On The Sublattice [I, C0] Of the Lattice of Cech Closure Operators

Baby Chacko
Associate Professor, Department of Mathematics, St. Joseph’s College, Devagiri, Calicut-8, and Kerala, India.

Abstract: The interval [I,C0] where I is the indiscrete closure operator and C0 is the co-finite closure operator on a set X is a complete sublattice of the lattice of all closure operators on X. In this paper, we determine a class of automorphisms of the lattice [I,C0] and characterize the group of automorphisms on [I,C0] when X is finite.

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I. Introduction

In 1958, Juris Hartmanis [4] determined the automorphisms of the lattice LT(X) of all topologies on a fixed set X as follows: for p ∈ S(X) and τ ∈ LT(X), define the mapping Ap by Ap(τ) = { p(U) : U ∈ τ }. Then Ap(τ) is a topology on X and Ap(τ) is an automorphism of LT(X). If X is infinite or X contains at most two elements, the group of automorphisms of LT(X) is precisely {Ap : p ∈ S(X)}. Otherwise, the group of automorphisms of LT(X) is {Ap : p ∈ S(X)} ∪ {Bp : p ∈ S(X)} where Bp : LT(X) → LT(X) is defined by Bp(τ) = { X-p(U) : U ∈ τ } for τ ∈ LT(X). From this result, we can conclude that, if X is an infinite set and P is any topological property, then the set of topologies in LT(X) possessing the property P may be identified simply from the lattice structure of LT(X), since the only automorphisms of LT(X) for infinite X are those which simply permute elements of X. Therefore any automorphism of LT(X) must map all the topologies in LT(X) onto their homeomorphic images. Thus the topological properties of elements of LT(X) must be determined by the position of the topologies in LT(X). In this paper, we determine a class of automorphisms of the lattice [I, C0] and characterize the group of automorphisms on [I, C0] when X is finite.

II. Preliminaries

Definition 2.1
Let φ(X) denotes the power set of a set X. A Cech closure operator on a set X is a function V : φ(X) → φ(X) such that,
(i) V(∅) = ∅
(ii) A ⊆ V(A) for all A ∈ φ(X)
(iii) V(∩B) = V(A) ∩ V(B) for all A,B ∈ φ(X).
For brevity, we call V a closure operator on X and the pair (X,V) a closure space.

Definition 2.2
Let (X,V) be a closure space. A subset A of X is said to be closed, if V(A) = A and open, if V(X - A) = X - A.

Definition 2.3
The set of all open sets in (X,V) is a topology on X, called the topology associated with V. On the other hand, to every topology τ on X, we can associate a closure operator V on X (the Kuratowski closure operator) defined by V(A) = cl(A) where cl(A) denotes the closure of A in (X,τ). We say that V is the closure operator associated with τ.

Remark 2.4
A closure operator on a set need not be the closure operator associated with the topology associated with it. In this sense Cech closure operators on a set X can be considered as generalization of topologies on X.

Definition 2.5
Let V1 and V2 be two closure operators on a set X. Then V1 is said to be coarser than V2 (or V2 is said to finer than V1) if V1(A) ⊆ V2 for all A ∈ φ(X). In this case we write V1 ≤ V2.

Example 2.6
Let V : φ(X) → φ(X) be defined by
V(A) = ∅ if A = ∅
= X otherwise.
Then \( V \) is a closure operator on \( X \), called the indiscrete closure operator. The indiscrete closure operator is usually denoted by \( I \).

**Example 2.7**

Let \( X \) be an infinite set. Define \( V : \wp(X) \rightarrow \wp(X) \) by

\[
V(A) = A \text{ if } A \text{ is finite,} \\
= X \text{ otherwise.}
\]

Then \( V \) is a closure operator on \( X \), called the co-finite closure operator. The co-finite closure operator is usually denoted by \( C_0 \).

**Remark 2.8**

The relation "coarser than" is a partial order on the set of all closure operators on \( X \). We denote the set of all closure operators on a set \( X \) by \( LC(X) \). Then \( LC(X) \) is a complete lattice under the relation "coarser than" and the least element of this lattice is \( I \).

**Definition 2.9**

A closure operator on \( X \) other than \( I \) is called an infra closure operator, if the only closure operator on \( X \) strictly smaller than it is \( I \), the indiscrete closure operator on \( X \). Note that the infra closure operators on \( X \) are precisely the atoms of the lattice \( LC(X) \).

**Definition 2.10**

For \( a, b \in X \), \( a \neq b \), define \( V_{(a,b)} \) by,

\[
V_{(a,b)}(A) = A \text{ if } A = \emptyset, \\
= X - \{b\} \text{ if } A = \{a\}, \\
= X \text{ otherwise.}
\]

Then \( V_{(a,b)} \) is an infra closure operator on \( X \).

**Theorem 2.11**

A closure operator on \( X \) is an infra closure operator if and only if it is of the form \( V_{(a,b)} \) for some \( a, b \in X \) such that \( a \neq b \). [6]

**Notation 2.12**

We use the notation \( \Omega \) to denote the atoms of the lattice \( LC(X) \). Then by the Theorem 1, the members of \( \Omega \) are of the form \( V_{(a,b)} \) where \( a, b \in X, a \neq b \).

**Remark 2.13**

The interval \([I,C_0]\) where \( I \) is the indiscrete closure operator and \( C_0 \) is the co-finite closure operator on a set \( X \) is a complete sublattice of the lattice \( LC(X) \).

**Remark 2.14**

The set of all closure operators \([I,C_0]\) on a set \( X \) under the partial order "\( \leq \)" defined by \( V_1 \leq V_2 \iff V_2(A) \subseteq V_1(A) \) for every \( A \in \wp(X) \) is a complete lattice.

**Notation 2.15**

We use the notation \( S(X) \) to denote the set of all bijections on \( X \).

### III. Main Results

**Lemma 3.1**

For a closure operator \( V \) on \( X \) such that \( V \leq C_0 \), define a relation \( \rho V \) on \( X \) by \( \rho V = \{ (x,y) : y \in V(\{x\}) \} \). Then \( \rho V \) is a reflexive relation on \( X \).

**Proof:** Obvious.

**Lemma 3.2**

For \( R \in LR(X) \), the lattice of all reflexive relation on \( X \), define \( \nu R : \wp(X) \rightarrow \wp(X) \) by \( \nu R(A) = \{ y \in X : xRy \text{ for some } x \in A \}, A \in \wp(X) \). Then \( \nu R \) is a closure operator in \([I,C_0]\).

**Proof:** Obvious.

**Remark 3.3**

It can be easily verified that, the mapping \( \nu : LR(X) \rightarrow [I,C_0] \) defined by \( \nu(R) = \nu R \) and the inverse mapping \( \rho : [I,C_0] \rightarrow LR(X) \) defined by \( \rho(V) = \rho V \) are dual isomorphisms.

**Theorem 3.4**

Let \( X \) be a non-empty set. For \( V \in [I,C_0] \) and \( p \in S(X \times X - \Delta) \), let \( R_{p,V} = p(\rho V - \Delta) \cup \Delta \). Then \( R_{p,V} \in LR(X) \). Further, let \( T_V = \nu R_{p,V} \). Then \( T_V \in [I,C_0] \) and the mapping \( T_p \) defined by \( T_p(V) = T_V \) for \( V \in [I,C_0] \) is an automorphism of \([I,C_0]\).

**Proof:**

From the Lemmas 3.1, 3.2 & Remark 3.3, it follows that for \( V_1, V_2 \in [I,C_0] \),

\[
V_1 \leq V_2 \iff \rho V_2 \subseteq \rho V_1 \\
\iff R_{p,V_2} \subseteq R_{p,V_1} \\
\iff T_{p}V_1 \leq T_{p}V_2 \\
\iff T_{p}(V_1) \leq T_{p}(V_2)
\]
Further, since the correspondences \( V \rightarrow \rho V \), \( \rho V \rightarrow \rho V - \Delta \), \( \rho V - \Delta \rightarrow p(\rho V - \Delta) \), \( p(\rho V - \Delta) \rightarrow p(\rho V - \Delta) \cup \Delta \) and \( R_{p,V} \rightarrow T_p V \) are bijections, it follows that \( T_p : V \rightarrow T_p V \) is a bijection.

Hence \( T_p \) is an automorphism of the lattice \([I,C_0]\).

**Remark 3.5**

Obviously the set of atoms of the lattice \([I,C_0]\) is precisely the set \( \Omega = \{ V_{(a,b)} : a, b \in X \text{ and } a \neq b \} \).

**Theorem 3.6**

Let \( X \) be a non-empty finite set. Then the lattice \([I,C_0]\) coincides with the lattice \( LC(X) \) of all Čech closure operators on \( X \) and hence the group of the lattice \([I,C_0]\) is precisely the set \( \{ T_p : p \in S((X \times X) - \Delta) \} \).

**Proof:**

Let \( A \) be any automorphism of the lattice \([I,C_0]\). We want to show that \( A = T_p \) for some \( p \in S((X \times X) - \Delta) \). For \( V_{(a,b)} \in \Omega \), let \( A(V_{(a,b)}) = V_{(a,b)'} \) for some \( (a,b)' \in (X \times X) - \Delta \). Then \( (a,b)' \) is unique. Define \( p(a,b) = (a,b)' \). Then \( p \in S((X \times X) - \Delta) \).

Now for \( V_{(a,b)} \in \Omega \),

\[
A(V_{(a,b)}) = V_{(a,b)'}
= \cup [ (\rho V_{p(a,b)} - \Delta) \cup \Delta ]
= \cup [ p(\rho V_{(a,b)} - \Delta) \cup \Delta ]
= T_p V_{(a,b)}
= T_p (V_{(a,b)})
\]

Hence \( A = T_p \) on \( \Omega \). Since \( X \) is finite, the lattice \([I,C_0] = LC(X)\) is atomistic and hence it follows that \( A = T_p \) on \([I,C_0]\).

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**References**


