

On Vibrations Of An Axially Moving Beam Under Material Damping

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Abstract: In this paper a materially damped linear homogeneous beam-like equation has been considered. The viscoelastic beam is simply supported at both ends, whereas general initial conditions are considered. From mechanical and physical point of view the problem describes a mathematical model of internally damped transversal vibrations of a moving conveyor belt or a viscoelastic chain drive. From Hamilton's principle, a fifth order partial differential equation (PDE) for axially moving continuum has been formulated. The axial speed of the beam is considered to be positive, constant and small compared to wave velocity, and it is also assumed that the introduced material damping is relatively small. The solutions of equation of motion are based upon a two timescales perturbation method. By application of this perturbation method, it has been shown that the material damping does in fact affect the solution responses, and it reduces the vibration and noise in the system. It has also been shown that the material damping generated in the system depends on the mode number n , which is obviously expected from mechanical point of view.

Keywords: Conveyor belt, Beam-like, Axially moving, Material damping, Two timescales

I. Introduction

In world around us, almost all physical, structural, and mechanical systems are generally appearing into class of the oscillatory systems. Axially translating systems, for example, are appearing in such category. Axially translating systems have received much research attention since last six decades. Axially translating systems have been observed in many practical and engineering applications. The energy dissipation, known as damping, can easily be associated to axially translating systems, see Refs.[1-3]. Axially translating systems have many engineering applications. For example, conveyor belt systems, as given in Refs. [4-6], magnetic tapes, pipes conveying fluids, data saving devices, and elevator cable systems, see Ref. [5], and all such kind of systems are bound to vibrations. The study of axially translating systems with constant or time-varying velocity with viscous, material or boundary damping have received much importance in manufacture and design. It is common experience that the vibration causes severe failures to many mechanical or physical structures. From this viewpoint, it becomes necessary to design systems where unnecessary noise and vibrations can be reduced by means of solid procedures. Tacoma Narrows bridge is a good example being taught in education and research institutes for a complete structural collapse. This collapse was due to winds at certain speed. Apart from damaging structures, the vibration also causes human problems, that is, vibrations create anxiety to society. Keeping in view the effects of vibrations, it is matter of necessity to formulate methods and devise procedures to decrease vibrations from the physical and mechanical systems.

In several cases, damping devices can be kept through the support conditions to control vibrations through boundaries as seen in Refs. [4-7]. In Ref. [8] the damping device is introduced through whole spatial domain of the translating system. The reflection and damping properties for a wave equation have been studied in Ref. [9], where the authors have provided interesting results for a semi-infinite string. For different boundary conditions, in Ref. [10], the authors have provided detailed analysis for the energetics of the elevator cable systems. The authors, in Ref. [11], studied the energetics of an axially translating continuum, where they studied the case for fixed supports for string-like problem and the case for simple supports in case of beam-like problem. In Ref. [12], authors have provided detailed analysis of dampers connected at middle of string as well as beam. But the position of the damper always plays a significant role. If the damper is introduced at wrong spatial position it may increase energy of motion and may destabilize the system, for details see Ref. [13].

In this article, a materially damped beam-type equation has been considered. The paper is organized in the following way. In Sec. 2 the governing equations of motions are formulated based upon the Hamilton's principle. In Sec. 3 the analytic approximations of the solutions of an initial-boundary value problem have been obtained by using method of multiple timescales. Sec. 4 discusses results obtained in Sec. 3, and Sec. 5 represents the concluding remarks and directions for future research.

II. The Governing Equations

The mathematical model of a traveling tensioned beam under material damping is based upon the following assumptions: $u(x, t)$ models the displacement field in vertical direction from equilibrium position, the mass of beam per unit length, ρ , is constant, the material damping coefficient $\bar{\beta}$ is constant, and the effects of gravity and other forces are neglected. Thus, the beam-type equation under material damping is given as follows,

$$\rho(u_{tt} + 2\bar{V}u_{xt} + \dot{\bar{V}}u_x + \bar{V}^2u_{xx}) - Tu_{xx} + Elu_{xxxx} + \bar{\beta}(u_{xxxxt} + \bar{V}u_{xxxxx}) = 0; 0 < x < L, t > 0 \quad (1)$$

With simply supported boundary conditions,

$$u(0, t) = u_{xx}(0, t) = u(L, t) = u_{xx}(L, t) = 0; t > 0 \quad (2)$$

and the general initial conditions,

$$u(x, 0) = f(x), \text{ and } u_t(x, 0) = g(x); 0 < x < L. \quad (3)$$

The terms in the bracket into Eq. (1) represent acceleration quantities. The equations contained in (1)-(3) can be put into a non-dimensional form by using following dimensionless quantities:

$u^* = \frac{u}{L}, x^* = \frac{x}{L}, t^* = \frac{ct}{L}, V_0 = \frac{\bar{V}}{c}, \mu = \frac{El}{\rho c^2 L^2}, \delta_0 = \frac{\bar{\beta}}{\rho c L^3}, f^* = \frac{f}{L}, g^* = \frac{g}{c}$, where $c = \sqrt{T/\rho}$ is the wave velocity. Thus, the equations (1)-(3) into non-dimensional form become:

$$u_{tt} + 2V_0u_{xt} + \dot{V}_0u_x + (V_0^2 - 1)u_{xx} + \mu u_{xxxx} + \delta_0(u_{xxxxt} + V_0u_{xxxxx}) = 0; 0 < x < 1, t \geq 0. \quad (4)$$

The boundary conditions are given as,

$$u(0, t) = u_{xx}(0, t) = 0, \text{ and } u(1, t) = u_{xx}(1, t) = 0; t \geq 0. \quad (5)$$

The initial conditions are given as,

$$u(x, 0) = f(x), \text{ and } u_t(x, 0) = g(x); 0 < x < 1. \quad (6)$$

III. The Analytic Approximations

In this section an approximation of the solutions to the IBVP (4)-(6) by using a multiple timescales perturbation method is constructed. For a complete overview of this method, see Refs. [14-16]. Following two assumptions are made to utilize a this method. The velocity \bar{V} of the beam is assumed to be small compared to wave velocity c and that the damping coefficient $\bar{\beta}$ is small compared to $\rho c L^3$. Thus, it is reasonable to write $V_0 = \frac{\bar{V}}{c} = O(\varepsilon)$, and $\delta_0 = \frac{\bar{\beta}}{\rho c L^3} = O(\varepsilon)$, that is, $V_0 = \varepsilon V$ and $\delta_0 = \varepsilon \delta$. The parameter ε is a dimensionless small parameter as described by $0 < \varepsilon \ll 1$. Utilizing these assumptions in Eqs. (4)-(6), it follows that

$$u_{tt} - u_{xx} + \mu u_{xxxx} = -2\varepsilon V u_{xt} - \varepsilon^2 V^2 u_{xx} - \varepsilon \delta u_{xxxxt} - \varepsilon^2 \delta V u_{xxxxx}, \quad (7)$$

$$u(0, t; \varepsilon) = u_{xx}(0, t; \varepsilon) = 0, \text{ and } u(1, t; \varepsilon) = u_{xx}(1, t; \varepsilon) = 0, \quad (8)$$

$$u(x, 0; \varepsilon) = f(x), \text{ and } u_t(x, 0; \varepsilon) = g(x). \quad (9)$$

According to a two timescales method a function $u(x, t; \varepsilon)$ is supposed to be a function of spatial variable x , the fast timescale $t=t$ and, the slow timescale $\tau=ct$. For this reason,

$$u(x, t; \varepsilon) = y(x, t, \tau; \varepsilon). \quad (10)$$

By using Eq. (10), the time derivatives can be transformed as follows,

$$u_t = y_t + \varepsilon y_\tau, \quad (11)$$

$$u_{tt} = y_{tt} + 2\varepsilon y_{t\tau} + \varepsilon^2 y_{\tau\tau}.$$

By substituting Eqs. (10)-(11) into Eqs. (7)-(9), the problem in y up to $O(\varepsilon)$ is given as follows,

$$\begin{aligned} y_{tt} - y_{xx} + \mu y_{xxxx} &= -2\varepsilon y_{t\tau} - 2\varepsilon V y_{xt} - \varepsilon \delta y_{xxxxt}, \\ y(0, t, \tau; \varepsilon) &= y_{xx}(0, t, \tau; \varepsilon) = y(1, t, \tau; \varepsilon) = y_{xx}(1, t, \tau; \varepsilon) = 0, \\ y(x, 0, 0; \varepsilon) &= f(x), y_t(x, 0, 0; \varepsilon) = g(x) - \varepsilon y_\tau(x, 0, 0; \varepsilon). \end{aligned} \quad (12)$$

It is assumed that not only $u(x, t; \varepsilon)$ can be approximated by the asymptotic expansion, but also $u(x, t; \varepsilon) = y(x, t, \tau; \varepsilon)$ can be approximated in the powers of ε in the asymptotic expansion as,

$$y(x, t, \tau; \varepsilon) = y_0(x, t, \tau) + \varepsilon y_1(x, t, \tau) + \varepsilon^2 \dots, \quad (13)$$

and that all the y_j 's for $j = 0, 1, 2, \dots$, are found in such a way that no unbounded (secular) terms arise. It is also assumed that the unknown functions y_j are $O(1)$. Now, by substituting Eq. (13) and its subsequent derivatives into Eq. (12), then by equating the powers of ε^0 and ε^1 , and neglecting the ε^2 and the higher powers of ε , the $O(1)$ -problem is followed as given by,

$$\begin{aligned} y_{0tt} - y_{0xx} + \mu y_{0xxxx} &= 0, \\ y_0(0, t, \tau) &= y_{0xx}(0, t, \tau) = y_0(1, t, \tau) = y_{0xx}(1, t, \tau) = 0, \end{aligned} \quad (14)$$

$$y_0(x, 0, 0) = f(x), y_{0_t}(x, 0, 0) = g(x).$$

The $O(\varepsilon)$ -problem is given as,

$$\begin{aligned} y_{1_{tt}} - y_{1_{xx}} + \mu y_{1_{xxxx}} &= -2y_{0_{t\tau}} - 2Vy_{0_{xt}} - \delta y_{0_{xxxxt}}, \\ y_1(0, t, \tau) = y_{1_{xx}}(0, t, \tau) = y_{1_t}(0, t, \tau) &= y_{1_{xx}}(1, t, \tau) = 0, \\ y_1(x, 0, 0) = 0, y_{1_t}(x, 0, 0) &= -y_{0_\tau}(x, 0, 0). \end{aligned} \tag{15}$$

It can be observed that the $O(1)$ -problem has solution only for the positive eigenvalues, $\lambda = \lambda_n = n^2\pi^2(1 + n^2\pi^2\mu)$, $n \in \mathbb{N}$, Ref. [17,18]. Thus, the solution of $O(1)$ -problem is given as,

$$y_0(x, t, \tau) = \sum_{n=1}^{\infty} (A_{n0}(\tau) \cos(\sqrt{\lambda_n}t) + B_{n0}(\tau) \sin(\sqrt{\lambda_n}t)) \phi_n(x), \tag{16}$$

where $\phi_n(x) = \sin(\gamma_n x) - \frac{\sin(\gamma_n)}{\sinh(\beta_n)} \sinh(\beta_n x)$ are the eigenfunctions with $\gamma_n = \sqrt{\frac{-1 + \sqrt{4\lambda_n\mu + 1}}{2\mu}}$, and $\beta_n = \sqrt{\frac{1 + \sqrt{4\lambda_n\mu + 1}}{2\mu}}$, and where A_{n0} and B_{n0} are undetermined functions. They can be obtained from the $O(\varepsilon)$ -problem. The values of the constants $A_{n0}(0)$ and $B_{n0}(0)$ can easily be obtained by the initial values as given in Eq. (14) and by using the orthogonality properties of the eigenfunctions. The eigenfunctions $\phi_p(x)$, $p \in \mathbb{N}$ satisfy the following orthogonality properties,

$$\int_0^1 \phi_p(x) \phi_q(x) dx = 0, \text{ for } p \neq q \tag{17}$$

$$\neq 0, \text{ for } p = q.$$

Thus, by using the initial values as given in Eq. (14) and the orthogonality properties of the eigenfunctions as given in Eq. (17), A_{n0} and B_{n0} are given as

$$A_{n0}(0) = \frac{\int_0^1 f(x) \phi_n(x) dx}{\int_0^1 \phi_n^2(x) dx}, \tag{18}$$

$$\sqrt{\lambda_n} B_{n0}(0) = \frac{\int_0^1 g(x) \phi_n(x) dx}{\int_0^1 \phi_n^2(x) dx}. \tag{19}$$

Now, the eigenfunction expansion method is introduced in solving the $O(\varepsilon)$ -problem. Following form for the solution $y_1(x, t, \tau)$ is assumed,

$$y_1(x, t, \tau) = \sum_{n=1}^{\infty} w_n(t, \tau) \phi_n(x), \tag{20}$$

where $w_n(t, \tau)$ are the unknown functions of t and τ , and where $\phi_n(x)$ are the given eigenfunctions. Thus, by substitution of Eq. (20) in the $O(\varepsilon)$ -equation, it yields

$$\sum_{n=1}^{\infty} (w_{n_{tt}}(t, \tau) + \lambda_n w_n(t, \tau)) \phi_n(x) = -2y_{0_{t\tau}} - 2Vy_{0_{xt}} - \delta y_{0_{xxxxt}}. \tag{21}$$

Now, by substitution of the solution $y_0(x, t, \tau)$ from Eq. (16) into Eq. (21), it follows that

$$\begin{aligned} \sum_{n=1}^{\infty} (w_{n_{tt}}(t, \tau) + \lambda_n w_n(t, \tau)) \phi_n(x) \\ = -2 \sum_{n=1}^{\infty} R_{n_{t\tau}}(t, \tau) \phi_n(x) - 2V \sum_{n=1}^{\infty} R_{n_t}(t, \tau) \phi_n'(x) \\ - \delta \sum_{n=1}^{\infty} R_{n_t}(t, \tau) \phi_n^{(iv)}(x), \end{aligned} \tag{22}$$

where $R_n(t, \tau)$ is given by,

$$R_n(t, \tau) = A_{n0}(\tau) \cos(\sqrt{\lambda_n}t) + B_{n0}(\tau) \sin(\sqrt{\lambda_n}t). \tag{23}$$

By multiplying both sides of Eq. (22) with $\phi_m(x)$, then by integrating so-obtained equation from $x = 0$ to $x = 1$ with application of the orthogonality property of eigenfunctions, it follows that

$$\begin{aligned} w_{m_{tt}}(t, \tau) + \lambda_m w_m(t, \tau) \\ = -2R_{m_{t\tau}} - 2VR_{m_t} \Theta_{mm} - \delta R_{m_t} \Delta_{mm} - \sum_{n=1, n \neq m}^{\infty} (2VR_{n_t} \Theta_{nm} + \delta R_{n_t} \Delta_{nm}), \end{aligned} \tag{24}$$

where Θ_{nm} and Δ_{nm} are constants depending on the indices m and n , and their values are given as follows

$$\Theta_{nm} = \frac{\int_0^1 \phi_n'(x) \phi_m(x) dx}{\int_0^1 \phi_m^2(x) dx}, \tag{25}$$

$$\Delta_{nm} = \frac{\int_0^1 \phi_n^{(iv)}(x) \phi_m(x) dx}{\int_0^1 \phi_m^2(x) dx}.$$

Note that $\Theta_{mm} = 0$ for all integers m , whereas $\Delta_{mm} = \int_0^1 \phi_n''^2(x) dx > 0$. Thus, by using this value of Θ_{mm} , changing index from m to n and making use of Eq. (23) into Eq. (24), it readily follows that

$$\begin{aligned} w_{n\tau\tau} + \lambda_n w_n &= \sqrt{\lambda_n} \left(2A_{n0}'(\tau) + \delta \Delta_{nn} A_{n0}(\tau) \right) \sin(\sqrt{\lambda_n} t) \\ &\quad - \sqrt{\lambda_n} \left(2B_{n0}'(\tau) + \delta \Delta_{nn} B_{n0}(\tau) \right) \cos(\sqrt{\lambda_n} t) \\ &\quad + \sum_{m=1, m \neq n}^{\infty} \{ (2V\Theta_{mn} + \delta \Delta_{mn}) \sqrt{\lambda_m} A_{m0}(\tau) \sin(\sqrt{\lambda_m} t) \\ &\quad - (2V\Theta_{mn} + \delta \Delta_{mn}) \sqrt{\lambda_m} B_{m0}(\tau) \cos(\sqrt{\lambda_m} t) \} \end{aligned} \tag{26}$$

On right hand side of Eq. (26) first two terms are solutions of the homogeneous equation. Such terms will give rise to secular terms in $w_n(t, \tau)$. Since it is assumed that the functions $y_0(x, t, \tau), y_1(x, t, \tau), \dots$ are bounded on timescale of $O(\varepsilon^{-1})$. Thus, to have secular free behavior, the following solvability conditions are imposed in Eq. (26),

$$\begin{aligned} A_{n0}'(\tau) + \Delta_{nn} \frac{\delta}{2} A_{n0}(\tau) &= 0, \\ B_{n0}'(\tau) + \Delta_{nn} \frac{\delta}{2} B_{n0}(\tau) &= 0. \end{aligned} \tag{27}$$

The solutions to above system of two uncoupled ordinary differential equations in (27) are given as follows

$$\begin{aligned} A_{n0}(\tau) &= A_{n0}(0) e^{-\Delta_{nn} \frac{\delta}{2} \tau}, \\ B_{n0}(\tau) &= B_{n0}(0) e^{-\Delta_{nn} \frac{\delta}{2} \tau}, \end{aligned} \tag{28}$$

where $A_{n0}(0)$ and $B_{n0}(0)$ are given in Eqs. (18) and (19), respectively. Thus, by using Eq. (28) into Eq. (16), the complete solution to the $O(1)$ -problem is given as follows,

$$y_0(x, t, \tau) = \sum_{n=1}^{\infty} e^{-\Delta_{nn} \frac{\delta}{2} \tau} \left(A_{n0}(0) \cos(\sqrt{\lambda_n} t) + B_{n0}(0) \sin(\sqrt{\lambda_n} t) \right) \phi_n(x). \tag{29}$$

Now, by substituting $\tau = \varepsilon t$ into the expression, $-\Delta_{nn} \frac{\delta}{2} \tau$, and then dividing the so-obtained expression by t , the damping parameter Γ_n for all the oscillation modes can be approximated by,

$$\Gamma_n = -\varepsilon \Delta_{nn} \frac{\delta}{2}. \tag{30}$$

From Eq. (26) with Eq. (27), it follows that

$$\begin{aligned} w_{n\tau\tau} + (n\pi)^2 w_n &= \sum_{m=1, m \neq n}^{\infty} (2V\Theta_{mn} + \delta \Delta_{mn}) \sqrt{\lambda_m} \left(A_{m0}(\tau) \sin(\sqrt{\lambda_m} t) \right. \\ &\quad \left. - B_{m0}(\tau) \cos(\sqrt{\lambda_m} t) \right). \end{aligned} \tag{31}$$

Now, it can be seen that the Eq. (31) is a second order nonhomogeneous ODE in $w_n(t, \tau)$. Thus, the solution to Eq. (31) is given as

$$\begin{aligned} w_n(t, \tau) &= A_{n1}(\tau) \cos(\sqrt{\lambda_n} t) + B_{n1}(\tau) \sin(\sqrt{\lambda_n} t) \\ &\quad + \sum_{m=1, m \neq n}^{\infty} \frac{(2V\Theta_{mn} + \delta \Delta_{mn}) \sqrt{\lambda_m}}{\lambda_n - \lambda_m} \left(A_{m0}(\tau) \cos(\sqrt{\lambda_m} t) \right. \\ &\quad \left. - B_{m0}(\tau) \sin(\sqrt{\lambda_m} t) \right), \end{aligned} \tag{32}$$

where $A_{n1}(\tau)$ and $B_{n1}(\tau)$ are yet undetermined functions of a slow variable τ , these functions can be obtained from the $O(\varepsilon^2)$ -problem. Thus, the Eq. (20) with Eq. (32) can be expressed as

$$y_1(x, t, \tau) = \sum_{n=1}^{\infty} \left\{ A_{n1}(\tau) \cos(\sqrt{\lambda_n} t) + B_{n1}(\tau) \sin(\sqrt{\lambda_n} t) + \sum_{m=1, m \neq n}^{\infty} \frac{(2V\theta_{mn} + \delta\Delta_{mn})\sqrt{\lambda_m}}{\lambda_n - \lambda_m} (A_{m0}(\tau) \cos(\sqrt{\lambda_m} t) - B_{m0}(\tau) \sin(\sqrt{\lambda_m} t)) \right\} \phi_n(x). \tag{33}$$

Now, by using the inner product (17) and the initial values in Eq. (15) with Eq. (16) into Eq. (33), it follows that $A_{n1}(0)$ and $B_{n1}(0)$ are given by

$$A_{n1}(0) = - \sum_{m=1, m \neq n}^{\infty} \frac{(2V\theta_{mn} + \delta\Delta_{mn})\sqrt{\lambda_m}}{\lambda_n - \lambda_m} A_{m0}(0), \tag{34}$$

$$\sqrt{\lambda_n} B_{n1}(0) = \sum_{m=1, m \neq n}^{\infty} \frac{(2V\theta_{mn} + \delta\Delta_{mn})\lambda_m}{\lambda_n - \lambda_m} B_{m0}(0) - A'_{n0}(0). \tag{35}$$

It can be seen that the solution $y_1(x, t, \tau)$ still contains infinitely many undetermined functions $A_{n1}(\tau)$ and $B_{n1}(\tau)$, for $n \in \mathbb{N}$. These unknown functions can be used to avoid secular terms in solution $y_2(x, t, \tau)$. At this time, it is not reasonable to construct the higher order calculations. This is a reason, we can take $A_{n1}(\tau) = A_{n1}(0)$ and $B_{n1}(\tau) = B_{n1}(0)$. So far, a asymptotic expansion $y(x, t, \tau) = y_0(x, t, \tau) + \varepsilon y_1(x, t, \tau)$ has been constructed for $u(x, t)$.

IV. Results and Discussion

This section is all about to comment, to interpret and to explain the results obtained in Sec. 3. Using the complete analytical solution of the $O(1)$ -problem, the influence of small parameter ε and the damping parameter on the axially moving system will be discussed in detail. By using the $O(1)$ -solution and the $O(\varepsilon)$ -solution, it yields

$$u(x, t) = \sum_{n=1}^{\infty} e^{-A_{nm} \frac{\delta}{2} \varepsilon t} (A_{n0}(0) \cos(\sqrt{\lambda_n} t) + B_{n0}(0) \sin(\sqrt{\lambda_n} t)) \phi_n(x),$$

where $A_{n0}(0)$ and $B_{n0}(0)$ are given by Eqs. (18) and (19). From physical point of view, all terms can be explained in above solution to the IBVP (4)-(6). The terms $A_{n0}(0) \cos(\sqrt{\lambda_n} t) + B_{n0}(0) \sin(\sqrt{\lambda_n} t)$ are the oscillation terms obtained from a time-dependent part of the equation. These terms oscillate with frequencies $n\pi\sqrt{1 + n^2\pi^2\mu}$ for $n \in \mathbb{N}$. The term $e^{-A_{nm} \frac{\delta}{2} \varepsilon t}$ is occurring due to material damping. This term shows that as the time parameter t will increase for fixed values of δ and ε the size of the oscillation amplitudes $A_{n0}(0)$ and $B_{n0}(0)$ will start to decrease and it is also shown that as mode number n starts to increase the oscillation amplitudes tend to decrease for fixed δ, ε , and t . The last term $\phi_n(x)$ is the solution of the space-dependent part which describes the shapes of the oscillation curves along x -axis.

V. Conclusions and Future Work

In this paper, an initial-boundary value problem (IBVP) for the materially damped axially translating continuum has been studied. Solving the IBVP a method of two timescales has successfully been applied to obtain the analytic solutions of a mathematical model which describes the transversal vibrations of a conveyor belt system. It has been shown, in this paper, that all oscillation modes are damped for the system. It has also been shown that damping rates are, in fact, depending on mode numbers n . From mechanical point of view, this response is reasonable because as oscillations increase the more heat is generated in the system which internally damps the vibratory energy of motion. As modes increase the oscillation amplitudes decrease and the belt system gets stable. This research problem can further be extended to internal damping of an axially translating ring as well as axially translating beam with time-dependent axial velocities.

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