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On A New Class of Numbers

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Abstract: *The present paper studies a new class of numbers. Results obtained in this paper are a table, recurrence relations, generating functions and Summation formulas for these new class of numbers . Many results reduce to their corresponding results for the Catalan numbers .*

I. Definition:

(1.1) Consider the quadratic equation

$x^2 = 2x + 1$

Its roots are

$\alpha = 1 + \sqrt{2}$

$\beta = 1 - \sqrt{2}$

$\alpha + \beta = 2$

$\alpha\beta = -1$

Let $V_n = \alpha^n + \beta^n$ for $n \geq 0$, then

$V_0 = 2, V_1 = 2, V_2 = 6, V_3 = 14, V_4 = 34, V_5 = 82, V_6 = 198, V_7 = 478,$

$V_8 = 1154$ etc.

These numbers satisfies the following recurrence relation

$V_{n+1} = 2V_n + V_{n-1}$; $n \geq 1$ with $V_0 = 2, V_1 = 2$.

In view of (1.1) we define a new class of numbers

(1.2) $F_{n,k} = (-1)^{n-k}\binom{2n+1}{n-k}V_{2k+1}$ where n is any non-negative integer and

$0 \leq k \leq n$.

Also these numbers generalize the Catalan numbers in a non trivial way.

The catalan numbers C_n are defined by means of the generating relations ([3],p.82)

(1.3) $$\sum_{n=0}^{\infty} C_n t^n = \frac{1-\sqrt{1-4t}}{2t}$$

Or by the explicit formula ([3],p.101)

(1.4) $$C_n = \frac{1}{n+1}\binom{2n}{n}$$

The following relationship is obvious

(1.5) $$C_n = \frac{(-1)^n}{2} \quad \frac{F_{n,0}}{2n+1}$$

As usual $(\alpha)_n$ is Pochhammer's symbol and is defined by

(1.6) $$(\alpha)_n = \begin{cases} 1 & \text{if} \quad n = 0 \\ \alpha(\alpha+1).....(\alpha+n-1), & \text{for all} \quad n \in \{1,2,3.......\} \end{cases}$$

${}_2F_1$ will denote the hypergeometric function defined by

(1.7) $${}_2F_1\left[\begin{matrix} a,b; \\ \quad x \\ c; \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b_n)}{(c)_n} \frac{x^n}{n!}; c \neq 0,1,-2,---$$

The Jacobi Polynomials are defined by

(1.8) $$P_n^{(\alpha,\beta)}(x) = \frac{(1+\beta)_n}{n!}\left(\frac{x-1}{2}\right)^n {}_2F_1\left[\begin{matrix} -n,-\alpha-n\ ; \\ \quad \frac{x+1}{x-1} \\ 1+\beta\ ; \end{matrix}\right]$$

(1.9) In [1] A.K.Agarwal studied a new kind ofnumbers

The New kind of Numbers are defined as $f(n,k) = (-1)^{n-k}\binom{2n+1}{n-k} L_{2k+1};\ 0 \le k \le n$

and n is a non-negative integers and L_{2k+1} is a Lucas number of order 2k+1.

These new kind of numbers have the interesting property that

(1.10) $$\sum_{k=0}^{n} f(n,k) = 1$$

Theorem 1 $$\sum_{k=0}^{n} F_{n,k} = 2^{2n+1}$$

Proof : Let n be an odd positive integer and α,β be the roots of $x^2 = 2x + 1$ as defined in (1.1) Then from the binomial expansion of $(\alpha + \beta)^n$ we get

(1.11) $$2^n = V_1 - \binom{n}{1} V_{n-2} + \binom{n}{2} V_{n-4} \dots + (-1)^{\frac{n-1}{2}} \binom{n}{\frac{n-1}{2}} V_1$$

where V_n is defined as in (1.1) Setting n=2m+1 in (1.11) we obtain

$$2^{2m+1} = \sum_{k=0}^{m} (-1)^{m-k} \binom{2m+1}{m-k} V_{2k+1}$$

Remark 1 : Theorem 1 is analogues with the following property of Sterling numbers of the first kind [see [10],(6) p. 145]

$$\sum_{k=0}^{m} S_n^k = 0$$

Remark 2: Also theorem 1 is analogues with (1.10)

II. . Table for $F_{n,k}$

n/k	0	1	2	3	4	5	6	7
0	2							
1	–6	14						
2	20	–70	82					
3	–70	294	–574	478				
4	252	–1176	2952	–4302	2786			
5	–924	4620	–13530	26290	–30646	16238		

III. Recurrence Relations

(4.1) (n+k+2)(n+k+1) $F_{n,k+1}$ + 6(n–k)(n+k+1) $F_{n,k}$ +(n–k+1)

(n–k) $F_{n,k-1}$ =0 for $k \ge 1$

(4.2) $$F_{n+r,k} = (-1)^r \frac{(2n+2)_{2r}}{(n-k+1)_r (n+k+2)_r} F_{n,k}$$

where r is a non negative integer.

Proof of (4.1)

The sequence $\{V_{2k+1}\}$ satisfies the following recurrence relations

(4.1.1) $V_{2n+3} = 6V_{2k+1} - V_{2k-1}$

Now using the definition (1.2) and (4.1.1) we arrive at (4.1).

Proof of (4.2)

By definition, we have

(4.2.1) $$F_{n,k} = (-1)^{n-k} \binom{2n+1}{n-k} V_{2k+1}$$

And

(4.2.2) $$F_{n+1,k} = (-1)^{n-k+1} \binom{2n+3}{n-k+1} V_{2k+1}$$

Eliminating V_{2k+1} we have

$$F_{n+1,k} = (-1)\frac{(2n+3)(2n+2)}{(n-k+1)(n+k+2)} F_{n,k}$$

Proceeding in similar manner and using (1.6) we arrive at (4.2)

IV. Generating Relations

(5.1) $$\sum_{n=0}^{\infty} \frac{F_{n+k,k}}{2(k+n)+1} y^n = \frac{x^{2k+1}}{2k+1} V_{2k+1}$$

(5.2) $$\sum_{n=0}^{\infty} F_{n+k,k}\, y^n = \frac{x^{2k+2}}{2-x} V_{2k+1}$$

Where $y = (1-x)x^{-2}$

Proof of (5. 1)

Here we shall use the identity

(5. 1.1) $$\sum_{n=0}^{\infty} \frac{\alpha}{\alpha+n\beta} \binom{\alpha+n\beta}{n} y^n = x^{\alpha} \;;\; y = (x-1)x^{-\beta}$$

(see[3]. P. 147)

Setting α=2k+1 and β =2 in (5.1.1) we arrive at (5. 1)

Proof of (5. 2)

Here we shall use the identity

(5.2.1) $$\sum_{n=0}^{\infty} \binom{\alpha+n\beta}{n} y^n = \frac{x^{\alpha+1}}{(1-\beta)x+\beta} \;;\; y = (x-1)x^{-\beta}$$

(see[3]. P. 147)

Setting α=2k+1 and β =2 we arrive at (5.2)

Remark 3

For k = 0 ; (5. 1) and (1.4) yields (1.3)

Remark 4 For k = 0 ; (5.2) yields the following generating relation for the Catalan numbers

$$\sum_{n=0}^{\infty} (2n+1)C_n y^n = \frac{x^2}{2-x} \; ; \; y = (x-1)x^{-2}$$

Theorem (5.3) : Let $F_{n,k}$ be defined as in (1.2) then

(5.3.1) $$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} F_{n+k,k}\, y^{n+k} = \frac{2x^3}{(2-x)(x^2-8x+8)}$$

(5.3.2) $$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{F_{n+k,k}}{2(n+k)+1} y^{n+k} = xF_4\left[1,\frac{1}{2};\frac{3}{2},\frac{1}{2};1-x,2(1-x)\right]$$

Where F_4 is Appell's double hypergeometric function of fourth kind defined by (see[4],p.14)

(5.3.3) $$F_4\left[a,b;c,c^{/};x,y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m (c^1)_n} \frac{x^m}{m!} \frac{y^n}{n!}$$

$\sqrt{|x|} + \sqrt{|y|} < 1.$

Proof of (5.3.1):

The numbers $\{V_{2k+1}\}$ Satisfy the following generating relation

(5.3.4) $$\sum_{k=0}^{\infty} V_{2k+1} x^k = \frac{2+2x}{x^2-6x+1} \; ; |x|<1$$

Using the definition of $F_{n,k}$ we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} F_{n+k,k}\, y^{n+k} = \sum_{k=0}^{\infty} V_{2k+1} y^k \sum_{n=0}^{\infty} \binom{2n+2k+1}{n} (-y)^n$$

Now summing the inner series with the help of (5.2.1) and using (5.3.4) we arrive at (5.3.1)

Proof of (5.3.2):

From the definition of the sequence $\{ V_n \}$ we have.

(5.3.5) $$V_n = \left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n$$

Which allows us to use the identity

$${}_2F_1\left[\begin{matrix} \frac{a}{2}, \frac{a}{2}+\frac{1}{2}; \\ \frac{1}{2}; \end{matrix} \; z^2\right] = \frac{1}{2}\left\{(1-z)^{-a} + (1+z)^a\right\}$$

Setting a = –n ,z = $\sqrt{2}$ we see that

(5.3.6) $${}_2F_1\left[\begin{matrix} -\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}; \\ \frac{1}{2}; \end{matrix} \; 2\right] = \frac{1}{2}\left\{\left(1-\sqrt{2}\right)^n + \left(1+\sqrt{2}\right)^n\right\}$$

Comparing (5.3.5) and (5.3.6) we get

$$V_n = 2\,{}_2F_1\left[\begin{matrix} -\frac{n}{2}-\frac{1}{2}, -n \;; \\ \frac{1}{2} \;; \end{matrix} \; 2\right]$$

For odd n we have

(5.3.7)
$$V_{2n+1} = 2\ {}_2F_1\left[\begin{matrix} -n-\frac{1}{2}, -n\ ; \\ \frac{1}{2}\ ; \end{matrix}\ 2\right]$$

Again in (1.7) setting $\alpha = \frac{1}{2}, \beta = -\frac{1}{2}$ and x=3 we see that

$$V_{2n+1} = \frac{n\,!}{\left(\frac{1}{2}\right)_n}\rho_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(3)$$

Again in (1.8) setting γ=1, $\delta = \frac{1}{2}$, $\alpha = \frac{1}{2}, \beta = -\frac{1}{2}$ and x=3 we get

$$\sum_{n=0}^{\infty}\frac{(1)_n\left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}\right)_n}\rho_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(3)t^n = F_4\left[1,\frac{1}{2};\frac{3}{2},\frac{1}{2};t,2t\right]$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{n!\left(\frac{1}{2}\right)_n}{\left(\frac{1}{2}\right)_n(2n+1)\left(\frac{1}{2}\right)_n}\rho_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(3)t^n = F_4\left[1,\frac{1}{2};\frac{3}{2},\frac{1}{2};t,2t\right]$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{V_{2n+1}}{2n+1}t^n = F_4\left[1,\frac{1}{2};\frac{3}{2},\frac{1}{2};t,2t\right]$$

Now, Starting with L.H.S. we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{F_{n+k,k}}{2(n+k)+1}y^{n+k} = \sum_{k=0}^{\infty}V_{2k+1}y^k$$

$$\sum_{n=0}^{\infty}\frac{1}{2(n+k)+1}\binom{2n+2k+1}{n}(-y)^n$$

Now summing the inner series with the help of (5.1.1) we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{F_{n+k}}{2(n+k)+1}y^{n+k} = \sum_{k=0}^{\infty}V_{2k+1}y^k\,\frac{x^{2k+1}}{2k+1}$$

$$= x\sum_{k=0}^{\infty}\frac{V_{2k+1}}{2k+1}\left(x^2y\right)^k$$

$$= xF_4\left[1,\frac{1}{2};\frac{3}{2};\frac{1}{2};1-x,2(1-x)\right]$$

V. Summation Formulae

(6.1)
$$\sum_{m=0}^{n-1}\left\{F_{n+k-m-1,k} + \frac{V_{2k+1}}{V_{2k-1}}F_{n+k-m-1,k-1}\right\}F_{m,0} = 2F_{n+k,k}$$

$F_{n,0}V_{2k+1}; k \geq 1$

(6.2)
$$\sum_{m=0}^{n-1}\frac{F_{m,o}}{2m+1}\,\frac{F_{n-m-1,o}}{2(n-m)-1} = \frac{2F_{n,0}}{2n+1}$$

(6.3) $$\sum_{m=0}^{n}\frac{n-2m(k+1)}{(2m+1)\{2(n+k-m)+1\}}F_{m,0}\ F_{n+k-m,k}=0$$

(6.4) $$\sum_{m=0}^{n}(-1)^k\left[\frac{F_{n,n-k}}{V_{2(n-k)+1}}+\frac{F_{n,n-k+1}}{V_{2(n-k)+3}}\right]=(2n+1)C_n$$

Proof of (6.1):

Replacing k by k–1 in (5. 2) we obtain

(6.1.1) $$\sum_{n=0}^{\infty}F_{n+k-1,k-1}y^n=\frac{x^{2k}}{2-x}V_{2k-1}$$

Again putting k = 0 in (5. 2) we obtain

(6.1.2) $$\sum_{n=0}^{\infty}F_{n,0}y^n=\frac{x^{2k}}{2-x}$$

From (5. 2) and (6.1.1) we obtain.

$$\sum_{n=0}^{\infty}F_{n+k,k}\ y^n=\frac{V_{2k+1}}{V_{2k-1}}x^2\ \sum_{n=0}^{\infty}F_{n+k-1,k-1}\ y^n$$

$$\Rightarrow\frac{1}{2-x}\sum_{n=0}^{\infty}F_{n+k,k}\ y^n=\frac{V_{2k+1}}{V_{2k-1}}\frac{x^2}{2-x}\ \sum_{n=0}^{\infty}F_{n+k-1,k-1}\ y^n$$

$$\Rightarrow 2\left(1-\frac{x^2y}{2-x}\right)\sum_{n=0}^{\infty}F_{n+k,k}\ y^n=\frac{V_{2k+1}}{V_{2k-1}}\frac{2x^2}{2-x}\ \sum_{n=0}^{\infty}F_{n+k-1,k-1}\ y^n$$

$$\Rightarrow\left[2-\sum_{m=0}^{\infty}F_{m,0}\ y^{m+1}\right]\sum_{n=0}^{\infty}F_{n+k,k}\ y^n$$

$$=\frac{V_{2k+1}}{V_{2k-1}}\sum_{m=0}^{\infty}F_{m,0}\ y^m\sum_{n=0}^{\infty}F_{n+k-1,k-1}\ y^n$$

Now equating the coefficients of y^n we arrive at (6.1)

Proof of (6.2)

Putting k = 0 in (5.1) we obtain

$$\sum_{n=0}^{\infty}\frac{F_{n,0}}{2n+1}y^n=2x$$

$$\Rightarrow\sum_{n=0}^{\infty}\frac{F_{n,0}}{2n+1}y^n=2(1-x^2y)$$

$$=2-\frac{y}{2}\sum_{m=0}^{\infty}\frac{F_{m,0}}{2m+1}y^m\sum_{n=0}^{\infty}\frac{F_{n,0}}{2n+1}y^n$$

Now equating the coefficients of y^n we arrive at (6.2)

Proof of (6.3):

From (5.1) and (5. 2) we obtain.

$$\sum_{n=0}^{\infty}F_{n+k,k}\ y^n=(2k+1)\frac{x}{2-x}\sum_{n=0}^{\infty}\frac{F_{n+k,k}}{2(n+k)+1}y^n$$

$$\Rightarrow 2x\sum_{n=0}^{\infty}F_{n+k,k}\ y^n=(2k+1)\frac{2x^2}{2-x}\sum_{n=0}^{\infty}\frac{F_{n+k,k}}{2(n+k)+1}y^n$$

$$\Rightarrow \sum_{m=0}^{\infty} \frac{F_{m,0}}{2m+1} y^m \sum_{n=0}^{\infty} F_{n+k,k}\, y^n$$

$$=(2k+1)\sum_{m=0}^{\infty} F_{m,0}\, y^m \sum_{n=0}^{\infty} \frac{F_{n+k,k}}{2(n+k)+1} y^n$$

Now equating the coefficients of y^n we arrive at (6.3)

Proof of (6.4):

Using the following identity (see[3] p.65)

$$\binom{2n+1}{n} = \sum_{k=0}^{n} \left[\binom{2n+1}{k} - \binom{2n+1}{k-1}\right]$$

We arrive at (6.4)

Remark 5 : In view of (1.5), (6.2) yields the following formula for Catalan numbers

$$\sum_{m=0}^{n-1} C_m\, C_{n-m-1} = C_n$$

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