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Characterization And Structural Properties Of (T,S)-Pythagorean Fuzzy Subnear-Rings

Prabhat Wats, Amal Kumar Adak
Department Of Mathematics,Marwari College Bhagalpur
Department Of Mathematics, Ganesh Dutt College,
(Lalit Narayan Mithila University, Darbhanga)
Begusarai, India, 851101.

Abstract
The Pythagorean fuzzy set extends the intuitionistic fuzzy set by defining membership and nonmembership degrees such that the sum of their squares is less than or equal to one. We propose the concept of a Pythagorean fuzzy subnear-ring (PyFSNR) of a near-ring using t-norm T and a t-conorm S. We then characterize these substructures using a special class of level sets $U(A; t, s])$, which generalizes the notion of classical level subsets. Furthermore, we examine the behavior of the structures associated with homomorphisms. Utilizing congruence relations on near-rings, we develop a (T,S)-PyFSNR on the corresponding quotient near-ring.

Keywords: *Pythagorean fuzzy sets, Intuitionistic fuzzy sets, Pythagorean Fuzzy Subnear-Rings, Level set.*

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I. Introduction

To address the inherent imprecision in decision-making, Zadeh [23] introduced the concept of fuzzy sets (FSs), characterized by a membership function ϱ. This function assigns to each element a value within the interval $[0,1]$, representing degree of membership. In FSs, membership function quantifies the extent to which an element belongs to a given class: 0 indicates no membership, 1 denotes full membership, and values in between reflect partial association.

The classical theory of fuzzy sets was found to be somewhat limited, primarily due to its exclusion of a non-membership function and absence of a formal treatment for hesitation.In order to overcome these limitations, Atanassov [10] developed notion of intuitionistic fuzzy sets (IFSs). An IFS specified by a membership function ϱ and a non-membership function ϑ, along with a hesitation margin π (representing the degree of indeterminacy, i.e., neither membership nor non-membership). These satisfy the conditions $\varrho + \vartheta \leq 1$ and $\varrho + \vartheta + \pi = 1$.

However, situations also arise where $\varrho + \vartheta \geq 1$, which are not adequately captured by IFSs. In order to address this limitation, Yager [21, 22] proposed the notion of Pythagorean fuzzy sets (PyFSs). In a PFS, membership degree ϱ and non-membership degree ϑ satisfy $\varrho^2 + \vartheta^2 \leq 1$, and consequently $\varrho^2 + \vartheta^2 + \pi^2 = 1$, where π represents Pythagorean fuzzy index. This construct offers a richer framework for modeling vagueness and uncertainty.

In 1971, Rosenfeld [14] extended “fuzzy set theory to group theory and defined fuzzy group and derived some properties". Kuroki [15] introduced “the concept of fuzzy semigroups as a generalization of classical semigroups". A comprehensive treatment of their theoretical foundations was later provided by Mordeson et al. [13], offering “a systematic exposition of fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines and fuzzy languages". Abou-Zaid [1] introduced the concepts of fuzzy subnear-rings and fuzzy ideals within the framework of near-rings. Using the concept of intuitionistic fuzzy sets, Biswas [7] proposed the idea of an intuitionistic fuzzy subgroup of a group. The notion of intuitionistic fuzzy ideals of semigroups was first proposed by Kim et al. [?]. Kim et al. [9] investigated intuitionistic fuzzy semigroup bi-ideals. Zhan Jianming et al. [24] covered the different characteristics of intuitionistic fuzzy ideals of close rings. Concept of “ normal intuitionistic fuzzy N-subgroup in a near-ring" was presented by Cho et al. in [8]. Several work done in this area of PyF ideals [2, 3, 4, 5].

The concept of PyFSNR of a near-ring with regard to t norm T and t conorm S is presented in this work. Then, using a unique kind of level sets $U(A; [t, s])$, which is a generalisation of traditional level subsets, we describe each one of them. Specifically, we build (T, S)-PyFSNR on the near-ring of quotient with the aid of the congruence relations on near-rings.

II. Preliminaries

We go over several basic elements that are essential to this article in this part.

Definition 2.1 *An algebra* $(N,+,\cdot)$ *is near-ring if*
(i) $(N,+)$ is a group.
(ii) $(N,\cdot)$ is a semigroup.
(iii) $\zeta_1(\zeta_2+\zeta_3)=\zeta_1\zeta_2+\zeta_1\zeta_3;\ \zeta_1,\zeta_2,\zeta_3\in N$.

Definition 2.2 $I\subseteq N$ *is a subnear-ring of* N *iff* $\zeta_1-\zeta_2,\zeta_1\zeta_2\in I;\ \zeta_1,\zeta_2\in I$.

Definition 2.3 $f:N_1\to N_2$ *is homomorphism if* $f(\zeta_1+\zeta_2)=f(\zeta_1)+f(\zeta_2)$ *and* $f(\zeta_1\zeta_2)=f(\zeta_1)f(\zeta_2)$ *;* $\zeta_1,\zeta_2\in N_1$.

Definition 2.4 *An ideal* I *of* N *such that*
(a) $(I,+)$ is a normal subgroup of $(N,+)$
(b) $NI\subset I$
(c) $(r+i)s-rs\in I$ for all $i\in I$ and $r,s\in N$.
Let I be a subset of N. I qualifies as a left ideal of N if it fulfils conditions (a) and (b).
Again, I is classified as a right ideal of N if it meets conditions (a) and (c).
If I satisfies the properties of being a left ideal and a right ideal, then I is classified as an ideal of N.

Definition 2.5 *A quotient near-ring is defined as a near-ring that results from the quotient operation applied to a near-ring and one of its ideals and represented as* N/I.

The coset of I associated with a is defined as the set $a+I=\{a+s|s\in I\}$. The collection of all cosets is represented by N/I.

Lemma 1 *If* I *is an ideal of* N*, then the quotient set* N/I *forms a near-ring under the operations defined by*
$$(a+I)+(b+I)=(a+b)+I \quad \text{and} \quad (a+I)\cdot(b+I)=(a\cdot b)+I,$$
for all $a,b\in N$.

Definition 2.6 *A fuzzy set* F *is*
$$F=\{\langle\zeta_1,\varrho_F(\zeta_1)\rangle:\zeta_1\in X\},$$
where $\varrho_F:X\to[0,1]$ is a membership function.
Complement of ϱ is $\bar{\varrho}(\zeta_1)=1-\varrho(\zeta_1)$ for all $\zeta_1\in X$.

Definition 2.7 *An intuitionistic fuzzy set (IFS)* I *is*
$$I=\{\langle\zeta_1,\varrho_I(\zeta_1),\vartheta_I(\zeta_1)\rangle:\zeta_1\in X\},$$
where $\varrho_I:\zeta_1\to[0,1]$, $\vartheta_I:X\to[0,1]$ and $0\le\varrho_I(\zeta_1)+\vartheta_I(\zeta_1)\le 1$, for all $\zeta_1\in X$.
Indeterminacy $\pi_I(\zeta_1)=1-\varrho_I(\zeta_1)-\vartheta_I(\zeta_1)$.

Definition 2.8 *A pythagorean fuzzy set (PyFS)* P *is*
$$P=\{\langle\zeta_1,\varrho_P(\zeta_1),\vartheta_P(\zeta_1)\rangle|\zeta_1\in X\},$$
where $\varrho_P(\zeta_1):X\to[0,1]$ and $\vartheta_P(\zeta_1):X\to[0,1]$ denotes degree of membership and non-membership respectively with $0\le(\varrho_P(\zeta_1))^2+(\vartheta_P(\zeta_1))^2\le 1$.
The degree of indeterminacy $\pi_P(\zeta_1)=\sqrt{1-(\varrho_P(\zeta_1))^2-(\vartheta_P(\zeta_1))^2}$.

Some Operations on PyFSs
Let P_1 and P_2 be two PyFSs, then
(i) $P_1\subseteq P_2\ \ iff\ \ (\rho_{P_1}(\zeta_1)\le\rho_{P_2}(\zeta_1))$ and $(\sigma_{P_1}(\zeta_1)\ge\sigma_{P_2}(\zeta_1))$
(ii) $P_1=P_2\ \ iff\ \ (\rho_{P_1}(\zeta_1)=\rho_{P_2}(\zeta_1))$ and $(\sigma_{P_1}(\zeta_1)=\sigma_{P_2}(\zeta_1))$
(iii) $P_1\cap P_2=\{\langle\zeta_1.\min(\rho_{P_1}(\zeta_1),\rho_{P_2}(\zeta_1)),\max(\sigma_{P_1}(\zeta_1),\sigma_{PP_2}(\zeta_1))\rangle:\zeta_1\in X\}$
(iv) $P_1\cup P_2=\{\langle\zeta_1.\max(\rho_{P_1}(\zeta_1),\rho_{P_2}(\zeta_1)),\min(\sigma_{P_1}(\zeta_1),\sigma_{P_2}(\zeta_1))\rangle:\zeta_1\in X\}$.

Definition 2.9 *A PyFS* $A=(\varrho_A,\vartheta_A)$ *of* N *is a Pythagorean fuzzy subnear-ring (PyFSNR) of* N *if*
(i) $\varrho_A(\zeta_1-\zeta_2)\ge\min\{\varrho_A(\zeta_1),\varrho_A(\zeta_2)\}$ and $\vartheta_A(\zeta_1-\zeta_2)\le\max\{\vartheta_A(\zeta_1),\vartheta_A(\zeta_2)\}$,
(ii) $\varrho_A(\zeta_1\zeta_2)\ge\min\{\varrho_A(\zeta_1),\varrho_A(\zeta_2)\}$ and $\vartheta_A(\zeta_1\zeta_2)\le\max\{\vartheta_A(\zeta_1),\vartheta_A(\zeta_2)\}$, for all $\zeta_1,\zeta_2\in N$.

Definition 2.10 *A PyFS* $A = (\varrho_A, \vartheta_A)$ *of* N *is Pythagorean fuzzy ideal (PyFI) of* N *if*
(i) $\varrho_A(\zeta_1 - \zeta_2) \geq \varrho_A(\zeta_1) \wedge \varrho_A(\zeta_2)$ and $\vartheta_A(\zeta_1 - \zeta_2) \leq \vartheta_A(\zeta_1) \vee \vartheta_A(\zeta_2)$,
(ii) $\varrho_A(\zeta_2 + \zeta_1 - \zeta_2) = \varrho_A(\zeta_1)$ and $\vartheta_A(\zeta_2 + \zeta_1 - \zeta_2) = \vartheta_A(\zeta_1)$.
(iii) $\varrho_A(r\zeta_1) \geq \varrho_A(\zeta_1)$ and $\vartheta(r\zeta_1) \leq \vartheta(\zeta_1)$,
(iv) $\varrho((\zeta_1 + i)\zeta_2 - \zeta_1\zeta_2) \geq \varrho_A(i)$ and $\vartheta((\zeta_1 + i)\zeta_2 - \zeta_1\zeta_2) \leq \vartheta_A(i)$ for all $\zeta_1, \zeta_2, r, i \in N$.

If $A = (\varrho_A, \vartheta_A)$ satisfies (i),(ii) and (iii) then A is called a Pythagorean fuzzy left ideal of N and if $A = (\varrho_A, \vartheta_A)$ satisfies (i),(ii) and (iv) then A is called a Pythagorean fuzzy right ideal of N.

Lemma 2 *If* ϱ *is a PyFI of* N*, then* $\varrho(0) \geq \varrho(\zeta_1)$ *and* $\vartheta(0) \leq \vartheta(\zeta_1)$ *for all* $\zeta_1 \in N$.

Definition 2.11 *A* t*-norm is a function* $T: [0,1] \times [0,1] \to [0,1]$ *if*
(T1) $T(\zeta_1, 1) = \zeta_1$,
(T2) $T(\zeta_1, \zeta_2) = T(\zeta_2, \zeta_1)$
(T3) $T(\zeta_1, T(\zeta_2, \zeta_3)) = T(T(\zeta_1, \zeta_2), \zeta_3)$.
(T4) $T(\zeta_1, \zeta_2) \leq T(\zeta_1, \zeta_3)$, whenever $\zeta_2 \leq \zeta_3$, for all $\zeta_1, \zeta_2, \zeta_3 \in [0,1]$.
An example of t-norm is $T(\zeta_1, \zeta_2) = \min(\zeta_1, \zeta_2)$.
In general case, $T(\zeta_1, \zeta_2) \leq \min(\zeta_1, \zeta_2)$ and $T(\zeta_1, 0) = 0$ for all $\zeta_1, \zeta_2 \in [0,1]$.

Definition 2.12 $S: [0,1] \times [0,1] \to [0,1]$ *is a* t*-co-norm if*
(S1) $S(\zeta_1, 0) = \zeta_1$,
(S2) $S(\zeta_1, \zeta_2) = S(\zeta_2, \zeta_1)$
(S3) $S(\zeta_1, S(\zeta_2, \zeta_3)) = S(S(\zeta_1, \zeta_2), \zeta_3)$, for all $\zeta_1, \zeta_2, \zeta_3 \in [0,1]$
(S4) If $\zeta_1 \leq p$ and $\zeta_2 \leq q$, then $S(\zeta_1, \zeta_2) \leq S(p, q)$, for all $\zeta_1, \zeta_2, p, q \in [0,1]$

Proposition 1 *Let* S *be a* t*-co-norm. Then*
(1) $S(\zeta_1, 1) = 1,\ \zeta_1 \in [0,1]$
(2) $S(\zeta_1, \zeta_2) \geq \max(\zeta_1, \zeta_2),\ \zeta_1, \zeta_2 \in [0,1]$.

Definition 2.13 *Let* $A = \langle \varrho_A, \vartheta_A \rangle$ *be a PyFS in* N *and let* $s, t \in [0,1]$. *Then,*
$U(A: t, s) = \{(\varrho_A(\zeta_1) \geq t, \vartheta_A(\zeta_1) \leq s): \zeta_1 \in N\}$ with $t + s \leq 1\}$ is defined as level set of A.

III. Main Results

This section provides a definition of (T, S) PyFIs of near-rings and establishes several fundamental properties.

Definition 3.1 *A PyFS is defined as* $A = (\varrho_A, \vartheta_A)$ *A Pythagorean fuzzy subnearing in* N *is defined in accordance to a* t*-norm and co-*t*-norm, shortly* (T, S)*-PyFSNR of* N *if*
$(TF1)$ $\varrho_A(\zeta_1 - \zeta_2) \geq T(\varrho_A(\zeta_1), \varrho_A(\zeta_2))$ and $\vartheta_A(\zeta_1 - \zeta_2) \leq S(\vartheta_A(\zeta_1), \vartheta_A(\zeta_2))$
$(TF2)$ $\varrho_A(\zeta_1\zeta_2) \geq T(\varrho_A(\zeta_1), \varrho_A(\zeta_2))$ and $\vartheta_A(\zeta_1\zeta_2) \leq S(\vartheta_A(\zeta_1), \vartheta_A(\zeta_2))$, for all $\zeta_1, \zeta_2 \in N$.

Definition 3.2 *A* (T, S) *Pythagorean fuzzy subnearring* $A = (\varrho_A, \vartheta_A)$ *in* N *is referred to as* (T, S)*-PyFI of* N *if:*
$(TF3)$ $\varrho_A(\zeta_2 + \zeta_1 - \zeta_2) \geq T(\varrho_A(\zeta_1))$ and $\vartheta_A(\zeta_2 + \zeta_1 - \zeta_2) \leq S(\vartheta_A(\zeta_1))$
$(TF4)$ $\varrho_A(\zeta_1\zeta_2) \geq T(\varrho_A(\zeta_2))$ and $\vartheta_A(\zeta_1\zeta_2) \leq S(\vartheta_A(\zeta_2))$
$(TF5)$ $\varrho_A((\zeta_1 + z)\zeta_2 - \zeta_1\zeta_2) \geq T(\varrho_A(z))$ and $\vartheta_A((\zeta_1 + z)\zeta_2 - \zeta_1\zeta_2) \leq T(\vartheta_A(z)),\ \zeta_1, \zeta_2, z \in N$.

$A = (\varrho_A, \vartheta_A)$ is a PyF left ideal of N if it satisfies (TF1), (TF2), (TF3) and $(TF4)$, and $A = (\varrho_A, \vartheta_A)$ is (T, S)-PyF right ideal of N if it satisfies $(TF1), (TF2), (TF3)$ and $(TF5)$.
$A = (\varrho_A, \vartheta_A)$ is called (T, S)-PyFI of N if it is both left and right (T, S)-PyFI of N.

Proposition 2 *Every PyFS* $A = (\varrho_A, \vartheta_A)$ *of* N *is* (T, S)*-PyFI of* N *iff the level set*

$$U(A; \alpha, \beta) = \{\zeta_1 \in N: \varrho_A(\zeta_1) \geq \alpha, \vartheta_A(\zeta_1) \leq \beta\}$$

is a PyFI of N when it is nonempty.

Proposition 3 *Every* (T, S)*-PyFI of* N *is a* (T, S)*-PyFSNR of* N. *The converse of this proposition does not necessarily hold in general.*

Example 1 *Let* $N = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$ *be a set with binary operations as*

$+$	κ_1	κ_2	κ_3	κ_4
κ_1	κ_1	κ_2	κ_3	κ_4
κ_2	κ_2	κ_1	κ_4	κ_3
κ_3	κ_3	κ_4	κ_2	κ_1
κ_4	κ_4	κ_3	κ_1	κ_2

$\cdot$	κ_1	κ_2	κ_3	κ_4
κ_1	κ_1	κ_1	κ_1	κ_1
κ_2	κ_1	κ_1	κ_1	κ_1
κ_3	κ_1	κ_1	κ_1	κ_1
κ_4	κ_1	κ_2	κ_3	κ_4

Then $(N, +, .)$ is a near-ring.

We define a Pythagorean fuzzy subset $A = (\varrho_A, \vartheta_A)$ and $\varrho_A: N \to [0,1]$ by $\varrho_A(\kappa_1) > \varrho_A(\kappa_2) > \varrho_A(\kappa_4) = \varrho_A(\kappa_3)$ and $\vartheta_A(\kappa_1) < \vartheta_A(\kappa_2) \le \vartheta_A(\kappa_3) = \vartheta_A(\kappa_4)$.

Let $T(\kappa_1, \kappa_2) = \max(\kappa_1 + \kappa_2 - 1, 0)$ which is a t-norm for all $\kappa_1, \kappa_2 \in [0,1]$ and $S(\kappa_1, \kappa_2) = \min(\kappa_1 + \kappa_2, 1)$ which is a t-conorm for all $\kappa_1, \kappa_2 \in [0,1]$.

Here A is a (T,S)-PyFSNR of N.

It is evident that A qualifies as a (T,S)-PyF left ideal of N, however, $A = (\varrho_A, \vartheta_A)$ does not satisfy the conditions to be classified as a (T,S)-PyF right ideal of N. Given that $\varrho_A((\kappa_3 + \kappa_2)\kappa_4 - \kappa_3\kappa_4) = \varrho_A(\kappa_4)$, it follows that $\varrho_A(\kappa_4) \le \varrho_A(\kappa_2)$.

Definition 3.3 *Let* N_1 *and* N_2 *represent two near-rings, and let* f *denote a function mapping elements from* N_1 *to* N_2*. If* B *is a PyFS in* N_2*, then image of* A *under* f *constitutes PyFS in* N_1 *is*

$$f(A)(\zeta) = \begin{cases} \left(\sup_{\zeta \in f^{-1}(\xi)} \varrho_A(\zeta), \quad \inf_{\zeta_1 \in f^{-1}(\xi)} \vartheta_A(\zeta) \right) & if \quad f^{-1}(\xi) \neq \emptyset, \\ 0, & otherwise \end{cases}$$

for each $\xi \in N_2$.

Theorem 3.1 *Assume that* $f: N_1 \to N_2$ *is an onto homomorphism. If* $A = (\varrho_A, \vartheta_A)$ *is a* (T,S)*-PyFI in* N_1*, then* $f(A)$ *is a* (T,S)*-PyFI in* N_2.

Proof. Let $\xi_1, \xi_2 \in N_2$. Then

$$\begin{aligned} f(A)(\xi_1 - \xi_2) &= (\sup\{\varrho_A(\zeta): \zeta \in f^{-1}(\xi_1 - \xi_2)\}, \inf\{\vartheta_A(\zeta): \zeta \in f^{-1}(\xi_1 - \xi_2)\}) \\ &\ge \sup\{T(\varrho_A(\zeta_1), \varrho_A(\zeta_2))\}, \inf\{S(\vartheta_A(\zeta_1), \vartheta_A(\zeta_2))\} \\ &= \sup\{T(\varrho_A(\zeta_1): \zeta_1 \in f^{-1}(\xi_1)\}, \sup\{\varrho_A(\zeta_2): \zeta_2 \in f^{-1}(\xi_2)\}), \\ &\quad \inf\{S(\vartheta_A(\zeta_1): \zeta_1 \in f^{-1}(\xi_1)\}, \sup\{\vartheta_A(\zeta_2): \zeta_2 \in f^{-1}(\xi_2)\}) \\ &= T(f(\varrho_A)(\xi_1), f(\varrho_A(\xi_2))), S(f(\vartheta_A)(\xi_1), f(\vartheta_A(\xi_2))) \\ &= (T,S)\{f(\varrho_A, \vartheta_A)(\xi_1), (\xi_2)\}, \ where \ \ \zeta_1 \in f^{-1}(\xi_1), \zeta_2 \in f^{-1}(\xi_2) \end{aligned}$$

Again,

$$\begin{aligned} f(A)(\xi_1\xi_2) &= \sup\{T(\varrho_A(\zeta): \zeta \in f^{-1}(\xi_1\xi_2))\}, \inf(S(\vartheta_A(\zeta): \zeta \in f^{-1}(\xi_1\xi_2))\} \\ &\ge \sup\{T(\varrho_A(\zeta_1), \varrho_A(\zeta_2): \zeta_1 \in f^{-1}(\xi_1), \zeta_2 \in f^{-1}(\xi_2)\}) \\ &\quad \inf\{S(\vartheta_A(\zeta_1), \vartheta_A(\zeta_2): \zeta_1 \in f^{-1}(\xi_1), \zeta_2 \in f^{-1}(\xi_2)\}) \\ &= \sup\{T(\varrho_A(\zeta_1): \zeta_1 \in f^{-1}(\xi_1), \varrho_A(\zeta_2): \zeta_2 \in f^{-1}(\xi_2))\}, \\ &\quad \inf\{S(\vartheta_A(\zeta_1): \zeta_1 \in f^{-1}(\xi_1), \vartheta_A(\zeta_2): \zeta_2 \in f^{-1}(\xi_2))\} \\ &= T(f(\varrho_A)(\xi_1), f(\varrho_A)(\xi_2)), S(f(\vartheta_A)(\xi_1), f(\vartheta_A)(\xi_2)). \\ &= (T,S)\{f(A)(\xi_1), f(A)(\xi_2)\}. \end{aligned}$$

Therefore, $f(\varrho_A, \vartheta_A)$ is a (T,S)-PyFSNR in N_2.

Let $\xi_1, \xi_2, \xi_3 \in N_2$. Then

$$\begin{aligned} &f(A)(\xi_1 + \xi_2 - \xi_1) \\ &= \sup\{\varrho_A(\zeta): \zeta \in f^{-1}(\xi_1 + \xi_2 - \xi_1)\}, \inf\{\vartheta_A(\zeta): \zeta \in f^{-1}(\xi_1 + \xi_2 - \xi_1)\} \\ &\ge \sup\{T(\varrho_A(\zeta_1), (\varrho_A(\zeta_2): \zeta_1 \in f^{-1}(\xi_1), \zeta_2 \in f^{-1}(\xi_2)\}) \\ &\ge \sup\{T(\varrho_A(\zeta_1 + \zeta_2 - \zeta_1))\}, \inf\{S(\vartheta_A(\zeta_1 + \zeta_2 - \zeta_1))\} \\ &\ge \sup\{T(\varrho_A(\zeta_1): \zeta_1 \in f^{-1}(\xi_1))\}, \inf S(\vartheta_A(\zeta_1): \zeta_1 \in f^{-1}(\xi_1)) \\ &= (T,S)f(A)(\xi_1), \ where \ \ \zeta_1 \in f^{-1}(\xi_1), \zeta_2 \in f^{-1}(\xi_2) \end{aligned}$$

$$\begin{aligned} f(A)(\xi_1\xi_2) &= \sup\{T(\varrho_A(\zeta): \zeta \in f^{-1}(\xi_1\xi_2))\}, \inf\{S(\vartheta_A(\zeta): \zeta \in f^{-1}(\xi_1\xi_2))\} \\ &\ge \sup\{T(\varrho_A(\zeta_1\zeta_2))\}, \inf\{S(\vartheta_A(\zeta_1\zeta_2))\} \\ &\ge \sup\{T(\varrho_A(\zeta_2): \zeta_2 \in f^{-1}(\xi_2))\}, \inf\{S(\vartheta_A(\zeta_2): \zeta_2 \in f^{-1}(\xi_2))\} \\ &= (T,S)f(A)(\xi_2), \zeta_1 \in f^{-1}(\xi_1), \zeta_2 \in f^{-1}(\xi_2) \end{aligned}$$

$$\begin{aligned} &f(A)((\xi_1 + \xi_2)\xi_3 - \xi_1\xi_3) \\ &= \sup\{T(\varrho_A(\zeta): \zeta \in f^{-1}((\xi_1 + \xi_2)\xi_3 - \xi_1\xi_3))\}, \inf\{S(\vartheta_A(\zeta): \zeta \in f^{-1}((\xi_1 + \xi_2)\xi_3\xi_1\xi_3))\} \\ &\ge \sup\{T(\varrho_A((\zeta_1 + \zeta_2)\zeta_3 - \zeta_1\zeta_3)): \zeta_1 \in f^{-1}(\xi_1), \zeta_2 \in f^{-1}(\xi_2), \zeta_3 \in f^{-1}(\xi_3)\} \end{aligned}$$

$$\geq \sup\{T(\varrho_A(\zeta_3): \zeta_3 \in f^{-1}(\xi_3))\}, \inf\{S(\vartheta_A(\zeta_3): \zeta_3 \in f^{-1}(\xi_3))\}$$
$$= (T,S)f(A)(\xi_3).$$

Therefore, $f(A)$ is a (T,S)-PyFI of N_2.

Definition 3.4 *A PyFI $A = (\varrho_A, \vartheta_A)$ of N is defined as normal if $\varrho_A(0) = 1$ and $\vartheta_A(0) = 0$.*

Theorem 3.2 *Let $A = (\varrho_A, \vartheta_A)$ be an (T,S)-PyFI of N and let $A^* = (\varrho_A^*, \vartheta_A^*)$ be a PyFS defined by $\varrho_A^*(\zeta) = \varrho_A(\zeta) + 1 - \varrho_A(0)$ and $\vartheta_A^*(\zeta) = \vartheta_A(\zeta) - \vartheta_A(0)$ for all $\zeta \in N$.*
Then A^* is a normal (T,S)-PyFI of N containing A.

Proof. Let $\zeta, \xi \in N$, then

$$(\varrho_A^*(\zeta - \xi), \vartheta_A^*(\zeta - \xi)) = (\varrho_A(\zeta - \xi) + 1 - \varrho_A(0), \vartheta_A(\zeta), \vartheta_A(0)) \geq T\{(\varrho_A(\zeta), \varrho_A(\xi)) + 1 - \varrho_A(0)\},$$
$$\geq T\{(\varrho_A(\zeta), \varrho_A(\xi)) + 1 - \varrho_A(0)\}, S\{(\vartheta_A(\zeta), \vartheta_A(\xi)) - \vartheta_A(0)\}$$
$$= T(\varrho_A(\zeta) + 1 - \varrho_A(0), \varrho_A(\xi) + 1 - \varrho_A(0)), s(\vartheta_A(\zeta) - \vartheta_A(0), \vartheta_A(\xi) - \vartheta_A(0))$$
$$= T(\varrho_A^*(\zeta), \varrho_A^*(\xi)), S(\vartheta_A^*(\zeta), \vartheta_A^*(\xi)) = (T,S)\{(\varrho_A^*(\zeta), \varrho_A^*(\xi), (\vartheta_A^*(\zeta), \vartheta_A^*(\xi)\}(\varrho_A^*(\zeta\xi), \vartheta_A^*(\zeta\xi))$$
$$= (\varrho_A(\zeta\xi) + 1 - \varrho_A(0), \vartheta_A(\zeta) - \vartheta_A(0)$$
$$\geq T(\varrho_A(\zeta), \varrho_A(\xi)) + 1 - \varrho_A(0), S(\vartheta_A(\zeta), \vartheta_A(\xi) - \vartheta_A(0)$$
$$= T(\varrho_A(\zeta) + 1 - \varrho_A(0), \varrho_A(\xi) + 1 - \varrho_A(0)), S(\vartheta_A(\zeta) - \vartheta_A(0), \vartheta_A(\xi) - \vartheta_A(0)) =$$
$$(T,S)\{(\varrho_A^*(\zeta), \varrho_A^*(\xi)), (\vartheta_A^*(\zeta), \vartheta_A^*(\xi))\}.$$

Thus, A^* is a (T,S)- PyFSNR of N.

Theorem 3.3 *If $A = (\varrho_A, \vartheta_A)$ is an (T,S)-PyFI of N, then for all $\zeta \in N$, $(A)(\zeta) = (\{sup\{t \in [0,1] | \varrho_A(\zeta) \geq t\}, inf\{s \in [0,1] | \vartheta_A(\zeta) \leq s\})$ such that $t + s \leq 1$.*

Proof. Define $p = \sup\{t \in [0,1] | \zeta \in U(A; t, s)\}$ and $q = \inf\{t \in [0,1] | \zeta \in U(A; t, s)\}$.

Let $\epsilon \geq 0$. It follows that $p - \epsilon \leq t$ and $q + \epsilon \geq s$, where t and s are elements of the interval $[0,1]$ and satisfy the condition $t + s \leq 1$.
Since ϵ is arbitrary $(p,q) \leq \langle \varrho_A(\zeta), \vartheta_A(\zeta) \rangle = A(\zeta)$.

Let $A(\zeta) = (\vartheta_1, \vartheta_1')$, then $\zeta \in U(A: \vartheta_1, \vartheta_1')$ and so $\vartheta_1 \in \{t \in [0,1] | \zeta \in U(A: t, s)\}$ and $\vartheta_1' \in \{s \in [0,1] | \zeta \in U(A: t, s)\}$.
Then $A(\zeta) = (\vartheta_1, \vartheta_1') \leq (p,q)$.
Hence $A(\zeta) = (p,q)$.
This completes the proof.

Theorem 3.4 *If A is a (T,S)-PyFI of N, then $\overline{A}$ of N/I defined by*

$$\bar{A}(a + I) = (\sup_{\zeta \in I}\{\varrho_A(a + \zeta)\}, \inf_{\zeta \in I}\{\vartheta_A(a + \zeta)\})$$

is a (T,S)-PyFI of N/I of N concerning I.

Proof. Let $a, b \in \mathbb{N}$ such that $a + I = b + I$. It follows that b can be expressed as $a + \xi$, where ξ is an element of the set I. Thus

$$\bar{A}(b + I) = \{sup_{\zeta \in I}(\varrho_A(b + \zeta)), \inf_{\zeta \in I}(\vartheta_A(b + \zeta))\}$$
$$= \{\sup_{\zeta \in I}(\varrho_A(a + \xi + \zeta)), \inf_{\zeta \in I}(\vartheta_A(a + \xi + \zeta))\}$$
$$= \{\sup_{\zeta + \xi = z \in I}(\varrho_A(a + z)), \inf_{\zeta \in I}(\vartheta_A(a + z))\}$$
$$= \{\bar{A}(a + I)\}$$

This demonstrates the clarity of A.
Let $\zeta + I, \xi + I \in N/I$, subsequently,

$$\bar{A}((\zeta + I) - (\xi + I)) = \bar{A}((\zeta - \xi) + I)$$
$$= \{\sup_{z \in I}(\varrho_A((\zeta - \xi) + z)), \inf_{z \in I}(\vartheta_A((\zeta - \xi) + z))\}$$
$$= \{\sup_{z = u - v \in I}(\vartheta_A((\zeta - \xi) + (u - v))), \inf_{z = u - v \in I}(\vartheta_A((\zeta - \xi) + (u - v)))\}$$
$$\geq \{\sup_{u,v \in I} T(\varrho_A(\zeta + u)), \varrho_A(\xi + v)\}, \inf_{u,v \in I} S\{\vartheta_A(\zeta + u), \vartheta_A(\xi + v)\}$$
$$= T\{\sup_{u \in I}\{\varrho_A(\zeta + u)\}, \sup_{v \in I}\{\varrho_A(\xi + v)\}\}\}, S\{\inf_{\varrho \in I}\vartheta_A(\zeta + u), \inf_{v \in I}\vartheta_A(\xi + v))\}$$
$$= (T,S)\{\bar{A}(\zeta + I), \bar{A}(\xi + I)\}\bar{A}(\zeta + I)(\xi + I)) = \bar{A}((\zeta\xi) + I)$$
$$= \{\sup_{z \in I}\{\varrho_A(\zeta\xi) + z)\}, \inf_{z \in I}\{\vartheta(\zeta\xi) + z))\}\}$$

$$= \{ \sup_{z=uv\in I} \{\varrho_A((\zeta\xi) + (uv))\}, \inf_{z=uv\in I} \{\vartheta_A((\zeta\xi) + (uv))\}\}$$
$$\geq \{\sup_{u,v\in I} T\{\varrho_A(\zeta + u), \varrho_A(\xi + v)\}, \inf_{u,v\in I} S\{\vartheta_A(\zeta + u), \vartheta_A(\xi + v)\}\}$$
$$= T\{\sup_{u\in I}\{\varrho_A(\zeta + u)\}, \sup_{v\in I}\{\varrho_A(\xi + v)\}, S\{\inf_{u\in I}\{\vartheta_A(\zeta + u)\}, \inf_{v\in I}(\xi + v))\}$$
$$= T\{\bar{A}(\zeta + I), \bar{A}(\xi + I)\}, S\{\bar{A}(\zeta + I), \bar{A}(\xi + I)\}$$
$$= (T,S)\{\bar{A}(\zeta + I), \bar{A}(\xi + I)\}$$

Thus $\bar{A}$ is a (T,S)-PyFI of N/I.

Theorem 3.5 *Let I denote an ideal of N. There exists a one-to-one correspondence between the set of (T,S)-PyFIs A of N for which $A(0) = A(s)$ holds for all $sinI$ and the set of all (T,S) PyFIs $barA$ of N/I.*

Proof. Let A be a (T,S)-PyFI of R. We prove that $\bar{A}$ defined by
$$\bar{A}(a + I) = (\sup\{(\varrho_A(a + \zeta))\}, \inf\{(\vartheta_A(a + \zeta))\})$$
is a (T,S)-PyFI of N/I.
Since $A(0) = A(s)$ for all $s \in I$,
$$\Rightarrow (1,0) = (\varrho_A(s), \vartheta_A(s))A(a + s)$$
$$\geq \{T(\varrho_A(a), \varrho_A(s)), S(\vartheta_A(a), \vartheta_A(s))\}$$
$= (\{\varrho_A(a), \vartheta_A(a)\}) = A(a)$.
Again,
$A(a) = (\varrho_A(a + s - s), \vartheta_A(a + s - s)) \geq \{T(\varrho_A(a + s), \varrho_A(s)), S(\vartheta_A(a + s), \vartheta_A(s))\} = A(a + s)$.
Thus $A(a + s) = A(a)$ for all $s \in I$, that is, $\bar{A}(a + I) = A(a)$.
Hence, the correspondence $A \rightarrow \bar{A}$ is one-to-one.
Let A be a (T,S) PyFI of N/I and define a PyFS A in N by $A(a) = \bar{A}(a + 1)$ for all $a \in I$.
For $\zeta, \xi \in N$, we have
$$A(\zeta - \xi) = \bar{A}((\zeta - \xi) + I) = \bar{A}((\zeta + I) - (\xi + I)) \geq \{T\{\varrho_A(\zeta + I), \varrho_A(\xi + I)\}, S\{\vartheta_A(\zeta + I), \vartheta_A(\xi + I)\}\}$$
$$= T\{\varrho_A(\zeta), \varrho_A(\xi)\}, S\{\vartheta(\zeta), \vartheta(\xi)\}A(\zeta\xi) = \bar{A}((\zeta\xi) + I) = \bar{A}((\zeta + I) \cdot (\xi + I))$$
$$\geq \{T\{\varrho_A(\zeta + I), \varrho_A(\xi + I)\}, S\{\vartheta_A(\zeta + I), \vartheta_A(\zeta + I)\}\} = \{T\{\varrho_A(\zeta), \varrho_A(\xi)\}, S\{\vartheta_A(\zeta), \vartheta_A(\xi)\}\}$$
Thus A is a (T,S) PyFI of N.

Furthermore, it is established that $A(z) = barA(z + I) = barA(I)$ holds for all $zinI$. This indicates that $A(z) = A(0)$ for every z within the interval I. Conclusion of the proof is hereby presented.

IV. Conclusion

This paper presents the concept of PyFSNR associated with a near-ring in relation to the t norm T and the t conorm S. We proceed to characterise all of them utilising a specific type of level sets $U(A; [t,s])$, which serves as a generalisation of traditional level subsets. The analysis that follows looks at how these structures behave when they undergo homomorphism. We develop a (T,S)-PyFSNR within the context of the near-ring of quotients.

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Data Availability

All data supporting the reported findings in this research paper are provided within the manuscript.

Conflicts of Interest

The authors declare that there is no conflict of interest concerning the reported research findings.

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