers encounter when simulating quantum mechanical systems. As compared to a classical computer, it has been shown that simulating quantum mechanics on a quantum computer will achieve exponential speed up. As a result, if finance is quantum mechanical, it might be possible to simulate it on a quantum computer at exponential speed. This research expands the binomial option pricing model of finance which is an option valuation method developed in 1979 by financial analysts named Cox-Ross-Rubinstein. It is used in specification of nodes between the time period of valuation date and the expiration date of options.

A quantum model for the binomial market is presented in this paper. Also, the Cox-Ross-Rubinstein binomial option pricing model will be recalled from a quantum perspective by considering Maxwell-Boltzmann statistics of a system of N distinct entities. As a consequence, it indicates that using quantum financial models in finance theory might be of concern. Among classical and quantum computers, a complete description of a system of n components needs only n bits, while in quantum physics, 2^n complex numbers are needed.

In this research, some basic mathematical tools called density matrix, quantum operator, Hilbert space etc. are used to construct a circuit which follows the quantum theories and based on quantum mechanics. Some quantum gates called rotation gate, addition gate and Hadamard gate are used to help the circuit to run the quantum binomial option pricing model. Also, the multistep binomial option pricing model will be converted to the quantum model which can be restricted to single step or two step models. It also generates much more effective circuit to price the options called Quantum Binomial Option Pricing Model with time complexity $O((N-T)\log_2(N-T))$ in the quantum computational complexity class whose risk-neutral system has an interesting structure as a disk in the unit ball of R^3, with two thresholds that eliminate arbitrage opportunities from the quantum field.

II. Literature Review

Accardi & Boukas (2007) mentioned a quantum extension of the Black-Scholes equation in the sense of Hudson-Parthasarathy quantum stochastic calculus and also used Quantum Brownian motion and the Poisson method to depict stock markets which was so much motivated by the work of Segal and Segal (1997) on Black-Scholes pricing formula in the quantum context.

Meyer (2009) presented most relevant article to understand quantum binomial model which gives useful information about time and space complexity in quantum sense.

Ivancevic (2009) proposed a bidirectional quantum associative memory structure for Black–Scholes–like option price progression. It comprised of a pair of coupled NLS equations, one controlling stochastic volatility as well as the other administering option price, both self-organizing in an adaptive 'market heat potential' trained by continuous Hebbian learning. By using approach of lines with adaptive step-size integrator, this stiff pair of NLS equations were solved numerically. He established a quantum neural composition approach for option price modelling.

Zhang & Huang (2010) used a cosine distribution to simulate the state of stock price in equilibrium. After adding an external field into the Hamiltonian to analytically calculate the wave function, the distribution and the average value of the rate of return were shown.

Aerts et. al. (2012) projected a sphere model that described the buying/selling mechanism of a stock. They also demonstrated that the stock does not have a definite price until it is exchanged. it concluded very essentially because it gives theoretical support for the use of quantum models in finance.

Wang and Wang (2014) replaced the Glosten–Jagannathan–Runkle (GJR) model's classical normal mechanism by a quantum wave-function distribution centred on a "one-dimensional infinitely deep square potential well" for the better results.

Gregory pelts (2018) presented quantum pricing, gave group theory and Hilbert space approach to build solvable financial models and also used Hamiltonian operators for certain calibrations.

Hao et. al. (2019) used Dirac-matrix formalism and the Feynman path integral approach to re-examine the Black-Scholes and Cox-Ross-Rubinstein model.

Focardi et. al. (2020) discussed, why quantum probability can be used in finance. The replacement of random variables with operators, business self-reflexivity, and the presence of incompatible findings are all crucial problems. Quantum probability theory, quantum stochastic processes, and the pricing of options in a quantum sense were all covered by the authors in their proposed article.

Baaquie (2020) approached linear algebra, functional analysis and many other pure mathematical tools to develop an idea about quantum mathematics to establish a bridge between quantum mechanics and finance.

III. Methodology

The expanded formula for pricing an option C which is:

$$C = e^{-rt}\left[\frac{e^{rt}-d}{u-d}c_u + \left(1-\frac{e^{rt}-d}{u-d}\right)c_d\right]$$

Where, $q = \frac{e^{-rt}-d}{u-d}$ = probability of an upward stock movement

$u = e^{\sigma\sqrt{\Delta T}}$ = upward movement amount

$d = e^{-\sigma\sqrt{\Delta T}}$ = downward movement amount

$\sigma =$ Historical volatility

Here for risk-free bank account B and stock S, an arbitrage free replicating portfolio could be set up as:

$$B_1 = B_0(1+r)$$

$$S = S_0(1+R)$$

A single step binomial option pricing model is taken, to understand the basic formula. It is assumed that the replicating portfolio taken here is arbitrage-free to avoid some tough calculations. Any particle in quantum state should have density matrix which describes the statistical state, whether pure or mixed, of a system in quantum mechanics.

Here a density matrix ρ is defined as:

$$\rho = \frac{1}{2}[wI_2 + x\sigma_x + y\sigma_y + z\sigma_z]$$

In quantum mechanics, density matrix is always constructed with the help of Pauli matrices.

Pauli matrices: Pauli matrices are normalized pairs of matrices used to identify spin in quantum mechanics and are always Hermitian in nature.

$$I_2 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \sigma_x = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \sigma_y = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}, \sigma_z = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

Moreover, density matrix is always Hermitian matrix. Here the density matrix is,

$$\rho = \frac{1}{2}\begin{bmatrix}w+z & x-iy\\x+iy & w-z\end{bmatrix}$$

Density matrix is defined in one quantum state, to transform a stock from one quantum state to another quantum state, a quantum operator will be used.
To define quantum operator A:

$$A = [x_0 I_2 + x_1\sigma_x + x_2\sigma_y + x_3\sigma_z]$$

Quantum operator can also be defined by the use of Pauli matrices, which is:

$$A = \begin{bmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{bmatrix}$$

Here variables used to construct density matrix and quantum operator say, x, y, z, w, x_0, x_1, x_2 and x_3 are real values which helps to identify stock in a quantum state. Let a and b be the Eigen-values of quantum operator A.

Calculations to find the position of density matrix and its rate of change:

To find the Eigen value of A,

$$|A - \lambda I| = 0$$

$$\therefore \begin{vmatrix} x_0 + x_3 - \lambda & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 - \lambda \end{vmatrix} = 0$$

$$\therefore ([x_0 + x_3 - \lambda][x_0 - x_3 - \lambda] - [x_1 - ix_2][x_1 + ix_2]) = 0$$

$$\therefore (x_0 - \lambda)^2 - x_3^2 - x_1^2 - x_2^2 = 0$$

Making λ a subject of the equation,

$$\therefore (x_0 - \lambda)^2 = x_1^2 + x_2^2 + x_3^2$$

$$\therefore \lambda = x_0 \pm \sqrt{x_1^2 + x_2^2 + x_3^2}$$

To denote both the values of Eigen values with variables a and b respectively,

$$a = x_0 - \sqrt{x_1^2 + x_2^2 + x_3^2} \quad \text{And} \quad b = x_0 + \sqrt{x_1^2 + x_2^2 + x_3^2}$$

Here all $x_j \neq 0$ and $\sqrt{x_1^2 + x_2^2 + x_3^2} \neq 0$ with $a, b > -1$. These are the required Eigen values for the matrix A.
If x_0 is made subject of both the equations;

$$x_0 = a + \sqrt{x_1^2 + x_2^2 + x_3^2} \quad \text{And} \quad x_0 = b - \sqrt{x_1^2 + x_2^2 + x_3^2}$$

Comparing both the equations,

$$\therefore a + \sqrt{x_1^2 + x_2^2 + x_3^2} = b - \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$\therefore \sqrt{x_1^2 + x_2^2 + x_3^2} = \frac{b-a}{2}$$

Now squaring both the sides in the above equation

$$\therefore x_1^2 + x_2^2 + x_3^2 = \frac{(b-a)^2}{4}$$

Putting back the value of $x_1^2 + x_2^2 + x_3^2$ in the equation of any Eigen value,

$$\therefore x_0 = a + \frac{b-a}{2}$$

$$\therefore x_0 = \frac{a+b}{2}$$

Now to find the situation of the density matrix of stock, trace of the matrix can be used.
Therefore

$$\rho = \frac{1}{2}\begin{bmatrix} w+z & x-iy \\ x+iy & w-z \end{bmatrix}$$

$$\therefore tr(\rho) = \frac{1}{2}(w + z + w - z)$$

Now trace of any density matrix is one.

$$\therefore tr(\rho) = \frac{1}{2}(w + z + w - z) = 1$$

$$\therefore w = 1$$

To find the rate of change which describes the changes of two quantities, trace of the product of quantum operator and density matrix can be calculated as,

$$\text{Rate of change} = tr(\rho A)$$

$$= \frac{1}{2}\begin{bmatrix} w+z & x-iy \\ x+iy & w-z \end{bmatrix}\begin{bmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{bmatrix}$$

$$= \frac{1}{2}[2wx_0 + 2zx_3 + 2xx_1 + 2x_2 y]$$

$$= wx_0 + xx_1 + yx_2 + zx_3$$

Putting the values of w=1 and $x_0 = \frac{a+b}{2}$ in above equation, we get
Rate of change = $r = \frac{a+b}{2} + (xx_1 + yx_2 + zx_3)$

$$\therefore r - \frac{a+b}{2} = xx_1 + yx_2 + zx_3$$

Calculations to Find the Radius and Risk-Neutral measure in Sample Space Ω={1+ a, 1+b}:

To calculate the Eigen-Values of density matrix for prediction of the radius of risk neutral state, where the density matrix is:

$$\rho = \frac{1}{2}\begin{bmatrix} w+z & x-iy \\ x+iy & w-z \end{bmatrix}$$

Now $|\rho - \lambda I| = 0$,

$$\therefore \frac{1}{2}\begin{bmatrix} w+z-\lambda & x-iy \\ x+iy & w-z-\lambda \end{bmatrix} = 0$$

$$\therefore \frac{1}{2}[w^2 - wz - w\lambda + wz - z^2 - z\lambda - w\lambda + z\lambda + \lambda^2] - \frac{1}{2}[x^2 + y^2] = 0$$

Therefore,

$$\therefore \frac{1}{2}[w^2 - 2wz - z^2 - \lambda^2] - \frac{1}{2}[x^2 + y^2] = 0$$

Simplifying the equation,

$$\therefore \lambda = w \pm \sqrt{x^2 + y^2 + z^2}$$

Separating both the Eigen values,

$$\lambda_1 = w + \sqrt{x^2 + y^2 + z^2} \text{ And } \lambda_2 = w - \sqrt{x^2 + y^2 + z^2}$$

Which are required Eigen values.
If $\lambda_i = 1$ for any $i = 1$ then

$$w + \sqrt{x^2 + y^2 + z^2} = 1$$

But the value of w is one. So, the value of $\sqrt{x^2 + y^2 + z^2}$ will be equal to zero. Here x, y and z are real valued variables so it is not possible.
∴ for $w = 1$, $\sqrt{x^2 + y^2 + z^2} < 1$
Now the equation of Risk Neutral state can be derived by

$$r - \frac{a+b}{2} = xx_1 + yx_2 + zx_3$$

Here Risk neutral state is a unit disc with some radius.
An equation of sphere which covers whole risk neutral state

$$x^2 + y^2 + z^2 = R^2$$

But the radius of sphere must be less than one.

$$\therefore x^2 + y^2 + z^2 = 1 - R^2$$

If radius is $r_1^2 = 1 - R^2$ than $r_1 = \sqrt{1 - R^2}$. Now compute the formula which establishes the relation between radius and its corresponding plane.

$$\frac{xx_1 + yx_2 + zx_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} = \text{radius} = \text{r}_1 = \sqrt{1 - R^2}$$

Putting the values of sphere and risk neutral state from previous equations,

$$\frac{\left(r - \frac{a+b}{2}\right)}{\left(\frac{b-a}{2}\right)} = \sqrt{1 - R^2}$$

$$\therefore \frac{(2r - a - b)^2}{(b-a)^2} = 1 - R^2$$

$$\therefore 1 - \frac{(2r - a - b)^2}{(b-a)^2} = R^2$$

$$\therefore \sqrt{1 - \frac{(2r - a - b)^2}{(b-a)^2}} = R$$

Let B and S are the Risk-free Bank Account and Stock respectively. Now the replicating portfolio can be setup as:

$B_1 = B_0(1+r)$ And $S_1 = S_0(A+I_2)$. Here $R = \sqrt{1-M^2}$ where M stands for risk neutral measure on sample space Ω = {1+ a, 1+b}. So, from previous equations,

$$M = \frac{2r-a-b}{b-a}$$

Now for rate of change $r = b$ and $r = a$, Value of M is: $M_b = \frac{r-a}{b-a}$ and $1 - M_b = M_a = \frac{b-r}{b-a}$ respectively.

An option on a stock, whose current price is C at time zero will be discounted by $\frac{1}{1+r}$.

Calculations to Find the Quantum Binomial Option Pricing Model:

Assuming the market is quantum,

$$C = \frac{1}{1+r}[M_b h_b + M_a h_a]$$

$$\therefore C = \frac{1}{1+r}\left[\left(\frac{r-a}{b-a}\right)h_b + \left(\frac{b-r}{b-a}\right)h_a\right]$$

Where h_b is the price of call option if, there is an upward movement in the stock of $(1+b)$,

h_a is the price of the call option if there is a downward movement in the stock of $(1+a)$.

Now from the classical binomial option pricing model

$$h_b = [s_0(1+b)-k]^+ \text{and } h_a = [s_0(1+a)-k]^+$$

Where s_0 = stock price and k = strike price.

similarly, the model for put options can also be developed with slight changes.

Describing Different Notations used in the Classical and Quantum Binomial Model:

Classical Meaning	*Classical Variable*	*Quantum Variable*	*Quantum Meaning*
Risk-free bank Account after steep n	$\boldsymbol{B_n}$	$\boldsymbol{B_n}$	Risk-free bank Account after steep n
Stock price at step n	$\boldsymbol{S_n}$	$\boldsymbol{S_n}$	Stock price at step n
Strike price	$\boldsymbol{K}$	$\boldsymbol{K}$	Strike price
Risk-free Rate	$\boldsymbol{r}$	$\boldsymbol{r}$	Risk-free Rate
Random Real Scalar Growth	$\boldsymbol{R}$	$\boldsymbol{A}$	Random Complex Unitary Matrix Growth
Downward Moment Amount	$\boldsymbol{d}$	$\boldsymbol{1+a}$	Downward Moment Amount
Upward Movement Amount	$\boldsymbol{u}$	$\boldsymbol{1+b}$	Upward Movement Amount
Downward Option Price	$\boldsymbol{C_d}$	$\boldsymbol{h_a}$	Downward Option Price
Upward Option Price	$\boldsymbol{C_u}$	$\boldsymbol{h_b}$	Upward Option Price
Upward Stock Movement Probability	$\boldsymbol{q}$	$\boldsymbol{q}$	Upward Stock Movement Probability
Stock Volatility	$\boldsymbol{\sigma}$	$\boldsymbol{\sigma}$	Stock Volatility
Time Step	$\boldsymbol{\Delta t}$	$\boldsymbol{\Delta t}$	Time Step
Total Time	$\boldsymbol{T}$	$\boldsymbol{T}$	Total Time

Pricing of the Call option by the Classical model is given as:

$$C = e^{-rt}\left[\frac{e^{rt}-d}{u-d}c_u + \left(1 - \frac{e^{rt}-d}{u-d}\right)c_d\right]$$

In this Model, q is estimated using volatility. Moment amounts u and d are estimated as a function of the historical stock volatility. Quantum model is priced as:

$$C = \frac{1}{1+r}\left[\left(\frac{r-a}{b-a}\right)h_b + \left(\frac{b-r}{b-a}\right)h_a\right]$$

The Quantum model represents probability with density matrices. Expected value of the quantum operator $\boldsymbol{A}$ is calculated with repetition of the measurements.

Numerical example of the given model:

Firstly, fix some parameters to find the call option value from the quantum binomial option pricing model because of the unavailability of quantum circuit simulators.

Here stock price is Rs. 50 and strike price is Rs. 25. The calculated value of call option for classical binomial option pricing model with the same dataset is 25.23. The various values obtained by the above discussion and formulae are given as:

$r = 0.00786, a = -0.1006,$
$b = 0.1119,$
$x_0 = 0.00565$
$M_a = 0.4896,$
$M_b = 0.5104,$
$h_b = 30.595,$
$h_a = 19.97,$
$w = 1,$
$(x, y, z) = \left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}\right),$
$(x_1, x_2, x_3) = (0.001, 0.232, -0.229),$

$$\mathrm{C} = \frac{1}{1+r}\left[\left(\frac{r-a}{b-a}\right)h_b + \left(\frac{b-r}{b-a}\right)h_a\right]$$

$$= \frac{1}{1+0.00786}\left[\left(\frac{0.00786+0.1006}{0.1119+0.1006}\right)(30.595) + \left(\frac{0.1119-0.00786}{0.1119+0.1006}\right)(19.97)\right]$$

$$\mathbf{C} = \mathbf{25.14}$$

So, one can easily observe that the calculated value from the quantum binomial option pricing model and already calculated value from the classical one is almost equal. Briefly,

1. Stock movement is based on the evolution of the quantum rate of return A until a measurement is made.
2. When a measurement is finally made, the equation collapses to the classical binomial model with the option value being based on the Eigen values of A.
3. Each Eigen values a and b has a specific probability of being the value returned measurement.

Calculations for Probability of measuring a specific Eigen Values:

First of all, assume u and v form an orthonormal basis in C^2. A can be written in terms of outer products of the basis and its Eigen values follows $A = a|u\rangle\langle u| + b|v\rangle\langle v|$,

Where a and b are Eigen Values of A and $|u\rangle\langle u|$ means the outer product of the vectors $|u\rangle$ and $\langle u|$ means if

$$|u\rangle = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} and \langle u| = [u_1^* \, u_2^*] \; then \; |u\rangle\langle u| = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} [u_1^* \quad u_2^*]$$

Here "outer product is nothing but matrix generated by each element of u and v". Also "Transposition of the matrix is nothing but the inverse of the given matrix".

Thus, the probability that measures a system in the state ρ will be resulted in the Eigen value a as $\langle u|\rho\,|u\rangle$ and similarly for b it is $\langle v|\rho\,|v\rangle$. The probability that A takes the value a or b after measurement can be expressed as follows:

$$p(a) = \langle u|\rho|u\rangle = \frac{1}{2}[u_1 u_1^*(w+z) + u_1 u_2^*(x-iy) + u_2^* u_1(x+iy) + u_2 u_2^*(w-z)]$$
$$p(b) = \langle v|\rho|v\rangle = \frac{1}{2}[v_1 v_1^*(w+z) + v_1 v_2^*(x-iy) + v_2^* v_1(x+iy) + v_2 v_2^*(w-z)]$$

It will be reduced to:

$$p(a) = \langle u|\rho|u\rangle = \frac{1}{2}[(w+z)]$$
$$p(b) = \langle v|\rho|v\rangle = \frac{1}{2}[(w-z)]$$

The QBOPM is just like any other quantum model which evolves from one state to another state. It still remains in a quantum state and that's why it must not be measured.

Thus, measurement of the QBOPM should not occur until it has reached the desired number of steps required for the desired accuracy of the option price.

This means that during the evolution of the stock from one to another quantum state without measurement, the probability of the stock moving up or down is not manifested. The density matrix preserves an entire statistical ensemble of the state whereas in the Classical binomial model, a measurement occurs after each step.

Base Algorithm for Simulating the Quantum Binomial Model:

Deriving the Eigen vectors of the Quantum operator Proposed by A. Meyer :

Eigen Vectors will be used as the canonical basis for the simulation of a given volatility and a number of periods N. Volatility is measured as:

$$\sigma = \frac{\ln\left(1 + x_0 + \sqrt{x_1^2 + x_2^2 + x_3^2}\right)}{\sqrt{\frac{1}{t}}}$$

It can be rewritten in terms of b by substituting in earlier equations of Eigen values.

$$\sigma = \frac{\ln(1 + b)}{\sqrt{\frac{1}{t}}}$$

Here $t = \frac{T}{N}$ is the time of each period. The Eigen value b can then be calculated by rearranging equation as follows:

$$b = e^{\sigma\sqrt{\frac{1}{t}}} - 1.$$

Base Algorithm has several steps:

Deriving the Eigen vectors of the Quantum operators:

Using the value of b and the risk-free rate t as constants, the rest of the parameters for quantum operator and density matrix are shown in the equation. Equation is chosen such that the risk-free equation can be satisfied as:

$$\sqrt{1 - \frac{(2r - a - b)^2}{(b - a)^2}}$$

With these parameters chosen, the Eigen value a is calculated and the operator A for period-1 is constructed. The Eigen Vectors u and v of A are derived using both the Eigen values a and b. The Eigen Vectors u and v are then used as the canonical basis for further calculations.

Initialize the Input Values:

The second phase is the preparation of the input values according to the density matrix as follows:

$$\rho = \bigotimes_{J=1}^{N} (|u\rangle\langle u|(1 - q) + |v\rangle\langle v|q)$$

Here q is Upward Stock Movement Probability which can be derived similarly as the Classical BOPM.

Build the Quantum Operator:

The third phase is to build the quantum operator A that will be used to evolve the stock price S through the N periods.

This is done by combining the Eigen vectors u and v which forms the basis of the vector-space which is the generalized form for the Multistep BOPM.

$$\sum_{|\sigma|=m} \bigotimes_{J=1}^{N} |w_{j\sigma}\rangle\langle w_{j\sigma}|$$

Note that, here all σ (not to be confused with volatility) are subsets of {1, 2, ..., N}, $w_{j\sigma} = \begin{cases} u_j & j\epsilon\sigma \\ v_j & otherwise \end{cases}$, m will be used to represent the number of subsets per period as follows:

$$m = |\sigma| = \frac{N!}{n!\,(N - n)!}$$

This can be represented by Pascal's triangle. To combine the subsets, (the subset number) m is converted to binary for each subset. Similarly, subsets with the same number of zero's in their binary subset numbers are grouped together. The groups are sorted in descending order of the number zeros in their respective binary subset number m.

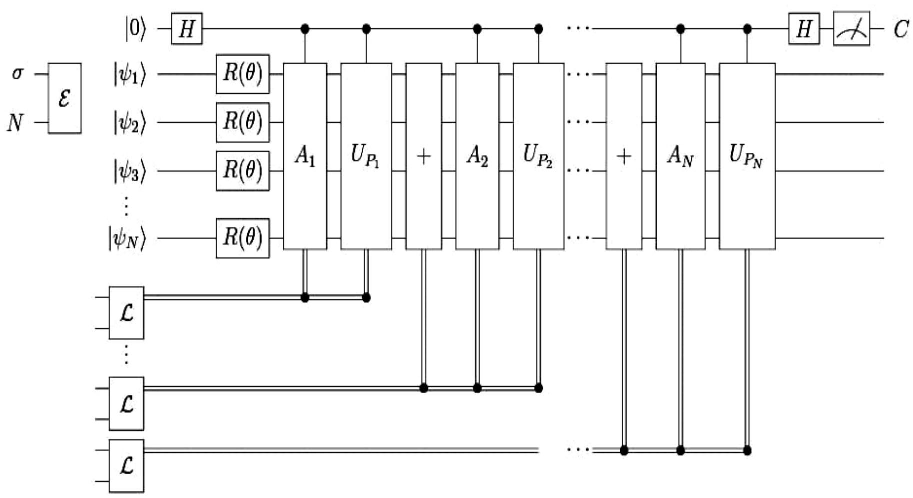
Finally for each binary subset number m, each zero is set to $|u\rangle\langle u|$ and each one is set to $|v\rangle\langle v|$ and then the tensor product for each of the resulting outer product is calculated.

Calculate the Payoff Formula for the Option Style:
First of all, to calculate the Payoff formula, for each n, the stock price $S_N = S_{N-1} + S_n$ is calculated. The next step is to implement the payoff formula and if the result is non-zero, multiply it by the quantum operator matrix created in the previous phase.

Calculate the Expected Value of the Quantum operator:
The Final phase is to determine the option price C_N by calculating the expected value of the quantum operator S_N as per the equation,

$$C_0^N = tr\left[\left(\bigotimes_{J=1}^{N} \rho_j\right)[S_N - k]^+\right]$$

Circuit for Quantum Binomial Option Pricing Model [11]:



The First part of the circuit above is the epsilon function call, which derives the Eigen vectors of the quantum operator based on volatility and the number of periods. Here epsilon requires four elementary calculations and solving for Eigen vectors of a matrix.
Second part of the circuit is the preparation of the input values $|\psi_1, \psi_2, \psi_3, \dots \dots, \psi_n\rangle$ according to the density matrix which can be done in the first step, using N rotation gates with appropriately selected values of theta (θ).

In the third part, each quantum operator is built as per the subset algorithm described in the section above which takes $O(2^N)$ time.

Subsequent simulations, where values other than volatility and the number of periods change, can be run without incurring the overhead associated with building the operators. However, it means that quantum binomial option pricing with stochastic volatility will incur this overhead every time. Now to implement the payoff function for option style being simulated where $\mathcal{L}$ contains the payoff condition for the option style being simulated, If the payoff is positive, the associated quantum operator $A_n$times U_{p_n}will act on the input qubits and add it to the value calculated in the previous step using the addition gate where the Addition gate will use the 2N qubits.

The final step is to calculate the expected value of the quantum operator. This is done using the gates on the wire with the zero ancillary qubits. The two Hadamard gates H and the control wire, calculate the expected value of the A operator $tr(\rho A)$ in constant time which is equal to the final value of the option C. The time complexity of the entire circuit, ignoring constant time operations, is therefore $O(2^N + Nlog_2N)$.

IV. Conclusion

In financial market, Binomial models are speculative and inaccurate because its assumption of no arbitrage and possibilities of early exercise. It is observed that a realistic model is a restriction of a large number of small binomial markets. However, there are some binomial problems in nature that cannot be represented using classical random variables. The risk-neutral world of quantum binomial markets has an interesting structure as a disk in the unit ball of R^3, whose radius is a function of the risk-free interest rate, with two thresholds that exclude market inefficiencies from this quantum market. This quantum algorithm belongs to the BQP quantum computational complexity class, according to analysis.

Despite the fact that the quantum algorithm used does not account for early exercise, it does retain all of the knowledge that would otherwise be lost in a complete binomial tree. Moreover, American options cannot be exercised because BQP does not allow us early exercise which can be developed in future. Besides that, since the matrix describing each cycle of the simulation is created, it seems fair to assume that early exercise could be incorporated into the algorithm without significantly increasing its complexity. If this is true, choice types requiring early exercise could be priced exponentially faster, as traditional binomial algorithms for these options have a space and time complexity of $O(N_2)$.

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