IOSR Journal of Mathematics (IOSRJM)
ISSN: 2278-5728 Volume 2, Issue 6 (Sep-Oct. 2012), PP 36-47
www.iosrjournals.org

K-derivation and symmetric bi-k-derivation on Gamma Banach Algebras

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Abstract: *In this paper, we define and study k-derivations and symmetric bi-k-derivations on a Γ-Banach algebra. We also define and study h⊗k-derivation d on the projective tensor product* $V \otimes_p V^{/}$ *for the h- and k-derivations* d_1 *and* d_2 *on Γ-Banach Algebras* $V(F)$ *and* $V^{/}(F')$ *respectively.*

AMS subject classification Code: 17D20 (γ, δ)
Key words: *k-derivation, symmetric bi-k-derivation, centroid.*

I. Introduction:

N. Nobusawa [7] introduced the notion of a Γ-ring, more general than a ring. W. E. Barnes [11] weakened slightly the condition in definition of Γ-ring in the sense of Nobusawa. W. E. Barnes [11], J. Luh [4] and S. Kyuno [10] studied the structure of Γ-rings and obtained various generalizations analogous to corresponding parts in ring theory. Bhattacharya and Maity [2] introduced the notion of a Γ-Banach algebra. In recent times, many far reaching results of general algebras have been extended to Γ-algebras by many outstanding research workers. In this paper, we study k-derivation on Γ-Banach algebras V and k⊗h- deviation on Γ⊗Γ-Banach algebra $V \otimes_p V^{/}$. We define symmetric bi-k-derivation on Γ-Banach algebras in which k: $\Gamma \to \Gamma$ is an additive map such that $k^n = k$, where n is a positive integers. Some important results relating to this concepts are proved. For example we show that **(a)** Let $V(F)$ and $V^{/}(F^{/})$ be two Γ-Banach algebra and $\Gamma^{/}$-Banach algebra respectively with $e\delta x = x\delta e = x(\forall x \in V)$, $e \in V, \delta \in \Gamma$ and $e^{/}\delta^{/}y = y\delta^{/}e^{/} = y$ $(\forall y \in V)$, $e' \in V', \delta' \in \Gamma'$. If d_1 and d_2 are k- and h-inner derivation on $V(F)$ and $V^{/}(F^{/})$ respectively implemented by (a, δ) and $(b, \delta^{/})$ respectively then d is a k⊗h- inner deviation on $V \otimes_p V^{/}$ implemented by $(a \otimes e^{/} + e \otimes b, \delta \otimes \delta^{/})$, **(b)** Let V be a 2-torsion free prime Γ-Banach algebra, $D_1(.,.)$, $D_2(.,.)$ and $D_3(.,.)$ and $D_4(.,.)$ the symmetric bi-k-derivations on V and d_1, d_2, d_3 and d_4 traces of $D_1(.,.), D_2(.,.), D_3(.,.)$ and $D_4(.,.)$ respectively. If $d_1(x)\gamma\, d_2(y) = d_3(x)\gamma\, d_4(y)$, for all x, y∈V and γ∈Γ and $d_1 \neq 0 \neq d_4$, then there exists $\lambda \in C_\Gamma$ such that $d_2(x) = \lambda\alpha d_1(x)$ for all α∈Γ, where C_Γ is the extended centroid of V, **(c)** Let V be a 2-torsion free prime Gamma Banach algebra and U be a non zero ideal of V. Suppose there exist symmetric bi-k-derivations $D_1 : V \times V \to V$ and $D_2 : V \times V \to V$ such that $D_1(d_2(x), x) = 0$ holds for all x∈U where d_2 denotes the trace of D_2. In this case $D_1 = 0$ or $D_2 = 0$, **(d)** Let V be a 2- and 3-torsion free prime Γ-Banach algebra. Let U be a non zero ideal of V and D_1:V×V→V and D_2:V×V→V be symmetric bi-k- derivations. Suppose further that there exists a symmetric bi-additive mapping B: V×V→V such that $d_1(d_2(x)) = f(x)$ holds, for all x∈ U, where D_1 and D_2 are the traces of D_1 and D_2 respectively and f is the trace of B. Then either D_1=0 and D_2=0.

II. Preliminaries

Let V and Γ be two additive abelian groups. If for all $x, y, z \in V; \gamma, \mu \in \Gamma$, the following conditions are satisfied, (a) $x\gamma y \in V$, (b) $(x+\gamma)\gamma z = x\gamma z + y\gamma z$, $x(\gamma+\mu)y = x\gamma y + x\mu y$,

$x\gamma(y+z)=x\gamma y+x\gamma z$ (c) $x\gamma(y\mu z)=(x\gamma y)\mu z$ then V is called a Γ- ring in the sense of Barnes[11]. If these conditions are strengthened to, (a') $x\gamma y\in V, \gamma x\mu\in\Gamma$, (b') is same as(b), (c) $(x+\gamma)\gamma z=x\gamma z+y\gamma z$, $x(\gamma+\mu)y=x\gamma y+x\mu y$, $x\gamma(y+z)=x\gamma y+x\gamma z$ (c) $x\gamma(y\mu z)=x(\gamma y\mu)z=(x\gamma y)\mu z$ (d) $x\gamma y=0, \forall x,y\in V$ implies $\gamma=0$, Then V is called a Γ-ring in the sense of Nobusawa.

A Γ-ring in the sense of Nobusawa V is called a Γ-Banach algebra over a field F if it satisfies the following postulates:

(a) a(xγy)=(ax)γy=xγ(ay), a∈Γ; x, y∈M; γ∈Γ.

(b) M is a Banach space over F with respect to a norm which satisfies
||xγy||≤||x|| ||γ|| ||y||, x, y∈V; γ∈Γ

A subset I of a Γ- Banach algebra V is said to be a right (left) ideal of V if

(a) I is a subspace of V (in the vector space sense).

(b) $x\gamma y\in I\,(y\gamma x\in I)$ for all $x\in I,\ \gamma\in\Gamma,\ y\in V$

i.e. $I\Gamma V\subseteq I\,(V\Gamma I\subseteq I)$.

A right Γ-ideal which is a left Γ-ideal as well as is called a two sided Γ-ideal or simply a Γ-ideal.

The notation $I\lhd V$ will mean I is an ideal of V.

A Γ-Banach algebra V is called 2-torsion free if $2x=0$ implies $x=0$, for all $x\in V$.

A Γ-ideal I of a Γ-Banach algebra V is said to be prime Γ-ideal if for any two Γ-ideals A and B, $A\Gamma B\subseteq P\Rightarrow A\subseteq P$ or $B\subseteq P$.

A Γ- Banach algebra V is said to have a left (right) strong unity if there exists some $d\in V,\delta\in\Gamma$ such that $d\delta x=x\,(x\delta d=x), \forall x\in V.$

The Projective tensor norm $\|.\|_\gamma$ on X⊗Y is defined as ||u||=inf{ $\sum_i \| x_i \| \| y_i \| : u=\sum_i x_i\otimes y_i;$ $x_i\in X, y_i\in Y$ }, where the infimum is taken over all (finite) representations of u. The completion of (X⊗Y, $\|.\|_\gamma$) is called the projective tensor product of X and Y and is denoted by $X\otimes_\gamma Y$.

Let V and $V^{/}$ be Γ-Banach algebras over the fields F_1 and F_2 respectively isomorphic to which are a field F. The projective tensor product $V\otimes_p V^{/}$ (with the projective tensor norm), is a Γ⊗Γ-Banach algebra over F, where multiplication is defined by the formula:

$(x\otimes y)(\alpha\otimes\beta)(x^{/}\otimes y^{/})=(x\alpha x^{/})\otimes(y\beta y^{/})$, where $x, y\in V$; $x^{/}, y^{/}\in V^{/}$;α, β∈Γ.

An additive operator d on the Γ-Banach algebra V over a field F into itself is called a k-derivation if d(xγy)=d(x)γy+xk(γ)y+xγd(y), for all x, y∈V; γ∈Γ,where k: Γ→Γ is also a additive map. If d(xγx)=d(x)γx+ x k(γ) x+ xγd(x) holds for all x∈V and γ∈Γ, then d is called a Jordan k-derivation on V.

Let a and γ be nonzero elements of V and Γ respectively. The d: V→V defined by $d(x)=[a, x]_\gamma$ and k: Γ→Γ defined by $k(\beta)=[\gamma,\beta]_a$ are two additive maps and d is a k-derivation on V. Then we call d is an inner k-derivation on V.

Let V be a Γ-Banach algebra. A mapping D(.,.) : V×V→V is said to be symmetric bi-additive if it is additive in both arguments and D(x, y)=D(y, x), for all x, y∈V. By the trace of D (.,.), we mean a map d: V→V defined by d(x)=D(x,x), ∀x∈V. A symmetric bi-additive map is called a symmetric bi-k-derivation if (a) D(xγy, z)=D(x, z)k(γ)y +x k(γ)D(y, x) (b) D(x, yγz)=D(x, y)k(γ)z +yk(γ)D(x, z), for all x, y, z∈V; γ∈Γ and k:Γ→Γ is a additive map. Since a map D(.,.) is symmetric bi-additive, the trace of D(.,.) satisfies the relation d(x+y)=d(x)+d(y)+2 D(x,y), for all x, y∈V and is an even function.

Let V be a prime Γ- Banach algebra such that VΓV≠V. Denote ₤= {(I, f): I (≠0) is an ideal of V and f: I→V is a Γ- Banach algebra homomorphism}

Define a relation ~ on £ by (I, f) ~ (J, g)⇔∃K(≠0)⊂I∩J such that f=g on K. Since V is a prime Γ-Banach algebra, it is possible to find such a K and so "~" is an equivalence relation on £. This gives a chance for us to get a partition of £. We then denote the equivalence class by $Cl(1, f) = \hat{f} = \{g : J \to V \mid (1, f) \sim (J, g)\}$ and denote by Q, the set of all equivalence Classes. Now we define an addition "+" and as follows $\hat{f} + \hat{g} = Cl(I, f) + Cl(J, g) = Cl(I \cap J, f + g)$.It can be easily shown that the addition "+" is well defined and (Q, +) is an abelian group.

Since VΓV≠V and since V is a prime Γ- Banach algebra VΓV(≠0) is an ideal of V. We can take the homomorphism $1_{V\Gamma} : V\Gamma V \to V$ as a **unit Γ**- Banach algebra homomorphism. Note that VβV≠0, for all 0≠β∈Γ. So that $1_{V\beta} : V\beta V \to V$ is a nonzero Γ- Banach algebra homomorphism. Define $\xi = \{(V\beta V, 1_{V\beta}) \mid 0 \neq \beta \in \Gamma\}$ and define a relation "≈" on ξ by $(V\beta V, 1_{V\beta}) \approx (V\gamma V, 1_{V_\gamma}) \Leftrightarrow$ ∃W=VαV(≠0)⊂VβV∩VγV such that $1_{V\beta} = 1_{V_\gamma}$ on W. we can easily check that "≈" is an equivalence relation on ξ. Denote by Cl(VβV,$1_{V\beta}$)=$\hat{\beta}$={(VγV,1_{V_γ}):(VβV,$1_{V\beta}$)≈(VγV,1_{V_γ})} and $\hat{\Gamma} = \{\hat{\beta} \mid 0 \neq \beta \in \Gamma\}$. Define an addition "+" on $\hat{\Gamma}$ as follows:

$$\hat{\beta} + \hat{\delta} = cl(V\beta V, 1_{V\beta}) + cl(V\delta V, 1_{V\delta})$$

$= cl(V\beta V \cap V\delta V, 1_{V\beta} + 1_{V\delta})$, for every β(≠0), δ(≠0)∈Γ. Then is an abelian group. Now we define a mapping (–,–,–): $Q \times \hat{\Gamma} \times Q \to Q, (\hat{f}, \hat{\beta}, \hat{g}) \mapsto \hat{f}\hat{\beta}\hat{g}$, as follows:

$\hat{f}\hat{\beta}\hat{g}$ =cl(I,f)cl(VβV,$1_{V\beta}$)cl(J,g)

=cl (IΓVβVΓJ, $f I_{V\beta} g$)

Where IΓVβVΓJ= $\left\{\sum_i u_i \alpha_i m_i \beta n_i \beta_i v_i : u_i \in I, v_i \in J; m_i, n_i \in V; \alpha_i \beta_i \in \Gamma\right\}$ is an ideal of V and $f 1_{V\beta}$ g: IΓVβVΓJ→V is a Γ- Banach algebra homomorphism which is define as $f 1_{V\beta}$ g $\left(\sum u_i \alpha_i m_i \beta n_i \beta_i v_i\right) = \sum_i f(u_i) \alpha_i 1_{V\beta}(m_i \beta n_i) \beta_i g(v_i)$ is a **Γ**- Banach algebra homomorphism. Then for $\hat{f}, \hat{g}, \hat{h} \in Q; \hat{\beta}, \hat{\gamma} \in \hat{\Gamma}$, we have

$$(\hat{f} + \hat{g})\hat{\beta}\hat{h} = \hat{f}\hat{\beta}\hat{h} + \hat{g}\hat{\beta}\hat{h}, \quad \hat{f}(\hat{\beta} + \hat{\gamma})\hat{g} = \hat{f}\hat{\beta}\hat{g} + \hat{f}\hat{\gamma}\hat{g}, \quad \hat{f}\hat{\beta}(\hat{g} + \hat{h}) = \hat{f}\hat{\beta}\hat{g} + \hat{f}\hat{\beta}\hat{h}$$

$$(\hat{f}\hat{\beta}\hat{g})\hat{\gamma}\hat{h} = \hat{f}(\hat{\beta}\hat{g}\hat{\gamma})\hat{h} = \hat{f}\hat{\beta}(\hat{g}\hat{\gamma}\hat{h}), \quad \hat{f}\hat{\beta}\hat{g} = \hat{0}, \forall \hat{f}, \hat{g}, \in Q \text{ implies } \hat{\beta} = \hat{0}.$$

Hence Q is a **Γ**-ring. Now we define scalar multiplication as $a\hat{f} = acl(U, \hat{f}) = cl(U, af)$,

$a \in F; \hat{f} \in Q$. Then for $\hat{f}, \hat{g} \in Q; a, b \in F$

$$a(\hat{f} + \hat{g}) = a\hat{f} + a\hat{g}, (a+b)\hat{f} = a\hat{f} + a\hat{g}, (ab)\hat{f} = a(b\hat{f}), 1.\hat{f} = \hat{f}$$

Hence $Q(F)$ is a vector space. Now for $\hat{f}, \hat{g} \in Q; \hat{\beta} \in \hat{\Gamma}; a \in F$ we can show that $a(\hat{f}\hat{\beta}\hat{g}) = (a\hat{f})\hat{\beta}\hat{g} = \hat{f}\hat{\beta}(a\hat{g})$.

Next define a norm on Q by $\|\hat{f}\| = \|(U, f)\| = \sup\{\|f(x)\| : x \in U, \|x\| \leq 1\}$

Then we find that $(\|.\|, Q)$ is a norm linear space. If $\{\hat{f}_n\}$ is a Cauchy sequence in Q, then for given $\epsilon < 0$, ∃ positive integer n_0 such that $m, n \geq n_0 \Rightarrow \|\hat{f}_n - \hat{f}_m\| < \epsilon$

$$\Rightarrow \sup\{\|\hat{f}_n(x) - \hat{f}_m(x)\| : x \in U_n \cap U_m and \|x\| \leq 1\} < \epsilon$$

$$\Rightarrow \|f_n(x) - f_m(x)\| < \epsilon, \ x \in U_n \cap U_m and \|x\| \leq 1$$

$\Rightarrow \{f_n(x)\}$ is a Cauchy sequence in M.

$\Rightarrow \exists(U_{0,} f_0) \in £$ such that $f_n(x) \to f_0(x)$, because the norm in Q is uniformly continuous. So we can prove easily that $\hat{f}_n \to \hat{f}_0 \in Q$. Therefore Q is a Banach Algebra over F.

Moreover, for $\hat{f}, \hat{g} \in Q$; $\hat{\beta} \in \hat{\Gamma}$, we have

$$\|\hat{f}\|\|\hat{\beta}\|\|\hat{g}\| = \|cl(I, f)\|\|cl(V\beta V, 1_{V\beta})\|\|cl(J, g)\|$$

$$= \|cl(I\Gamma V\beta V\Gamma J, f1_{V\beta} g)\|$$

$$= \sup\{\|f(u)\| : u \in I, \|u\| \le 1\}.\sup\{\|1_{V\beta}(x\beta y)\| : x\beta y \in V\beta V, \|x\beta y\| \le 1\}.\sup\{\|v\| : v \in J, \|v\| \le 1\}$$

$$= \sup\{\|f(u)\|\|1_{V\beta}(x\beta y)\|\|g(v)\| : u \in I, x\beta y \in V\beta V, v \in J; \|u\| \le 1, \|x\beta y\| \le 1, \|v\| \le 1\}$$

$$\ge \sup\{\|f(u)\|\|\gamma\|\|1_{V\beta}(x\beta y)\|\|\gamma'\|\|g(v)\| : u \in I, x\beta y \in V\beta V, v \in J; \|u\| \le 1, \|x\beta y\| \le 1, \|v\| \le 1, \|\gamma\| \le 1, \|\gamma'\| \le 1\}$$

$$= \sup\{\|f(u)\gamma 1_{V\beta}(x\beta y)\gamma' g(v)\| : u \in I, x\beta y \in V\beta V, v \in J; \|u\| \le 1, \|x\beta y\| \le 1, \|v\| \le 1, \|\gamma\| \le 1, \|\gamma'\| \le 1\}$$

$$= \sup\{\|f1_{V\beta} g(u\gamma x\beta y\gamma' v)\| : u \in I, x\beta y \in V\beta V, v \in J; \|u\gamma x\beta y\gamma' v\| \le 1\}$$

$$= \|cl(I\Gamma V\beta V\Gamma J, f1_{V\beta} g)\|$$

$$= \|cl(I, f)cl(V\beta V, 1_{V\beta})cl(J, g)\|$$

$$= \|\hat{f}\hat{\beta}\hat{g}\|$$

Thus Q is a $\hat{\Gamma}$-Banach algebra over F. Noticing that the mapping $\eta(\beta) = \hat{\beta}$ for every $0 \ne \beta \in \Gamma$ is an isomorphism. Therefore $\hat{\Gamma}$-Banach algebra Q is a Γ-Banach algebra.

The set $C_\Gamma = \{g \in Q \mid g\gamma f = f\gamma g, \forall f \in Q and \gamma \in \Gamma\}$, is called the extended centroid of Γ-Banach algebra V over F. If aγxβb=bγxβa, for all x∈V and β,γ∈Γ, where a(≠0), b∈V are fixed, then there exists $\lambda \in C_\Gamma$ such that b=λαa for α∈Γ.

III. The main results:

Theorem3.1. If d_1 and d_2 be bounded k- and h-derivations on Γ-Banach algebras $V(F_1)$ and $V'(F_2)$ respectively then there exists a bounded h⊗k-derivation d on the projective tensor product $V \otimes_p V'$ defined by the relation $d(u) = \sum_i [d_1(x_i) \otimes y_i + x_i \otimes d_2(y_i)]$, where $u = \sum_i x_i \otimes y_i \in V \otimes_p V$.

Proof: Since $d : V(F_1) \otimes_p V'(F_2) \to V(F_1) \otimes_p V'(F_2)$ is define as $d(u) = \sum_i [d_1(x_i) \otimes y_i + x_i \otimes d_2(y_i)]$, where $u = \sum_i x_i \otimes y_i \in V(F_1) \otimes_p V'(F_2)$. Clearly d is well defined. For any arbitrary element $u = \sum_i x_i \otimes y_i \in V(F_1) \otimes_p V'(F_2)$ and $\in$ >0 we have

$\|u\|_p + \in \ge \sum_{i=1}^{n} \|x_i\| \|y_i\|$, from the definition of projective norm. Now

$\|d(u)\|_p = \left\| \sum_i [d_1(x_i) \otimes y_i + x_i \otimes d_2(y_i)] \right\|_p$. Thus

$$\le \sum_i \left[\|d_1(x_i) \otimes y_i\| + \|x_i \otimes d_2(y_i)\| \right]$$

$= \sum_i \left[\|d_1(x_i)\| \, \|y_i\| + \|x_i\| \, \|d_2(y_i)\| \right]$

$\leq (\|d_1\| + \|d_2\|) \sum_i \|x_i\| \, \|y_i\|$

$\leq k(\|u\|_p + \in)$, where $k = \|d_1\| + \|d_2\|$

Thus $\|d(u)\|_p \leq k(\|u\|_p + \in)$. Since the left hand side is independent of $\in$ and $\in$ was arbitrary, it follows that $\|d(u)\|_p \leq k\|u\|_p$, for every $u \in V(F_1) \otimes_p V'(F_2)$. Consequently D is bounded.

To show that d is a k⊗h-derivation, we suppose that u=x⊗y and $v = x' \otimes y'$ are two elements of $V(F_1) \otimes_p V'(F_2)$. Then

$d((x \otimes y)(\alpha \otimes \beta)(x' \otimes y'))$, where $\alpha \otimes \beta \in \Gamma \otimes_p \Gamma'$

$= d(x\alpha x' \otimes y\beta y')$

$= d_1(x\alpha x') \otimes y\beta y' + x\alpha x' \otimes d_2(y\beta y')$

$= [d_1(x)\alpha x' + xk(\alpha)x' + x\alpha d_1(x')] \otimes y\beta y' + x\alpha x' \otimes [d_2(y)\beta y' + yh(\beta)y' + y\beta d_2(y')]$

$= d(x \otimes y)(\alpha \otimes \beta)(x' \otimes y') + (x \otimes y)(k \otimes h)(\alpha \otimes \beta)(x' \otimes y') + (x \otimes y)(\alpha \otimes \beta)d(x' \otimes y')$,

where $(k \otimes h)(\alpha \otimes \beta) = k(\alpha) \otimes \beta + \alpha \otimes h(\beta)$

Similarly, if $u = \sum_i x_i \otimes y_i$ and $v = \sum_j x'_j \otimes y'_j$ be two element of $V(F_1) \otimes_p V'(F_2)$ then summing over i and j we can prove easily that

$d(u(\alpha \otimes \beta)v) = d(u)(\alpha \otimes \beta)v + u[(k \otimes h)(\alpha \otimes \beta)]v + u(\alpha \otimes \beta)d(v)$

So, d is a k⊗h-derivation.

Theorem3.2. Let $V(F)$ and $V'(F')$ be two Γ-Banach algebra and Γ'-Banach algebra respectively with eδx = xδe=x, ∀x∈V, e∈V, δ∈Γ and $e'\delta'y = y\delta'e' = y$, $\forall y \in V', e' \in V', \delta' \in \Gamma'$. If d_1 and d_2 are k-and h-inner derivation on $V(F)$ and $V'(F')$ respectively implemented by (a,δ) and (b, δ') respectively then d is a k⊗h- inner deviation on $V \otimes_p V'$ implemented by $(a \otimes e' + e \otimes b, \delta \otimes \delta')$.

Proof: Let d_1 and d_2 be k- and h-inner derivations on $V(F)$ and $V'(F')$ respectively implemented by (a,δ) and (b, δ') i.e.

$d_1(x) = [a, x]_\delta$, where $k(\alpha) = [\delta, \alpha]_a$

$d_2(y) = [b, y]_{\delta'}$, where $h(\beta) = [\delta', \beta]_b$

Now $d(u) = d\left(\sum_i x_i \otimes y_i\right)$

$= \sum_i [d_1(x_i) \otimes y_i + x_i \otimes d_2(y_i)]$

$= \sum_i [\, [a, x_i]_\delta \otimes y_i + x_i \otimes [b, y_i]_{\delta'} \,]$

$= \sum_i [(a\delta x_i - x_i\delta a) \otimes y_i + x_i \otimes (b\delta' y_i - y_i\delta' b)]$

$= \sum_i [a\delta x_i \otimes y_i - x_i\delta a \otimes y_i + x_i \otimes b\delta' y_i - x_i \otimes y_i\delta' b]$

$$= \sum_i \left[a\delta x_i \otimes e'\delta' y_i - x_i\delta a \otimes y\delta' e' + e\delta x_i \otimes b\delta' y_i - x_i\delta e \otimes y_i\delta' b \right]$$

$$= \sum_i \left[(a \otimes e')(\delta \otimes \delta')(x_i \otimes y_i) - (x_i \otimes y_i)(\delta\otimes\delta')(a\otimes e') + (e\otimes b)(\delta\otimes\delta') \right.$$

$$\left. (x_i \otimes y_i) - (x_i \otimes y_i)(\delta\otimes\delta')(e\otimes b) \right]$$

$$= \sum_i \left[(a \otimes e' + e \otimes b)(\delta \otimes \delta')(x_i \otimes y_i) - (x_i \otimes y_i)(\delta\otimes\delta')(a\otimes e' + e\otimes b) \right]$$

$$= \sum_i \left[a \otimes e' + e \otimes b, x_i \otimes y_i \right]_{\delta\otimes\delta'}$$

$$= \sum_i \left[a \otimes e' + e \otimes b, u \right]_{\delta\otimes\delta'}$$

Now, we define $k\otimes h : \Gamma\otimes\Gamma \to \Gamma\otimes\Gamma$ by

$$(k\otimes h)(\alpha\otimes\beta) = \left[\delta\otimes\delta', \alpha\otimes\beta\right]_{a\otimes e' + e\otimes b}$$

It can be easily prove that

$$d\left((x\otimes y)(\alpha\otimes\beta)(x'\otimes y')\right) = d(x\otimes y)(\alpha\otimes\beta)(x'\otimes y') + (x\otimes y)(k\otimes h)(\alpha\otimes\beta)$$

$$(x'\otimes y') + (x\otimes y)(\alpha\otimes\beta)d(x'\otimes y')$$

Therefore d is a $k\otimes h$-inner derivation on $V \otimes_p V'$ implemented by $\left(a\otimes e' + e\otimes b, \delta\otimes\delta'\right)$

Theorem3.3. If d_1 and d_2 are k-and h-Jordan derivations, then d is an $h\otimes k$-Jordan derivation.

Proof: Obvious.

Remarks:

(i) The converse of the above three theorems are also true.

(ii) If $u = a \otimes e' \in V(F) \otimes_p V'(F')$, then from the definition of d in theorem 3.2 , we get

$d(u) = d_1(a)\otimes e'$, because $d_2(e') = 0$

(iii) If $u' = e \otimes b \in V(F) \otimes_p V'(F')$, then from the definition of d in theorem 3.2, we get

$d(u) = e\otimes d_2(b)$, because $d_1(e) = 0$

Theorem3.4. If d_1, d_2 and d are k- ,h- and $k\otimes h$-derivations respectively related as in theorem 3.1,3.2 and 3.3, then

$$\|d\| \le \|d_1\| + \|d_2\| \le 2\|d\|$$

Proof: we already proof in theorem 3.1 is that

$$\|d\| \le \left(\|d_1\| + \|d_2\| \right)(1 + \in)$$

Since $\in$ was arbitrary, it follows that

$$\|d\| \le \|d_1\| + \|d_2\| \qquad (1)$$

Next let $x \in V$ such that $\|x\| = 1$ .Then

$$\left\| \frac{x}{k} \otimes e' \right\| = \left\| \frac{x}{k} \right\| \left\| e' \right\| = 1 \text{ where } \left\| e' \right\| = k \ne 0$$

Now $\|d\| = \sup_u \left\{ \|d(u)\|_p : \|u\|_p = 1 \right\}$

$$\ge \left\| d\left(\frac{x}{k} \otimes e' \right) \right\|_p$$

$$= \left\| d_1\left(\frac{x}{k} \right) \otimes e' \right\|_p, \text{ since } d_2(e') = 0$$

$=\|d_1(x)\|$

Thus $\| d \| \geq \| d_1(x) \|$, for every $x \in V(F)$ with $\| x \| = 1$ and which implies $\| d \| \geq \| d_1 \|$

Similarly, we can prove that $\| d \| \geq \| d_2 \|$

$\therefore \|d_1\| + \|d_2\| \leq 2\| d \|$ (2)

The inequalities (1) and (2) together implies that

$\|d\| \leq \|d_1\| + \|d_2\| \leq 2\| d \|$.

Example1. Let V be the set of all 2×2 matrices of the type $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ where a, b are complex numbers and $\bar{a}$, $\bar{b}$ are their conjugates respectively and Γ be the set of all 2×2 matrices of the type $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$, where x is a real number . Then V be a Γ-Banach algebra over F=R with respect to usual matrix addition and multiplication and the norm is defined by

$\left\| \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \right\| = \max\{ |a|, |b| \}$ and $\left\| \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \right\| = |x|$.

Let $V^/$ be the set of all 2×2 matrix of the type $\begin{pmatrix} x & y \\ u & v \end{pmatrix}$, where x, y , u, v are real numbers and $\Gamma^/ = \Gamma$ and norm is defined as the norm is defined for V and Γ.Then $V^/(F^/)$ is a $\Gamma^/$ - Banach algebra, where $F' = R$.

Let d_1 and d_2 are k- and h-derivations implemented by (A, δ) and (B, δ') respectively, where

$A = \begin{pmatrix} 3i & 2i \\ 2i & -3i \end{pmatrix} \in V$, $B = \begin{pmatrix} 5 & 3 \\ 2 & 1i \end{pmatrix} \in V^/$

$\delta = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma$, $\delta^/ = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \in \Gamma^/$

Since $B_1 = \left\{ e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, e_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\}$ is a basis for V(F).

Now $d_1(e_1) = [A, e_1]_\delta$

$= A\delta e_1 - e_1\delta A$

$= \begin{pmatrix} 3i & 2i \\ 2i & -3i \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3i & 2i \\ 2i & -3i \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Similarly, $d_1(e_2) = \begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix}, d_1(e_3) = \begin{pmatrix} -4i & 6i \\ 6i & 4i \end{pmatrix}, d_1(e_4) = \begin{pmatrix} 0 & 6 \\ -6 & 0 \end{pmatrix}$.

Hence the matrix representation of d_1 with respect to B_1 is

$$[d_1]_{B_1} = \begin{pmatrix} 0 & 0 & -4i & 0 \\ 0 & -4 & 6i & 6 \\ 0 & 4 & 6i & -6 \\ 0 & 0 & 4i & 0 \end{pmatrix}$$

Therefore $\|\mathrm{d}_1\| = 6$

Also $B_2 = \left\{ e_1' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_3' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_4' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a basis for V'. Similarly, with respect to this basis, we find $[d_2]_{B_2} = \begin{pmatrix} 0 & -6 & 4 & 0 \\ -4 & 8 & 0 & 4 \\ 6 & 0 & -8 & -6 \\ 0 & 6 & -4 & 0 \end{pmatrix}$ and $\|d_2\| = 8$.

Next we wish to find the matrix representation of k⊗h-derivation d. Clearly $B_3 = \{e_i \otimes e_j : i = 1,2,3,4;\ j = 1,2,3,4\}$ is a basis for $V(F) \otimes_p V'(F')$.

$$d\left(e_1 \otimes e_1'\right) = d_1(e_1) \otimes e_1' + e_1 \otimes d_2\left(e_1'\right)$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -6 \\ 4 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -6 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 \\ 0 & 4 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -6 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 \\ 0 & -4 & 0 & 0 \end{pmatrix}$$

Similarly,

$$d\left(e_4 \otimes e_2'\right) = \begin{pmatrix} 0 & -4i & 0 & 6+8i \\ 0 & 0 & 0 & 4i \\ -4i & 0 & -6+8i & 0 \\ 0 & 0 & 4i & 0 \end{pmatrix}, d\left(e_4 \otimes e_3'\right) = \begin{pmatrix} 0 & 6i & 0 & 0 \\ 0 & 6-8i & 0 & -6i \\ 6i & 0 & 0 & 0 \\ -6-8i & 0 & -6i & 0 \end{pmatrix}, d\left(e_4 \otimes e_4'\right) = \begin{pmatrix} 0 & 0 & 0 & 6i \\ 0 & -4i & 0 & 6 \\ 0 & 0 & 6i & 0 \\ -4i & 0 & -6 & 0 \end{pmatrix}$$

$$d\left(e_1 \otimes e_2'\right) = \begin{pmatrix} -4 & 0 & 8 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & -4 & 0 & 8 \\ 0 & 0 & 0 & 4 \end{pmatrix}, d\left(e_1 \otimes e_3'\right) = \begin{pmatrix} 6 & 0 & 0 & 0 \\ -8 & 0 & -6 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & -8 & 0 & -6 \end{pmatrix}, d\left(e_1 \otimes e_4'\right) = \begin{pmatrix} 0 & 0 & 6 & 0 \\ -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & -4 & 0 & 0 \end{pmatrix}$$

$$d\left(e_2 \otimes e_1'\right) = \begin{pmatrix} 0 & -4 & -6i & 0 \\ 4i & 0 & 0 & 0 \\ 4 & 0 & 0 & 6i \\ 0 & -4i & 0 & 0 \end{pmatrix}, d\left(e_2 \otimes e_2'\right) = \begin{pmatrix} -4i & 0 & 8i & -4 \\ 0 & 0 & 4i & 0 \\ 0 & 4i & 4 & 0 \\ 0 & 0 & 0 & -4i \end{pmatrix}, d\left(e_2 \otimes e_3'\right) = \begin{pmatrix} 6i & 0 & 0 & 0 \\ -8i & -4 & -6i & 0 \\ 0 & -6i & 0 & 0 \\ 4 & 8i & 0 & 6i \end{pmatrix}$$

$$d\left(e_2 \otimes e_4'\right) = \begin{pmatrix} 0 & 0 & 6i & 0 \\ -4i & 0 & 0 & 0 \\ 0 & 0 & 0 & -6i \\ 0 & 4i & 4 & 0 \end{pmatrix}, d\left(e_3 \otimes e_1'\right) = \begin{pmatrix} -4i & 6i & 0 & 6 \\ 0 & -4 & 0 & 0 \\ 6i & 4i & -6 & 0 \\ 4 & 0 & 4 & 4 \end{pmatrix}, d\left(e_3 \otimes e_2'\right) = \begin{pmatrix} 0 & 4 & -4i & 6i-8 \\ 0 & 0 & 0 & -4 \\ -4 & 0 & 8+6i & 4i \\ 0 & 0 & 4 & 0 \end{pmatrix}$$

$$d\left(e_3 \otimes e_3'\right) = \begin{pmatrix} 0 & -6 & 0 & 0 \\ -4i & 8+6i & 0 & -6 \\ 6 & 0 & 0 & 0 \\ -8+6i & 4i & -6 & 0 \end{pmatrix}, d\left(e_3 \otimes e_4'\right) = \begin{pmatrix} 0 & 0 & 0 & -6 \\ 0 & 4 & -4i & 6i \\ 0 & 0 & 6 & 0 \\ -4 & 0 & 6i & 4i \end{pmatrix}, d\left(e_4 \otimes e_1'\right) = \begin{pmatrix} 0 & 6 & 0 & -6i \\ 0 & 4i & 0 & 0 \\ -6 & 0 & -6i & 0 \\ 4i & 0 & 0 & 0 \end{pmatrix}$$

The matrix representation of d with respect to the basis B_3 is

$$[d]_{B_3} = \begin{pmatrix}
0 & -4 & 6 & 0 & 0 & -4i & 6i & 0 & -4i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & -8 & -4 & -4i & 0 & -8i & -4i & 0 & 0 & -4i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 6i & -4 & 6 & 0 & -6 & -4i & 6i & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 4 & 0 & -8+6i & -4 & 4i & 0 & -6-8i & -4i \\
0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 6i & 4 & -6 & 0 & 6 & -4i & 6i & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & -4 & 0 & 8+6i & 4 & 4i & 0 & 6-8i & -4i \\
0 & -4 & 6 & 0 & 0 & 4i & -6i & 0 & 4i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4 & 0 & -8 & -4 & -4i & 0 & 8i & 4i & 0 & 0 & 4i & 0 & 0 & 0 & 0 & 0 \\
-6 & 8 & 0 & 6 & -6i & 8i & 0 & 6i & 0 & -4i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & -6 & 0 & 0 & 4i & -6i & 0 & 0 & 0 & 0 & -4i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & -6 & 8+6i & 0 & 6 & -6i & -6+8i & 0 & 6i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 4 & -6 & 6i & 0 & 4i & -6i & -6 \\
0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 6 & -8+6i & 0 & -6 & -6i & 6+8i & 0 & 6i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & -6 & 6i & 0 & 4i & -6i & 6 \\
-6 & 4 & 0 & 6 & 6i & -8i & 0 & -6i & 0 & 4i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 8 & -6 & 0 & 0 & -4i & 6i & 0 & 0 & 0 & 0 & 4i & 0 & 0 & 0 & 0
\end{pmatrix}_{16\times 16}$$

Therefore $\|d\| = 10$. Thus the strict inequality $\|d\| < \|d_1\| + \|d_2\| < 2\|d\|$ holds.

Lemma3.5 Let V be a 2-torsion free prime Γ-Banach algebra, D(.,.) the symmetric bi-k-derivation of V and d the trace of D(.,.) . If aγd(x) =0 for all x∈V and γ∈Γ, where a is a fixed element of V, then a=0 or D=0.

Lemma3.6 Let V be a 2-torsion free prime Γ-Banach algebra, $D_1(.,.)$ and $D_2(.,.)$ the symmetric bi-k-derivations on V and d_1 and d_2 the traces of $D_1(.,.)$ and $D_2(.,.)$ respectively. If $d_1(x)\gamma\, d_2(y) = d_2(x)\gamma\, d_1(y)$ for all x, y∈V and γ∈Γ and $d_1 \neq 0$, then there exists $\lambda \in C_\Gamma$ such that $d_2(x)=\lambda\alpha d_1(x)$ for α∈Γ, where C_Γ is the extended centroid of V.

Theorem3.7. Let V be a 2-torsion free prime Γ-Banach algebra, $D_1(.,.)$, $D_2(.,.)$ and $D_3(.,.)$ and $D_4(.,.)$ the symmetric bi-k-derivations on V and d_1, d_2, d_3 and d_4 traces of $D_1(.,.)$, $D_2(.,.)$, $D_3(.,.)$ and $D_4(.,.)$ respectively. If

$$d_1(x)\gamma\, d_2(y) = d_3(x)\gamma\, d_4(y) \qquad (3)$$

for all x, y∈V and γ∈Γ and $d_1 \neq 0 \neq d_4$, then there exists $\lambda \in C_\Gamma$ such that $d_2(x)=\lambda\alpha d_1(x)$ for all α∈Γ, where C_Γ is the extended centroid of V.

Proof: Let z∈V. Replacing y by y+z in (3), we get

$$d_1(x)\gamma\, D_2(y, z) = d_3(x)\gamma D_4(y, z) \qquad (4)$$

If we replace z by zδy In (4) , then

$d_1(x)\gamma z k(\delta) d_2(y) = d_3(x)\gamma z k(\delta) d_4(y)$ (5)

If $w \in V$ then

$d_1(x)\gamma z k(\alpha) d_2(w) = d_3(x)\gamma z k(\alpha) d_4(w)$ (6)

Substituting $zk(\alpha) d_4(w)$ for z in (5), we have

$d_1(x)\gamma z k(\alpha) d_4(w)k(\delta) d_2(y) = d_3(x)\gamma z k(\alpha) d_4(w)k(\delta) d_4(y)$ (7)

From (4) and (6), we have

$d_1(x)\gamma z k(\alpha) d_4(w)k(\delta) d_2(y) = d_1(x)\gamma z k(\alpha) d_2(w)k(\delta) d_4(y)$

$\Rightarrow d_1(x)\gamma zk(\alpha)(d_4(w)k(\delta) d_2(y) - d_2(w)k(\delta) d_4(y)) = 0$ (8)

Since $d_1 \neq 0$ and V is a prime Γ-Banach algebra. So equation (8) implies that

$d_4(w)k(\delta) d_2(y) = d_2(w)k(\delta) d_4(y)$

It follows from $d_4 \neq 0$ and lemma 3.5 that $d_2(y) = \lambda\alpha d_4(y)$ for some $\lambda \in C_\Gamma$. Hence by (4), we have $d_1(x)\gamma z k(\delta)\lambda\alpha d_4(y) = d_3(x)\gamma z k(\delta) d_4(y)$

Since $\lambda \in C_\Gamma$. Therefore

$d_1(x)\gamma k(\delta) d_4(y)\alpha\lambda = d_3(x)\gamma z k(\delta) d_4(y)$

$\Rightarrow \lambda\alpha d_1(x)\gamma k(\delta) d_4(y) = d_3(x)\gamma z k(\delta) d_4(y)$

$\Rightarrow [\lambda\alpha d_1(x) - d_3(x)]\gamma z k(\delta) d_4(y) = 0$

It follows from $d_4 \neq 0$ that $d_3(x) = \lambda\alpha d_1(x)$.

Lemma3.8. Let V be a 2-torsion free prime Gamma Banach algebra and let U be a nonzero ideal of V. Let a, $b \in V$ be fixed elements. If $a\gamma x\beta b + b\gamma x\beta a = 0$ is fulfilled for all $x \in U, \alpha, \beta \in \Gamma$ then either a=0 or b=0.

Theorem3.9. Let V be a 2-torsion free prime Gamma Banach algebra and U be a non zero ideal of V. Suppose there exist symmetric bi-k-derivations $D_1 : V \times V \to V$ and $D_2 : V \times V \to V$ such that

$$D_1(d_2(x), x) = 0 \quad (9)$$

holds for all $x \in U$ where d_2 denotes the trace of D_2. In this case $D_1 = 0$ or $D_2 = 0$.

Proof: let $y \in U$. Replacing x by x+y in (9) we get

$$D_1(d_2(x), y) + 2D_1(D_2(x, y), x) + D_1(d_2(y), x) + 2D_1(D_2(x, y), y) = 0 \quad (10)$$

substituting x by -x in (10) we get,

$$D_1(d_2(x), y) + 2D_1(D_2(x, y), x) = D_1(d_2(y), x) + 2D_1(D_2(x, y), y) \quad (11)$$

Comparing (10) and (11) we have

$$D_1(d_2(x), y) + 2D_1(D_2(x, y), x) = 0 \quad (12)$$

Let us replace in (12) y by $x\alpha y, \alpha \in \Gamma$ and use (10) and (12) we get,

$$d_2(x)k(\alpha)D_1(y, x) + d_1(x)k(\alpha)D_2(x, y) = 0 \quad (13)$$

Let us write in (13) $y\beta x$ instead of y we have

$$d_2(x)k(\alpha)vk(\beta)d_1(x) + d_1(x)k(\alpha)vk(\beta)d_2(x) = 0 \quad (14)$$

Let $d_1 \neq 0, d_2 \neq 0$. Then there exist elements $x_1, x_2 \in U$ such that $d_1(x_1) \neq 0$ and $d_2(x_2) \neq 0$. From (14) and lemma3.8 it follows that $d_1(x_2) = d_2(x_1) = 0$. Since $d_1(x_2) = 0$ the relation (13) reduces to $d_2(x_2)k(\alpha)D_1(y, x_2) = 0$. Using this relation and lemma3.5, we obtain that $D_1(y, x_2) = 0$ holds for all $y \in U$ since $d_2(x_2) \neq 0$. In particular, we have $D_1(x_1, x_2) = 0$ and so

$$d_1(x_1 + x_2) = d_1(x_1) + d_1(x_2) + 2D_1(x_1, x_2)$$
$$= d_1(x_1) \neq 0$$

Similarly we obtain $d_2(y) \neq 0$. But $d_1(y)$ and $d_2(y)$ cannot be both different from zero according to (14) and lemma 3.8. Therefore we have either $d_1 = 0$ or $d_2 = 0$.

Corollary3.10. Let V be a 2-torsion free semi-prime gamma banach algebra and U be non zero ideal of V. Suppose there exists a symmetric bi-k-derivation $D:V\times V\to V$ such that D(d(x),x)=0 holds for all $x\in U$, where d denotes the trace of D. In this case we have d=0.

Theorem3.11. Let V be a 2- and 3-torsion free prime Γ-Banach algebra. Let U be a non zero ideal of V and $D_1: V\times V\to V$ and $D_2: V\times V\to V$ be symmetric bi-k- derivations. Suppose further that there exists a symmetric bi-additive mapping $B:V\times V\to V$ such that $d_1(d_2(x))=f(x)$ holds, for all $x\in U$, where d_1 and d_2 are the traces of D_1 and D_2 respectively and f is the trace of B. Then either D_1=0 and D_2=0.

Proof: Putting x + y in place of x in $d_1(d_2(x))=f(x)$, we get

$$2d_1(D_2(x,y))+D_1((d_2(x),d_2(y))+2D_1(d_2(x),D_2(x,y))$$
$$+2D_1(d_2(y),D_2(x,y))=B(x,y) \quad (15)$$

Let us replace in (15) x by –x. We have

$$2d_1(D_2(x,y))+D_1((d_2(x),d_2(y))-2D_1(d_2(x),D_2(x,y))$$
$$-2D_1(d_2(y),D_2(x,y))=-B(x,y) \quad (16)$$

Adding (15) and (16) we get,

$$2d_1(D_2(x,y))+D_1((d_2(x),d_2(y))=0$$

Then equation (15) reduces

$$2D_1(d_2(x),D_2(x,y))+2D_1(d_2(y),D_2(x,y))=B(x,y) \quad (17)$$

Let us write in (17) 2x instead of x. we get

$$8D_1(d_2(x),D_2(x,y))+2D_1(d_2(y),D_2(x,y))=B(x,y) \quad (18)$$

Now substitute (17) from (18) we get

$$6D_1(d_2(x),D_2(x,y))=0$$
$$\Rightarrow D_1(d_2(x),D_2(x,y))=0 \quad (19)$$

Since V is a 2- and 3-torsion free Gamma Banach algebra. It follows that both terms on the left side of relation (17) are zero, which means that B=0.Then

$d_1(d_2(x))=0$, $x\in U$.

Substituting $y\alpha x, \alpha\in\Gamma$ for y in (19) we get

$$D_1(d_2(x),y)k(\alpha)d_2(x)+D_2(x,y)k(\alpha)D_1(d_2(x),x)=0 \quad (20)$$

Let us write $x\beta y, \beta\in\Gamma$ instead of y we have

$$D_1(d_2(x),x)k(\beta)yk(\alpha)d_2(x)+d_2(x)k(\beta)yk(\alpha)D_1(d_2(x),x)=0$$

From the relation above one can conclude that

$D_1(d_2(x),x)=0$ or $d_2(x)=0$

If $D_1(d_2(x),x)\neq 0$ for some $x\in U$, then $d_2(x)=0$

Contrary to the assumption $D_1(d_2(x),x)\neq 0$

There $D_1(d_2(x),x)=0$, for all $x\in U$, the proof of the theorem is complete since all the requirements of theorem 3.10 are fulfilled.

Corollary3.12 Let V be a semi-prime gamma ring which is 2-and 3-torsion free. Let U be a nonzero ideal of V and $d(U)\subset U$. Let $D:V\times V\to V$ be a symmetric bi-k-derivation and $B:V\times V\to V$ be a symmetric bi-additive mapping. Suppose that d(d(x))=f(x) holds for all $x\in U$, where d is the trace of D and f is the trace of B. In this case we have D=0.

Example2. Let $V=\left\{\begin{pmatrix} x & 0 \\ 0 & 0\end{pmatrix} : x\in R\right\}$ and $\Gamma=\left\{\begin{pmatrix} n & 0 \\ 0 & n\end{pmatrix} : x\in \mathbb{R}\right\}$

Then V(F=R) is a Γ-Banach algebra.

$D:V\times V\rightarrow V$ define as $D(A,B)=AB$, for all $A,B\in V$ and $k:\Gamma\rightarrow\Gamma$ define as $k(\alpha)=\frac{1}{n}\alpha$, where

$\alpha=\left\{\begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}\in\Gamma\right\}$.Then it is easy to show that D is a symmetric bi-k-derivation.

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