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An Integral Concerning H-Functions

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Abstract: *The aim of the present paper is to establish Mellin transform of the product concerning Fox's H-function and the multivariable H-function. The result established here are quit general in nature and a large number of known and new integrals can be obtained by specializing the parameters suitably of the various functions involved in them. The present integral generalizes most of the infinite integrals derived earlier by various authors.*

I. Introduction

The H-function introduced by Fox [3, p.408] will be represented and defined as follows:

$$H[z] = H_{r,e}^{k,f}\left[z \,\middle|\, \begin{matrix} (A_j, B_j)_{1,r} \\ (L_j, M_j)_{1,e} \end{matrix} \right] = \frac{1}{2\pi\omega}\int_L \phi(\xi)\, z^{\xi}\, d\xi\,, \tag{1.1}$$

where $\omega = \sqrt{-1}$,

$$\phi(\xi) = \frac{\prod_{j=1}^{k}\Gamma(L_j - M_j\xi)\prod_{j=1}^{f}\Gamma(1 - A_j + B_j\xi)}{\prod_{j=k+1}^{e}\Gamma(1 - L_j + M_j\xi)\prod_{j=f+1}^{r}\Gamma(A_j - B_j\xi)}\,, \tag{1.2}$$

where an empty product is to be interpreted as unity; $0 \le k \le e;\ 0 \le f \le r;\ B_j(j = 1,\dots,r)$ *and* $M_j(j = 1,\dots,e)$ are positive numbers. L is a suitable contour of Barnes type such that the poles of $\Gamma(L_h - M_h\xi)(h = 1,\dots,k)$ lie to the right of it and those of $\Gamma(1 - A_i + B_i\xi)(i = 1,\dots,f)$ lie to the left of it.

Braaksma [2] has obtained the conditions of convergence of the integral in (1.1) and the asymptotic expansion of the H-function.

In what follows for the sake of brevity,

$$T \equiv \sum_{i=1}^{f} B_i - \sum_{i=f+1}^{r} B_i + \sum_{i=1}^{k} M_i - \sum_{i=k+1}^{e} M_i\,. \tag{1.3}$$

The H-function of several complex variables is defined and represented in the following form [9, p.251, Eq.(C.1)]:

$$H[z_1,\dots,z_t] = H^{0,N:M_1,N_1;\dots;M_t,N_t}_{P,Q:P_1,Q_1;\dots;P_t,Q_t}\left[z_1,\dots,z_t \,\middle|\, \begin{matrix} (a_j:\alpha_j^{(1)},\dots,\alpha_j^{(t)})_{1,P} : (c_j^{(1)},\gamma_j^{(1)})_{1,P_1};\dots;(c_j^{(t)},\gamma_j^{(t)})_{1,P_t} \\ (b_j:\beta_j^{(1)},\dots,\beta_j^{(t)})_{1,Q} : (d_j^{(1)},\delta_j^{(1)})_{1,Q_1};\dots;(d_j^{(t)},\delta_j^{(t)})_{1,Q_t} \end{matrix} \right]$$

$$= \frac{1}{(2\pi\omega)^t}\int_{L_1}\cdots\int_{L_t} \psi(\xi_1,\dots,\xi_t)\,\phi_1(\xi_1)\dots\phi_t(\xi_t)\, z_1^{\xi_1}\dots z_t^{\xi_t}\, d\xi_1\dots d\xi_t\,, \tag{1.4}$$

where $\omega = \sqrt{-1}$,

$$\phi_i(\xi_i) = \frac{\prod_{j=1}^{M_i}\Gamma(d_j^{(i)} - \delta_j^{(i)}\xi_i)\prod_{j=1}^{N_i}\Gamma(1 - c_j^{(i)} + \gamma_j^{(i)}\xi_i)}{\prod_{j=M_i+1}^{Q_i}\Gamma(1 - d_j^{(i)} + \delta_j^{(i)}\xi_i)\prod_{j=N_i+1}^{P_i}\Gamma(c_j^{(i)} - \gamma_j^{(i)}\xi_i)} \quad (i = 1,\dots,t) \tag{1.5}$$

and

$$\psi(\xi_1,\dots,\xi_t) = \frac{\prod_{j=1}^{N}\Gamma\left(1 - a_j + \sum_{i=1}^{t}\alpha_j^{(i)}\xi_i\right)}{\prod_{j=N+1}^{P}\Gamma\left(a_j - \sum_{i=1}^{t}\alpha_j^{(i)}\xi_i\right)\prod_{j=1}^{Q}\Gamma\left(1 - b_j + \sum_{i=1}^{t}\beta_j^{(i)}\xi_i\right)}\,, \tag{1.6}$$

For the sake of brevity,

$$T_i \equiv -\sum_{j=N+1}^{P} \alpha_j^{(i)} + \sum_{j=1}^{N_i} \gamma_j^{(i)} - \sum_{j=N_i+1}^{P_i} \gamma_j^{(i)} - \sum_{j=1}^{Q} \beta_j^{(i)} + \sum_{j=1}^{M_i} \delta_j^{(i)} - \sum_{j=M_i+1}^{Q_i} \delta_j^{(i)} > 0, (\forall \quad i = 1,\ldots,t) \tag{1.7}$$

All the Greek letters occurring on the left hand side of (1.4) are assumed to be positive real numbers for standardization purposes; the definition of the multivariable H-function will, however, be meaningful even if some of these quantities are zero. For the convergence and existence conditions of the multivariable H-function we refer to the book by Srivastava et al. [9, pp.252-253, Eqs.(C.4)-(C.8)]. Throughout the paper it is assumed that this function satisfies the above-cited conditions.

The series representation of Fox's H-function is defined as follows [8]:

$$H_{p,q}^{m,n}\left[x\left|\begin{matrix}(e_p,E_p)\\(f_q,F_q)\end{matrix}\right.\right] = \sum_{g=1}^{m}\sum_{G=0}^{\infty}\frac{(-1)^G\,\phi(\eta_G)\,x^{\eta_G}}{G!\;F_g}, \tag{1.8}$$

where

$$\phi(\eta_G) = \frac{\prod_{j=1,\,j\neq g}^{m}\Gamma(f_j - F_j\eta_G)\prod_{j=1}^{n}\Gamma(1-e_j+E_j\eta_G)}{\prod_{j=m+1}^{q}\Gamma(1-f_j+F_j\eta_G)\prod_{j=n+1}^{p}\Gamma(e_j-E_j\eta_G)} \tag{1.9}$$

and $$\eta_G = \frac{(f_g+G)}{F_g}, \tag{1.10}$$

Also

$$\mathrm{T}' = \sum_{i=1}^{n}\mathrm{E_i} - \sum_{i=n+1}^{p}\mathrm{E_i} + \sum_{i=1}^{m}\mathrm{F_i} - \sum_{i=m+1}^{q}\mathrm{F_i} > 0\,. \tag{1.11}$$

II. MAIN INTEGRAL

$$\int_0^\infty x^{\alpha-1}\; H_{p,q}^{m,n}\left[yx^\sigma\left|\begin{matrix}(e_j,E_j)_{1,p}\\(f_j,F_j)_{1,q}\end{matrix}\right.\right] H_{r,e}^{k,f}\left[zx^\rho\left|\begin{matrix}(A_j,B_j)_{1,r}\\(L_j,M_j)_{1,e}\end{matrix}\right.\right] H\left[s_1x^\gamma,\ldots,s_tx^\gamma\right]dx$$

$$=\sum_{g=1}^{m}\sum_{G=0}^{\infty}\frac{(-1)^G\phi(\eta_G)}{G!\;F_g\;\rho}\,y^{\eta_G}\,z^{-\frac{(\alpha+\sigma\eta_G)}{\rho}}$$

$$\times H_{P+e,Q+r:P_1,Q_1;\ldots;P_t,Q_t}^{f,N+k:M_1,N_1;\ldots;M_t,N_t}\left[s_1z^{-\frac{\gamma}{\rho}},\ldots,s_tz^{-\frac{\gamma}{\rho}}\left|\begin{matrix}(a_j:\alpha_j^{(1)},\ldots,\alpha_j^{(t)})_{1,N},\left(1-L_j-(\alpha+\sigma\eta_G)\frac{M_j}{\rho}:\right.\\ \left(1-A_j-(\alpha+\sigma\eta_G)\frac{B_j}{\rho}:\frac{\gamma}{\rho}B_j,\ldots,\frac{\gamma}{\rho}B_j\right)_{1,r},\end{matrix}\right.\right.$$

$$\left.\left.\begin{matrix}\left.\frac{\gamma}{\rho}M_j,\ldots,\frac{\gamma}{\rho}M_j\right)_{1,e},(a_j:\alpha_j^{(1)},\ldots,\alpha_j^{(t)})_{N+1,P} & :(c_j^{(1)},\gamma_j^{(1)})_{1,P_1};\ldots;(c_j^{(t)},\gamma_j^{(t)})_{1,P_t}\\ (b_j:\beta_j^{(1)},\ldots,\beta_j^{(t)})_{1,Q} & :(d_j^{(1)},\delta_j^{(1)})_{1,Q_1};\ldots;(d_j^{(t)},\delta_j^{(t)})_{1,Q_t}\end{matrix}\right.\right], \tag{2.1}$$

provided that

$$\mathrm{Re}(\alpha) > 0; \quad (\sigma,\rho,\gamma) > 0,$$

$$\mathrm{Re}\left[\alpha+\sigma\left(\frac{f_j}{F_j}\right)+\rho\left(\frac{L_{j'}}{M_{j'}}\right)+\sum_{i=1}^{t}\gamma\left(\frac{d_{j''}^{(i)}}{\delta_{j''}^{(i)}}\right)\right] > 0,$$

$$\mathrm{Re}\left[\alpha+\sigma\left(\frac{e_l-1}{E_l}\right)+\rho\left(\frac{A_{l'}-1}{B_{l'}}\right)+\sum_{i=1}^{t}\gamma\left(\frac{c_{l''}^{(i)}-1}{\gamma_{l''}^{(i)}}\right)\right] < 0,$$

$$|\arg y| < \frac{1}{2}T'\pi, \quad |\arg z| < \frac{1}{2}T\pi, \quad |\arg s_i| < \frac{1}{2}T_i\pi \quad (i=1,\ldots,t),$$

where $j = 1,\ldots,m;\; j' = 1,\ldots,k;\; j'' = 1,\ldots,M_i\; (i=1,\ldots,t);$
$l = 1,\ldots,n;\; l' = 1,\ldots,f;\; l'' = 1,\ldots,N_i\; (i=1,\ldots,t).$

III. PROOF

To establish the integral (2.1), we first express the series representation of Fox's H-function and the multivariable H-function occurring in the left-hand side of (2.1) with the help of equations (1.8) and (1.4) respectively and then interchange the order of summation and integrations (which is permissible under the conditions stated with (2.1)), we find that left-hand side of (2.1)

$$=\sum_{g=1}^{m}\sum_{G=0}^{\infty}\frac{(-1)^G\phi(\eta_G)\,y^{\eta_G}}{G!\;F_g}\frac{1}{(2\pi i)^t}\int_{L_1}\cdots\int_{L_t}\psi(\xi_1,\ldots,\xi_t)\,\phi_1(\xi_1)\ldots\phi_t(\xi_t)$$

$$\times\left\{\int_0^{\infty}x^{\alpha+\sigma\eta_G+\sum_{i=1}^{t}\gamma\xi_i-1}\;H_{r,e}^{k,f}\left[zx^{\rho}\left|\begin{matrix}(A_j,B_j)_{1,r}\\(L_j,M_j)_{1,e}\end{matrix}\right.\right]dx\right\}s_1^{\xi_1}\ldots s_t^{\xi_t}\,d\xi_1\ldots d\xi_t\,, \tag{3.1}$$

now evaluating the inner x-integral in (3.1) with the help of the following integral:
and then reinterpreting the resulting Mellin-Barnes contour integral in terms of H-function of t-variables, we

$$\int_0^{\infty}x^{\alpha-1}\;H_{r,e}^{k,f}\left[zx^{\rho}\left|\begin{matrix}(A_j,B_j)_{1,r}\\(L_j,M_j)_{1,e}\end{matrix}\right.\right]dx=\frac{1}{\rho\,z^{\alpha/\rho}}\,\frac{\prod_{j=1}^{k}\Gamma\left(L_j+\frac{\alpha}{\rho}M_j\right)\;\prod_{j=1}^{f}\Gamma\left(1-A_j-\frac{\alpha}{\rho}B_j\right)}{\prod_{j=k+1}^{e}\Gamma\left(1-L_j-\frac{\alpha}{\rho}M_j\right)\;\prod_{j=f+1}^{r}\Gamma\left(A_j+\frac{\alpha}{\rho}B_j\right)},$$

$$\textit{provided}\quad \operatorname{Re}\left(\alpha+\rho\frac{L_j}{M_j}\right)>0,\quad \operatorname{Re}\left(\alpha+\rho\left(\frac{A_{j'}-1}{B_{j'}}\right)\right)<0\;(j=1,\ldots,k;\;j'=1,\ldots,f),\quad \rho>0,$$

$$T>0,\quad |\arg z|<\frac{1}{2}T\pi$$

arrive at the desired result (2.1).

IV. Particular Cases

(a) On taking N = P = Q = 0, the multivariable H-function reduces to the product of 't' Fox's H-function in our integral formula (2.1), we arrive at the following integral:

$$\int_0^{\infty}x^{\alpha-1}\;H_{p,q}^{m,n}\left[yx^{\sigma}\left|\begin{matrix}(e_j,E_j)_{1,p}\\(f_j,F_j)_{1,q}\end{matrix}\right.\right]\;H_{r,e}^{k,f}\left[zx^{\rho}\left|\begin{matrix}(A_j,B_j)_{1,r}\\(L_j,M_j)_{1,e}\end{matrix}\right.\right]$$

$$\times\prod_{i=1}^{t}\left\{H_{P_i,Q_i}^{M_i,N_i}\left[s_ix^{\gamma}\left|\begin{matrix}(c_j^{(i)},\gamma_j^{(i)})_{1,P_i}\\(d_j^{(i)},\delta_j^{(i)})_{1,Q_i}\end{matrix}\right.\right]\right\}dx$$

$$=\sum_{g=1}^{m}\sum_{G=0}^{\infty}\frac{(-1)^G\phi(\eta_G)}{G!\;F_g\;\rho}\,y^{\eta_G}\;z^{-\frac{(\alpha+\sigma\eta_G)}{\rho}}$$

$$\times H_{e,r:P_1,Q_1;\ldots;P_t,Q_t}^{f,k:M_1,N_1;\ldots;M_t,N_t}\left[s_1z^{-\frac{\gamma}{\rho}},\ldots,s_tz^{-\frac{\gamma}{\rho}}\left|\begin{matrix}\left(1-L_j-(\alpha+\sigma\eta_G)\frac{M_j}{\rho}:\frac{\gamma}{\rho}M_j,\ldots,\frac{\gamma}{\rho}M_j\right)_{1,e}\\\left(1-A_j-(\alpha+\sigma\eta_G)\frac{B_j}{\rho}:\frac{\gamma}{\rho}B_j,\ldots,\frac{\gamma}{\rho}B_j\right)_{1,r}\end{matrix}\right.\right.$$

$$\left.\begin{matrix}:(c_j^{(1)},\gamma_j^{(1)})_{1,P_1};\ldots;(c_j^{(t)},\gamma_j^{(t)})_{1,P_t}\\:(d_j^{(1)},\delta_j^{(1)})_{1,Q_1};\ldots;(d_j^{(t)},\delta_j^{(t)})_{1,Q_t}\end{matrix}\right], \tag{4.1}$$

valid under the same conditions as required for (2.1).

(b) If we take $\alpha_{j_1}^{(i)}=\beta_{j_2}^{(i)}=\frac{\gamma}{\rho}B_{j_3}=\frac{\gamma}{\rho}M_{j_4}=\gamma_{k_1}^{(i)}=\ldots=\gamma_{k_t}^{(i)}=\delta_{l_1}^{(i)}=\ldots=\delta_{l_t}^{(i)}=\sigma$

$\left(j_1 = 1,\ldots,P;\ j_2 = 1,\ldots,Q;\ j_3 = 1,\ldots,r;\ j_4 = 1,\ldots,e;\ k_1 = 1,\ldots,P_1;\ldots;\ k_t = 1,\ldots,P_t;\ l_1 = 1,\ldots,Q_1;\right.$ $\left.\ldots;\ l_t = 1,\ldots,Q_t;\ \ i = 1,\ldots,t\ \right)$, The multivariable H-function reduces to the G-function of several variables [6] in our integral formula (2.1), we arrive at the following integral:

$$\int_0^\infty x^{\alpha-1}\ H_{p,q}^{m,n}\left[yx^\sigma \left| \begin{matrix} (e_j, E_j)_{1,p} \\ (f_j, F_j)_{1,q} \end{matrix} \right.\right]\ H_{r,e}^{k,f}\left[zx^\rho \left| \begin{matrix} (A_j, B_j)_{1,r} \\ (L_j, M_j)_{1,e} \end{matrix} \right.\right]$$

$$\times G_{P,Q:P_1,Q_1;\ldots;P_t,Q_t}^{0,N:M_1,N_1;\ldots;M_t,N_t}\left[\left(s_1 x^\gamma\right)^{\frac{1}{\sigma}},\ldots,\left(s_t x^\gamma\right)^{\frac{1}{\sigma}} \left| \begin{matrix} (a_j)_{1,P} : (c_j^{(1)})_{1,P_1};\ldots;(c_j^{(t)})_{1,P_t} \\ (b_j)_{1,Q} : (d_j^{(1)})_{1,Q_1};\ldots;(d_j^{(t)})_{1,Q_t} \end{matrix} \right.\right] dx$$

$$= \sum_{g=1}^{m}\ \sum_{G=0}^{\infty} \frac{(-1)^G \phi(\eta_G)}{G!\ F_g\ \rho}\, y^{\eta_G}\ z^{-\frac{(\alpha+\sigma\eta_G)}{\rho}}$$

$$\times G_{P+e,Q+r:P_1,Q_1;\ldots;P_t,Q_t}^{f,N+k:M_1,N_1;\ldots;M_t,N_t}\left[\left(s_1 z^{-\frac{\gamma}{\rho}}\right)^{\frac{1}{\sigma}},\ldots,\left(s_t z^{-\frac{\gamma}{\rho}}\right)^{\frac{1}{\sigma}} \left| \begin{matrix} (a_j)_{1,N}, \left(1-L_j-(\alpha+\sigma\eta_G)\frac{M_j}{\rho}\right)_{1,e}, \\ \left(1-A_j-(\alpha+\sigma\eta_G)\frac{B_j}{\rho}\right)_{1,r}, \end{matrix} \right.\right.$$

$$\left.\begin{matrix} (a_j)_{N+1,P} : (c_j^{(1)})_{1,P_1};\ldots;(c_j^{(t)})_{1,P_t} \\ (b_j)_{1,Q}\ \ : (d_j^{(1)})_{1,Q_1};\ldots;(d_j^{(t)})_{1,Q_t} \end{matrix}\right], \tag{4.2}$$

conditions of existence of this result can easily be derived from those mentioned with (2.1).

(c) If we take $\sigma \to 0$ in the main integral (2.1), the series representation of Fox's H-function reduces into unity. Further, reduce the multivariable H-function to product of 't' Fox's H-function (by taking N = P = Q = 0), then on taking t = 1 and $\rho = 1,$ we arrive at the integral obtained by Gupta and Jain [4, p.601].

(d) On taking $\sigma \to 0$, the series representation of Fox's H-function reduces to unity in (2.1) and reducing the multivariable H-function to product of 't' Fox's H-function (by taking N = P = Q = 0), then on taking t = 1, $\rho = 1,$ z = 1 and replacing x by (x + a), we arrive at the integral evaluated earlier by Jain [5, p.375] after a little simplification.

Further, on taking $\gamma = 1, B_i = M_j = \gamma_l = \delta_h = 1\ \left(i = 1,\ldots,r;\ j = 1,\ldots,e;\ l = 1,\ldots,P_1;\ h = 1,\ldots,Q_1\right)$ in (2.1), we arrive at the result earlier given by saxena [7, p.47].

(e) For $\sigma \to 0$, the series representation of Fox's H-function reduces to unity in (2.1) and reducing the multivariable H-function to product of 't' Fox's H-function (by taking N = P = Q = 0), then on taking t = 2 in (2.1), we arrive at the integral earlier given by Anandani and Srivastava [1, p.37].

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