Owop, \( \Gamma m, \Gamma n, \Gamma b \) is the prescribed traction on natural boundaries, \( u \) is the prescribed displacement on the essential boundaries, and \( n \) is the vector of unit outward normal at a point on the natural boundary.

The unconstrained Galerkin weak form of Equation 1 is as follows (Liu, 2002),

\[
\int_{\Omega} (L^T \sigma) (D \sigma) \, d\Omega - \int_{\Omega} \sigma^T b \, d\Omega - \int_{\Gamma_e} \sigma^T t \, d\Gamma = 0
\]

For linear elasticity, the material matrix \( D \) is expressed as follows,
\[
D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad \text{for plane stress problem}
\]

\[
D = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \quad \text{for plane strain problem}
\]

where \( E \) is Young's modulus and \( \nu \) is Poisson's ratio. Substituting the RPIM approximation equation [1],

\[
u^h(\mathbf{x}) = [\mathbf{R}^T(\mathbf{x})\mathbf{S}_\alpha + \mathbf{P}^T(\mathbf{x})\mathbf{S}_\delta] \mathbf{U}_\delta = \mathbf{\Phi}(\mathbf{x}) \mathbf{U}_\delta
\]

into equation 4 we obtain,

\[
Kd = f
\]

where \( d \) is the vector of nodal displacement at all the unconstrained nodes and

\[
K_{ij} = \int_{\Omega} \mathbf{B}_i^T \mathbf{D} \mathbf{B}_j d\Omega
\]

\[
f_i = \int_{\Gamma} \mathbf{\varphi}_i \mathbf{t} d\Gamma + \int_{\Omega} \mathbf{\varphi}_i \mathbf{b} d\Omega
\]

In which

\[
\mathbf{B}_i = \begin{bmatrix} \mathbf{\varphi}_{i,x} & 0 \\ 0 & \mathbf{\varphi}_{i,y} \end{bmatrix}
\]

To define the global stiffness matrix integration calculus becomes necessary, see equation 9. Usually, Gauss points method is used. The nodal integration method can be an alternative technique to carry out the integration calculus and it’s more adapted to meshless approach, as it will be presented in following.

### III. Nodal Integration Scheme

We have an integral

\[
I = \int_{\Omega} \mathbf{f}(\mathbf{x}) d\Omega
\]

where \( \mathbf{f}(\mathbf{x}) \) is an integrable function, which is, for example, a component of matrix \( \mathbf{B}_i^T \mathbf{D} \mathbf{B}_j \) given in Equation 9; \( \Omega \) is the domain of the problem, that is represented by a set of \( \mathbf{\varphi} \) nodes distributed in the problem domain. In a nodal integration scheme, the domain \( \Omega \) is divided into a set of non-overlapping sub-domains \( \Omega_\mathbf{\varphi} = \sum_{i=1}^N \Omega_\mathbf{\varphi}_i \), each of them includes a node, and \( \Omega_\mathbf{\varphi} = \sum_{i=1}^N \Omega_\mathbf{\varphi}_i \) Then the integrated, Equation 12, can then be expressed as

\[
I = \sum_{i=1}^N \int_{\Omega_\mathbf{\varphi}_i} \mathbf{f}(\mathbf{x}) d\Omega_\mathbf{\varphi}_i
\]

In a meshfree method based on weak-form, a background mesh is needed for the implementation of numerical integration. For the present method, a background mesh is used for constructing the nodal integration domain for each node. The background mesh is not used for shape function construction which is constructed using a same set of nodes located in a local support domain. The independence of mesh from shape function construction has many advantages including the improvement in accuracy, which will be observed later in examples. This fact has also been found in many other existing works (Belytschko and al., 1994; Atluri and Zhu, 1998; Chen et al., 2001). It is very clear that one does not have to use mesh for shape function construction. In the present work, the mesh is only used for integration purpose. The question now is how
to evaluate \( \int_{\Omega_i} f(x) d\Omega \) over the nodal integration domain \( \Omega_i \). Here a novel and simple approach based on the Taylor series extension is presented.

The basic idea of this approach is to extend the integral function to some terms of Taylor series, and the integration will be approximately performed on these terms. Note that the integrand \( f(x) \) is required to be differentiable within the integration domain when it is extended to be terms of Taylor series. Therefore, RPIM shape functions are constructed using the same set of nodes in each integration domain. A shape function constructed is one-piece, and hence is differentiable to any order in the integration domain. Note that the discontinuity will occur on the interfaces of the integration domains, and hence causes the non-conformability, which is omitted in this work, as it is controlled by the use of RBF shape functions with proper shape parameters (Liu, 2002). Note that this kind of non-conformability exists for all the meshfree methods based on weak-form and nodal integration even the ones using MLS shape functions, unless strain smoothing technique is used (Chen and al., 2001; Liu and al., 2005a; Liu and Zhang, 2006).

For comparison, the EFG method based on nodal integration is also coded, in which shape functions are obtained using the MLS method (Belytschko and al., 1994). For convenience, this method is named as NI-MLS.

It is known that the MLS shape functions can be constructed to satisfy the compatibility condition and the continuity of the field function approximation is ensured (Liu, 2002).

IV. Formulation Of Nodal Integration

4.1. For 1D problems

To explain the method more clearly, the formulations for one-dimensional problems will be first presented. Based on Taylor series extension, a continuous function \( f(x) \) can be approximated in the vicinity of a point \( x_0 \) as follows [2]

\[
    f(x) \approx f(x_0) + f'(x_0)x + \frac{f''(x_0)}{2!}x^2
\]  

(14)

The 3rd order and above are truncated.

The integration for the function \( f(x) \) in the domain \( x_1 \leq x \leq x_2 \) can then be evaluated as:

\[
    \int_{x_1}^{x_2} f(x) dx \approx \int_{x_1}^{x_2} \left( f(x_0) + f'(x_0)x + \frac{f''(x_0)}{2!}x^2 \right) dx
\]

\[
= \int_{x_1}^{x_2} f(x_0) dx + \int_{x_1}^{x_2} f'(x_0)x dx + \int_{x_1}^{x_2} \frac{f''(x_0)}{2!}x^2 dx
\]

\[
= f(x_0)(x_2 - x_1) + \frac{1}{2}f'(x_0)(x_2^2 - x_1^2) + \frac{1}{6}f''(x_0)(x_2^3 - x_1^3)
\]  

(15)

Considering now a one-dimensional problem, the problem domain is presented by a set of nodes, as shown in Figure 1 and Figure 2. The integrand of \( f(x) \) is now a component of the matrix \( B_i^T D B_i \) (see Equation 9). When the field-nodes are regularly distributed, by using Equation (15), the numerical integration for the \( i^{th} \) node can be performed as follows.

![Figure 1 Integration domain with regular nodal distribution.](www.irosjournals.org)
Figure 2 Integration domain with irregular nodal distribution

Case 1: regularly distributed nodes

For an internal node

For an internal node, the integration can be applied as:

\[
\int_{-\frac{a}{2}}^{\frac{a}{2}} f(x) \, dx = f(x_i) \left( \frac{a}{2} + \frac{a}{2} \right) + \frac{1}{2} f'(x_i) \left[ \left( \frac{a}{2} \right)^2 - \left( -\frac{a}{2} \right)^2 \right] + \frac{1}{6} f''(x_i) \left[ \left( \frac{a}{2} \right)^3 - \left( -\frac{a}{2} \right)^3 \right]
\]

\[
= f(x_i) a + \frac{1}{24} f''(x_i) a^3
\]

(16)

where \( a \) is the nodal spacing as shown in Figure 1.

For the node at the left end

For the node located at the left end, the integration will as:

\[
\int_{0}^{\frac{a}{2}} f(x) \, dx = f(x_i) \left( \frac{a}{2} - 0 \right) + \frac{1}{2} f'(x_i) \left[ \left( \frac{a}{2} \right)^2 - 0 \right] + \frac{1}{6} f''(x_i) \left[ \left( \frac{a}{2} \right)^3 - 0 \right]
\]

\[
= \frac{1}{2} f(x_i) a + \frac{1}{8} f'(x_i) a^2 + \frac{1}{48} f''(x_i) a^3
\]

(17)

For the end at the right end

For the node located at the right end of the 1D domain,

\[
\int_{-\frac{a}{2}}^{0} f(x) \, dx = f(x_i) \left( 0 + \frac{a}{2} \right) + \frac{1}{2} f'(x_i) \left[ 0 - \left( \frac{a}{2} \right)^2 \right] + \frac{1}{6} f''(x_i) \left[ 0 - \left( -\frac{a}{2} \right)^3 \right]
\]

\[
= \frac{1}{2} f(x_i) a - \frac{1}{8} f'(x_i) a^2 + \frac{1}{48} f''(x_i) a^3
\]

(18)

Case 2: irregularly distributed nodes

When the field nodes are irregularly distributed, Equations (16), (17) and (18) can be expressed as follows.

For an internal node:

For internal node, the integration will be applied as:

\[
\int_{-\frac{b}{2}}^{\frac{b}{2}} f(x) \, dx = f(x_i) \left( \frac{b}{2} + \frac{a}{2} \right) + \frac{1}{2} f'(x_i) \left[ \left( \frac{b}{2} \right)^2 - \left( -\frac{a}{2} \right)^2 \right] + \frac{1}{6} f''(x_i) \left[ \left( \frac{b}{2} \right)^3 - \left( -\frac{a}{2} \right)^3 \right]
\]

\[
= \frac{1}{2} f(x_i) (a + b) + \frac{1}{8} f'(x_i) (b^2 - a^2) + \frac{1}{48} f''(x_i) (a^3 + b^3)
\]

(19)
For the node at the left end

\[
\int_{0}^{x_i} f(x)dx = f(x_i)(\frac{c}{2} - 0) + \frac{1}{2}f'(x_i) \left[ \frac{c}{2} \right]^2 - 0 + \frac{1}{6}f''(x_i) \left[ \frac{c}{2} \right]^3 - 0
\]
\[
= \frac{1}{2}f(x_i)c + \frac{1}{8}f'(x_i)c^2 + \frac{1}{48}f''(x_i)c^3
\]

(20)

For the end at the right end

\[
\int_{0}^{x_i} f(x)dx = f(x_i)(0 + \frac{d}{2}) + \frac{1}{2}f'(x_i) \left[ 0 - \left( \frac{d}{2} \right)^2 \right] + \frac{1}{6}f''(x_i) \left[ 0 - \left( \frac{d}{2} \right)^3 \right]
\]
\[
= \frac{1}{2}f(x_i)d - \frac{1}{8}f'(x_i)d^2 + \frac{1}{48}f''(x_i)d^3
\]

(21)

where a, b, c and d are nodal spacing for the irregularly distributed nodes as shown in Figure 2.

4.2 Numerical example

We calculate \( \int_{0}^{10} 3x^2 dx \) in domain [0,10]. The domain is divided in eleven nodes in which the integrals are calculated as shows table 1. Figure 3 shows the analytical and approximated results which coincide very good. These results have been used to plot figure 3.

<table>
<thead>
<tr>
<th>Node</th>
<th>Analytical value</th>
<th>Value by Taylor’s series</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \int_{0}^{0.5} 3x^2 dx ) = 0.125</td>
<td>( \int_{0}^{0.5} 3x^2 dx = \frac{1}{3}(3 \cdot 0^2) - \frac{1}{3}(6 \cdot 0) \cdot 1^2 + \frac{1}{24}(6) \cdot 1^3 = 0.125 )</td>
</tr>
<tr>
<td>1</td>
<td>( \int_{0.5}^{1} 3x^2 dx ) = 3.25</td>
<td>( \int_{0.5}^{1} 3x^2 dx = (3 \cdot 1^2) + \frac{1}{24}(6) \cdot 1^3 = 3.25 )</td>
</tr>
<tr>
<td>2</td>
<td>( \int_{1}^{1.5} 3x^2 dx ) = 12.25</td>
<td>( \int_{1}^{1.5} 3x^2 dx = (3 \cdot 2^2) + \frac{1}{24}(6) \cdot 1^3 = 12.25 )</td>
</tr>
<tr>
<td>3</td>
<td>( \int_{1.5}^{2} 3x^2 dx ) = 27.25</td>
<td>( \int_{1.5}^{2} 3x^2 dx = (3 \cdot 3^2) + \frac{1}{24}(6) \cdot 1^3 = 27.25 )</td>
</tr>
<tr>
<td>4</td>
<td>( \int_{2}^{2.5} 3x^2 dx ) = 48.25</td>
<td>( \int_{2}^{2.5} 3x^2 dx = (3 \cdot 4^2) + \frac{1}{24}(6) \cdot 1^3 = 48.25 )</td>
</tr>
<tr>
<td>5</td>
<td>( \int_{2.5}^{3} 3x^2 dx ) = 75.25</td>
<td>( \int_{2.5}^{3} 3x^2 dx = (3 \cdot 5^2) + \frac{1}{24}(6) \cdot 1^3 = 75.25 )</td>
</tr>
<tr>
<td>6</td>
<td>( \int_{3}^{3.5} 3x^2 dx ) = 108.25</td>
<td>( \int_{3}^{3.5} 3x^2 dx = (3 \cdot 6^2) + \frac{1}{24}(6) \cdot 1^3 = 108.25 )</td>
</tr>
<tr>
<td>7</td>
<td>( \int_{3.5}^{4} 3x^2 dx ) = 147.25</td>
<td>( \int_{3.5}^{4} 3x^2 dx = (3 \cdot 7^2) + \frac{1}{24}(6) \cdot 1^3 = 147.25 )</td>
</tr>
<tr>
<td>8</td>
<td>( \int_{4}^{4.5} 3x^2 dx ) = 192.25</td>
<td>( \int_{4}^{4.5} 3x^2 dx = (3 \cdot 8^2) + \frac{1}{24}(6) \cdot 1^3 = 192.25 )</td>
</tr>
<tr>
<td>9</td>
<td>( \int_{4.5}^{5} 3x^2 dx ) = 243.25</td>
<td>( \int_{4.5}^{5} 3x^2 dx = (3 \cdot 9^2) + \frac{1}{24}(6) \cdot 1^3 = 243.25 )</td>
</tr>
<tr>
<td>10</td>
<td>( \int_{5}^{5.5} 3x^2 dx ) = 142.63</td>
<td>( \int_{5}^{5.5} 3x^2 dx = \frac{1}{3}(3 \cdot 10^2) - \frac{1}{3}(6 \cdot 10) \cdot 1^2 + \frac{1}{24}(6) \cdot 1^3 = 142.63 )</td>
</tr>
</tbody>
</table>

Table 1 Analytical and approximated integral values for 1D case
4.3. for 2D problems

Applying Taylor series extension, a two-dimensional (2D) continuous function \( f(x, y) \) can be approximated in the vicinity of point \( (x_0, y_0) \) as follows,

\[
f(x, y) \approx f(x_0, y_0) + (x - x_0) \frac{\partial f}{\partial x}(x_0, y_0) + (y - y_0) \frac{\partial f}{\partial y}(x_0, y_0) + \frac{1}{2!} [(x - x_0) \frac{\partial^2 f}{\partial x^2}(x_0, y_0) + (y - y_0) \frac{\partial^2 f}{\partial y^2}(x_0, y_0)] f(x_0, y_0)
\]

The integration for function \( f(x, y) \) over the nodal integration domain \( \Omega_i \) can be expressed as,

\[
\int \int_{\Omega_i} f(x, y) d\Omega \approx \int \int_{\Omega_i} \left( f(x_0, y_0) + (x - x_0) \frac{\partial f}{\partial x}(x_0, y_0) + \frac{1}{2!} [(x - x_0) \frac{\partial^2 f}{\partial x^2}(x_0, y_0) + (y - y_0) \frac{\partial^2 f}{\partial y^2}(x_0, y_0)] f(x_0, y_0) \right) d\Omega
\]

\[
= f(x_0, y_0) \int \int_{\Omega_i} d\Omega + f_x(x_0, y_0) \int \int_{\Omega_i} x d\Omega + f_y(x_0, y_0) \int \int_{\Omega_i} y d\Omega + \frac{1}{2} f_{xx}(x_0, y_0) \int \int_{\Omega_i} x^2 d\Omega + f_{xy}(x_0, y_0) \int \int_{\Omega_i} xy d\Omega + \frac{1}{2} f_{yy}(x_0, y_0) \int \int_{\Omega_i} y^2 d\Omega
\]

\[
= f(x_0, y_0) A_i + f_x(x_0, y_0) M_{y_i} + f_y(x_0, y_0) M_{x_i} + \frac{1}{2} f_{xx}(x_0, y_0) M_{y_i} + \frac{1}{2} f_{yy}(x_0, y_0) M_{x_i}
\]

where:

\[
A_i = \int \int_{\Omega_i} d\Omega, \quad \text{Area of domain of } i^{th} \text{ node}
\]

\[
M_{x_i} = \int \int_{\Omega_i} x d\Omega, \quad M_{y_i} = \int \int_{\Omega_i} y d\Omega, \quad M_{xy_i} = \int \int_{\Omega_i} xy d\Omega, \quad M_{xx_i} = \int \int_{\Omega_i} x^2 d\Omega, \quad M_{yy_i} = \int \int_{\Omega_i} y^2 d\Omega
\]

the area moments of 1st order for the domain of the \( i^{th} \) node,

\[
M_{x_i} = \int \int_{\Omega_i} y^2 d\Omega, \quad M_{y_i} = \int \int_{\Omega_i} x^2 d\Omega, \quad M_{xy_i} = \int \int_{\Omega_i} xy d\Omega
\]

the area moments of 2nd order for the domain of the \( i^{th} \) node.

4.4. The integration along the boundary line

The integration for function \( f(x, y) \) along the boundary line can be formulated as:
To apply the nodal integration technique, a background cell is needed to divide the problem domain into nodal integration domains, each of which includes a node. When the nodes are regularly distributed, a rectangular domain can be used as the nodal integration domain (illustrated in Figure 4), and the union of all the rectangles forms the problem domain. As shown in 5.1, when the nodes are irregularly distributed, a tessellation can always be generated automatically by joining the centroids of the triangles and the mid-edge points (Ferzige and Peric, 1999).

\[
\int_{\Gamma} f(x,y) dl = \int_{\Gamma} \left( f(x_0, y_0) + \frac{\partial}{\partial x} \frac{\partial}{\partial y} f(x_0, y_0) \right) dl + \frac{1}{2} \int_{\Gamma} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f(x_0, y_0) dl
\]

4.5. Numerical example: two-dimensional case

Area test A is chosen, a 2x2 plate with 9 nodes distributed as shown in Figure 6, regular (a) and irregular (b) manner.

As a benchmark test, the following integral is chosen:

\[
I = \int_A 2x^2 y^2 \, dx \, dy
\]

A Fortran code, written in our laboratory, for each area calculates all necessary geometrical characteristics and evaluates the integral values in regular and irregular cases. The following Table 2 gives the cumulus results...
calculated by Maple, Analytical method and by present methods in both cases: regular and irregular nodal distribution. The following plot (figure 7) gives comparison between analytical and present methods in regular and irregular nodal distribution.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.128470</td>
<td>0.128470</td>
<td>0.166666</td>
<td>0.070400</td>
</tr>
<tr>
<td>2</td>
<td>0.218748</td>
<td>0.218677</td>
<td>0.250000</td>
<td>0.218600</td>
</tr>
<tr>
<td>3</td>
<td>0.222220</td>
<td>0.218677</td>
<td>0.250000</td>
<td>0.218600</td>
</tr>
<tr>
<td>4</td>
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<td>3.562419</td>
<td>3.666667</td>
<td>4.752256</td>
</tr>
<tr>
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<td>5.909720</td>
<td>5.909641</td>
<td>6.000000</td>
<td>6.227656</td>
</tr>
<tr>
<td>6</td>
<td>5.9999982</td>
<td>5.999918</td>
<td>6.083333</td>
<td>6.378456</td>
</tr>
<tr>
<td>7</td>
<td>10.753492</td>
<td>10.75339</td>
<td>10.91597</td>
<td>9.983253</td>
</tr>
</tbody>
</table>

Table 2 Analytical and approximated integral values for 2D case,(by Maple, Analytical calculus, Taylor’s series with regular nodal distribution, with irregular distribution)

Figure 7 Integration Results : Red, Analytical - Blue, Present method with regular nodal distribution- Black, Present method with irregular nodal distribution.

V. Discussion

The results given by present integration method, named in this paper nodal integration, are very accurate. The error’s calculation gives:

- For regular nodal distribution:
  \[ e = \left| \frac{I_{ex} - I_{code}}{I_{ex}} \right| \times 100\% = \left| \frac{14.222139 - 14.49931}{14.222139} \right| \times 100 = 1.94\% \]  (27)

- For irregular nodal distribution:
  \[ e = \left| \frac{I_{ex} - I_{code}}{I_{ex}} \right| \times 100\% = \left| \frac{14.222139 - 14.574774}{14.222139} \right| \times 100 = 2.47\% \]  (28)

What proves to pay more attention to this method which will replace classical Gauss one. Don’t forget to note that in irregular nodal distribution case it is necessary to determine all geometrical area characteristics in the beginning stage and then to proceed to integral calculation.
References

Books:

Theses: