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On Fuzzy Complex Derivatives

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ABSTRACT: *In this paper, we define and study the concepts of bounded closed complex complement normalized fuzzy numbers, and generalized rectangular valued bounded closed complex complement normalized fuzzy numbers, so that some basic properties and some characterizations are presented. Some important theorems of a fuzzy derivative for fuzzy complex functions which map a regular complex numbers into bounded closed complex complement normalized fuzzy numbers are proved. All this may be a foundation for researching fuzzy complex analysis.*
Keywords− *Fuzzy numbers, Fuzzy complex numbers, Fuzzy derivatives.*

I. Introduction

It is well known that fuzzy complex numbers and fuzzy complex analysis were first introduced by Buckley [1-3] in 1989 – 1992. He chose the definition of the derivative of a fuzzy mapping from open intervals into the set of fuzzy subsets of the reals in [5] to extend to the complex case, Buckley in [3] suggested that introducing a metric on the space of fuzzy complex numbers provide to study convergence, continuity and differentiation of fuzzy complex function (see [4, 7-8, 11-13]). Several scholars have extensively studied the theory of fuzzy complex numbers and fuzzy complex analysis (see [6, 9-10, 14-15, 17-19]).

In section two, we first review the definitions and basic properties related to fuzzy complex sets. We will also present the notations needed in the rest of the paper. In the next section, we give our definitions of the complement normalized fuzzy numbers (CNFNs), bounded closed complex CNFNs (BCCCNFNs), generalized rectangular valued BCCCNFNs (GRVBCCCNFNs) and discuss some of their basic properties. The last section contains main results and conclusions related to the fuzzy derivative of complex fuzzy functions.

II. Priliminaries

A fuzzy set $\tilde{A}$ defined on the universal set X is a function $\mu(\tilde{A},x) : X \to [0,1]$. Frequently, we will write $\mu_{\tilde{A}}(x)$ instead of $\mu(\tilde{A},x)$. The family of all fuzzy sets in X is denoted by $\mathcal{F}(X)$. The strong α−level of a fuzzy set $\tilde{A}$, denoted by ${}^{\alpha+}\tilde{A}$, is the non-fuzzy set of all elements of the universal set that belongs to the fuzzy set $\tilde{A}$ at least to the degree $\alpha \in [0,1]$. The weak α−level ${}^{\alpha-}\tilde{A}$ of a fuzzy set $\tilde{A} \in \mathcal{F}(X)$ is the crisp set that contains all elements of the universal set whose membership grades in the given set are greater than but do not include the specified value of α. The largest value of α for which the α-level is not empty is called the height of a fuzzy set $\tilde{A}$ denoted $\alpha_{\tilde{A}}^{max}$. The core of a fuzzy set $\tilde{A}$ is the non-fuzzy set of all points in the universal set X at which $\sup_x \mu_{\tilde{A}}(x)$ is essentially attained.

Let $\tilde{A}_i \in \mathcal{F}(X)$. Then the union of fuzzy sets $\tilde{A}_i$, denoted $\bigcup_i \tilde{A}_i$, is defined by $\mu_{\bigcup_i \tilde{A}_i}(x) = \sup_x \mu_{\tilde{A}_i}(x) = \bigvee_x \mu_{\tilde{A}_i}(x)$, the intersection of fuzzy sets $\tilde{A}_i$, denoted $\bigcap_i \tilde{A}_i$, is defined by $\mu_{\bigcap_i \tilde{A}_i}(x) = \inf_x \mu_{\tilde{A}_i}(x) = \bigwedge_x \mu_{\tilde{A}_i}(x)$, and the complement of $\tilde{A}_i$, denoted $\neg\tilde{A}_i$, is defined by $\mu_{\tilde{A}_i}(x) + \mu_{\neg\tilde{A}_i}(x) = 1$, for all x in the universal set X.

A fuzzy number $\tilde{a}$ is a fuzzy set defined on the set of real numbers R^1 characterized by means of a membership function $\tilde{a}(x): R^1 \to [0,1]$, which satisfies: (1) $\tilde{a}$ is upper semicontinuous, (2) $\mu_{\tilde{a}}(x) = 0$ outside some interval $[c,d]$, (3) There are real numbers a, b such that $c \le a \le b \le d$ and $\mu_{\tilde{a}}(x)$ is increasing on $[c,a]$, $\mu_{\tilde{a}}(x)$ is decreasing on $[b,d]$, $\mu_{\tilde{a}}(x) = 1$, $a \le x \le b$. We denote the set of all fuzzy numbers by $\mathcal{F}^{\star}$. $\hat{Z}$ is a fuzzy complex number if and only if (1) $\mu_{\hat{Z}}(z)$ is continuous; (2) ${}^{\alpha-}\hat{Z}, 0 \le \alpha < 1$, is open, bounded, connected and simply connected; and (3) ${}^{1+}\hat{Z}$ is non-empty, compact, arcwise connected, and simply connected. We use $\mathcal{F}^{\star\star}$ to the set of all fuzzy complex numbers.

Let $f(z',z'') = w$ be any mapping from $\mathbb{C} \times \mathbb{C}$ into $\mathbb{C}$. Buckley in [3] extend f to $\mathcal{F}^{\star\star} \times \mathcal{F}^{\star\star}$ into $\mathcal{F}^{\star\star}$ and write $f(\widehat{Z'},\widehat{Z''}) = \widehat{W}$ if $\mu_{\widehat{W}}(w) = \bigvee_{f(z',z'')=w} (\mu_{\widehat{Z'}}(z') \wedge \mu_{\widehat{Z''}}(z''))$. One obtains $\widehat{W} = \widehat{Z'} \oplus \widehat{Z''}$ or $\widehat{W} = \widehat{Z'} \odot \widehat{Z''}$ by using $f(\widehat{Z'},\widehat{Z''}) = \widehat{Z'} \oplus \widehat{Z''}$ or $f(\widehat{Z'},\widehat{Z''}) = \widehat{Z'} \odot \widehat{Z''}$, respectively. Proved that $\widehat{Z'} \circledast \widehat{Z''} \in \mathcal{F}^{\star\star}$ for the extended basic operation $*$ in $\mathbb{C}$.

III. Basic Definitions and Properties

In this section, the concepts of BCCCNFNs, GRVBCCCNFNs, and other related objects are introduced and some characterizations are given. The properties of extended operations have been investigated.

Definition 1. CNFN $\tilde{\mu}$ is a fuzzy set $\tilde{\mu}$ of the real line, such that core of $\neg\tilde{\mu}$ is empty and $^{\gamma+}\neg\tilde{\mu} = \begin{cases} \{u: \mu_{\neg\tilde{\mu}}(u) \geq \gamma\} & \text{if } \gamma \in (0, \alpha^{max}_{\neg\tilde{\mu}}] \\ \overline{\bigcup_{0\leq\gamma\leq\alpha^{max}_{\neg\tilde{\mu}}} {}^{\gamma-}\neg\tilde{\mu}} & \text{if } \gamma = 0 \end{cases}$ is compact. We use $\tilde{\mathcal{F}}^{*}_{\neg N}$ for the fuzzy power set of CNFNs.

Definition 2. For CNFNs $\tilde{\mu}$ and $\tilde{\lambda}$ with membership functions $\mu(\tilde{\mu},\mu_1)$ and $\mu(\tilde{\lambda},\lambda_1)$, respectively, we call $\tilde{Z} = \tilde{\mu} \boxplus i\,\tilde{\lambda}$ a BCCCNFN with membership function $\mu(\tilde{Z},z) = \mu(\tilde{\mu},\mu_1) \wedge \mu(\tilde{\lambda},\lambda_1)$, where $z = \mu_1 + i\,\lambda_1$. We denote the class of all the BCCCNFNs by $\tilde{\mathcal{F}}^{**}_{\neg N}$.

Definition 3. Let $\tilde{Z}$ be a BCCCNFN and f be an unary operation from complex field $\mathbb{C}$ into $\mathbb{C}$. Based on extension principle we define $f(\tilde{Z})$ as

$$f(\tilde{Z}) = \left(f(\neg\tilde{Z})\right)_{\neg} = \left(\left(f(N\neg\tilde{Z})\right)_N\right)_{\neg}$$

here N denotes the normalized set.

Theorem 4. The extended operation $\boxed{*}$ for BCCCNFNs in $\tilde{\mathcal{F}}^{**}_{\neg N}$ is associative.

Proof: Let $\tilde{Z}', \tilde{Z}'', \tilde{Z}''' \in \tilde{\mathcal{F}}^{**}_{\neg N}$. We have

$$\begin{aligned}
\tilde{Z}' \boxed{*} (\tilde{Z}'' \boxed{*} \tilde{Z}''') &= \left(\neg\tilde{Z}' \boxed{*}_N \neg(\tilde{Z}'' \boxed{*} \tilde{Z}''')\right)_{\neg} \\
&= \left(\neg\tilde{Z}' \boxed{*}_N \neg\left(\left(\neg\tilde{Z}'' \boxed{*}_N \neg\tilde{Z}'''\right)_{\neg}\right)\right)_{\neg} \\
&= \left(\left(N\neg\tilde{Z}' \circledast N\neg\left(\left(\neg\tilde{Z}'' \boxed{*}_N \neg\tilde{Z}'''\right)_{\neg}\right)\right)_N\right)_{\neg} \\
&= \left(\left(N\neg\tilde{Z}' \circledast N\neg\left(\left(\left(N\neg\tilde{Z}'' \circledast N\neg\tilde{Z}'''\right)_N\right)_{\neg}\right)\right)_N\right)_{\neg} \\
&= \left(\left(N\neg\tilde{Z}' \circledast \left(N\neg\tilde{Z}'' \circledast N\neg\tilde{Z}'''\right)\right)_N\right)_{\neg} \\
&= \left(\left(\left(N\neg\tilde{Z}' \circledast N\neg\tilde{Z}''\right) \circledast N\neg\tilde{Z}'''\right)_N\right)_{\neg} \\
&= \left(\left(N\neg\tilde{Z}' \circledast N\neg\tilde{Z}''\right)_N \boxed{*}_N \neg\tilde{Z}'''\right)_{\neg} \\
&= \left(\left(\neg\tilde{Z}' \boxed{*}_N \neg\tilde{Z}''\right) \boxed{*}_N \neg\tilde{Z}'''\right)_{\neg} \\
&= \left(\tilde{Z}' \boxed{*} \tilde{Z}''\right) \boxed{*} \tilde{Z}'''
\end{aligned}$$

Theorem 5. Let $\tilde{Z}$ be a BCCCNFN and f be an unary operation from complex field $\mathbb{C}$ into $\mathbb{C}$, then $\mu(f(\tilde{Z}),w) = \bigwedge_{w=f(z)} \mu(\tilde{Z},z)$.

Proof: We have

$$\begin{aligned}
\mu\left(\left(\left(f(N\neg\tilde{Z})\right)_N\right)_{\neg}, w\right) &= 1 - \mu\left(f(N\neg\tilde{Z}),w\right) \\
&= 1 - \bigvee_{w=f(z)} \mu(N\neg\tilde{Z},z) \\
&= 1 - \bigvee_{w=f(z)} \left(1 - \mu(\tilde{Z},z)\right) \\
&= 1 - \left(1 - \bigwedge_{w=f(z)} \mu(\tilde{Z},z)\right) \\
&= \bigwedge_{w=f(z)} \mu(\tilde{Z},z)
\end{aligned}$$

Theorem 6. Let $\tilde{Z}, \tilde{W} \in \tilde{\mathcal{F}}^{**}_{\neg N}$ and $f(z',z'') = w$ be any mapping from $\mathbb{C}\times\mathbb{C}$ into $\mathbb{C}$, then $\mu(\tilde{Z} \boxed{*} \tilde{W}, w) = \bigwedge_{f(z',z'')=w} \left(\mu(\tilde{Z},z') \vee \mu(\tilde{W},z'')\right)$.

Proof: Suppose that $\mu(N\neg\tilde{Z} \circledast N\neg\tilde{W}, w)$ attains its value at (z_0', z_0''). That is, $\mu(N\neg\tilde{Z} \circledast N\neg\tilde{W}, w) = \bigvee_{f(z_0',z_0'')=w} \left(\mu(N\neg\tilde{Z},z_0') \wedge \mu(N\neg\tilde{W},z_0'')\right) = \mu(N\neg\tilde{Z},z_0') \wedge \mu(N\neg\tilde{W},z_0'')$.

If $\mu(N\neg\tilde{Z},z_0') \wedge \mu(N\neg\tilde{W},z_0'') = \mu(N\neg\tilde{W},z_0'')$ then $\mu(N\neg\tilde{W},z_0'') \leq \mu(N\neg\tilde{Z},z_0')$ and for each (z',z'') so that $f(z',z'') = w$, we have $\mu(N\neg\tilde{W},z_0'') \geq \mu(N\neg\tilde{Z},z') \wedge \mu(N\neg\tilde{W},z'')$. Implies $1 - \mu(N\neg\tilde{W},z_0'') \geq 1 - \mu(N\neg\tilde{Z},z_0')$ and for each (z',z'') so that $f(z',z'') = w$, we have $1 - \mu(N\neg\tilde{W},z_0'') \leq \left(1 - \mu(N\neg\tilde{Z},z') \vee 1 - \mu(N\neg\tilde{W},z'')\right)$. That is, $1 - \mu(N\neg\tilde{W},z_0'') = \bigwedge_{f(z',z'')=w} \left(\left(1 - \mu(N\neg\tilde{Z},z')\right) \vee \left(1 - \mu(N\neg\tilde{W},z'')\right)\right)$.

If $\mu(N\neg\tilde{Z},z_0') \wedge \mu(N\neg\tilde{W},z_0'') = \mu(N\neg\tilde{Z},z_0')$, a similar proof can be give.

Now, $\mu(\tilde{Z} \boxed{*} \tilde{W}, w) = \mu\left(\left(\left(N\neg\tilde{Z} \circledast N\neg\tilde{W}\right)_N\right)_{\neg}, w\right) = 1 - \mu(N\neg\tilde{Z} \circledast N\neg\tilde{W}, w)$

$$= 1 - \left(1 - \bigwedge_{f(z',z'')=w} \left(\left(1 - \mu(N\neg\tilde{Z},z')\right) \vee \left(1 - \mu(N\neg\tilde{W},z'')\right)\right)\right)$$

$$= \wedge_{f(z',z'')=w}\left(\left(1-\mu(N\neg\tilde{Z},z')\right)\vee\left(1-\mu(N\neg\tilde{W},z'')\right)\right)$$
$$= \wedge_{f(z',z'')=w}\left(\mu(\tilde{Z},z')\vee\mu(\tilde{W},z'')\right)$$

Definition 7. The fuzzy conjugate $\tilde{Z}^\circ$ of BCCCNFN $\tilde{Z}=\tilde{\mu}\boxplus i\,\tilde{\lambda}$ is defined as $\mu(\tilde{Z}^\circ=\tilde{\mu}\boxminus i\,\tilde{\lambda},z)=\mu(\tilde{Z},z^\circ=\mu-i\,\lambda)$.

Theorem 8. Let $\tilde{Z},\tilde{W}\in\tilde{\mathcal{F}}^{**}_{\neg N}$ and $*$ is the four basic arithmetic operations, then $\left(\tilde{Z}\boxed{*}\tilde{W}\right)^\circ=\tilde{Z}^\circ\boxed{*}\tilde{W}^\circ$ and $\tilde{W}^{\circ\circ}=\tilde{W}$.

Proof: $\mu\left(\left(\tilde{Z}\boxed{*}\tilde{W}\right)^\circ,w\right)=\mu\left(\tilde{Z}\boxed{*}\tilde{W},w^\circ\right)$
$$=\wedge_{z'^\circ * z''^\circ=w^\circ}\left(\mu(\tilde{Z},z'^\circ)\vee\mu(\tilde{W},z''^\circ)\right)$$
$$=\wedge_{z' * z''=w}\left(\mu(\tilde{Z}^\circ,z')\vee\mu(\tilde{W}^\circ,z'')\right)$$
$$=\mu\left(\tilde{Z}^\circ\boxed{*}\tilde{W}^\circ,w\right)$$

Second, $\mu\left(\tilde{W}^{\circ\circ},w\right)=\mu\left(\tilde{W}^\circ,w^\circ\right)=\mu\left(\tilde{W},w^{\circ\circ}\right)=\mu(\tilde{W},w)$.

Definition 9. The fuzzy modulus $\tilde{W}^*$ of BCCCNFN $\tilde{W}$ is defined by $\mu(\tilde{W}^*,w)=\wedge\{\mu(\tilde{W},z=\mu+i\,\lambda):\ w=z^*=(\mu^2+\lambda^2)^{0.5}\}$.

Theorem 10. For $\circledast\in\{\cdot,\div\}$ and BCCCNFNs $\tilde{Z},\tilde{W}$ we have $\left(\tilde{Z}\boxplus\tilde{W}\right)^*\preccurlyeq\tilde{Z}^*\boxplus\tilde{W}^*$ and $\left(\tilde{Z}\boxed{\circledast}\tilde{W}\right)^*=\tilde{Z}^*\boxed{\circledast}\tilde{W}^*$.

Proof: $\mu(\tilde{Z}^*\boxplus\tilde{W}^*,w)=\wedge\left\{\mu(\tilde{Z}^*,z_1)\vee\mu(\tilde{W}^*,z_2):\ w=z_1+z_2\right\}$
$$=\wedge\left\{\wedge\{\mu(\tilde{Z},z'):\ z_1=z'^*\}\vee\wedge\{\mu(\tilde{W},z''):\ z_2=z''^*\}:w=z_1+z_2\right\}$$
$$=\wedge\left\{\wedge\{\mu(\tilde{Z},z_1)\vee\mu(\tilde{W},z_2):\ z_1=z'^*,z_2=z''^*\}:\ w=z_1+z_2\right\}$$
$$=\wedge\left\{\wedge\left\{\left(\mu(\tilde{Z},z_1)\vee\mu(\tilde{W},z_2)\right):z_1=z'^*,z_2=z''^*\right\},w=z'^*+z''^*\right\}$$
$$\geq\wedge\left\{\wedge\left\{\left(\mu(\tilde{Z},z_1)\vee\mu(\tilde{W},z_2)\right):z_1=z'^*,z_2=z''^*\right\},w=(z'+z'')^*\right\}$$
$$=\wedge\{\wedge\{\mu(\tilde{Z},z_1)\vee\mu(\tilde{W},z_2):\ z_1=z'^*,z_2=z''^*\},w=(z'+z'')^*\}$$
$$=\wedge\{\mu(\tilde{Z}\boxplus\tilde{W},z'+z''):\ w=(z'+z'')^*\}=\mu\left(\left(\tilde{Z}\boxplus\tilde{W}\right)^*,w\right)$$

The proof of the other part is similar.

Definition 11. We call $\mu_{[\tilde{\mu}]}(\delta):R^1\to\{[\gamma^-,\gamma^+];\ \gamma^-\leq\gamma^+$ and $(\gamma^-,\gamma^+)\in[0,\alpha_Z^{max}]^2\}$ as a generalized CNFNs (GCNFNs) if $\mu_{[\tilde{\mu}]}(\delta)=\left[\mu_{\tilde{\mu}^l}(\delta),\mu_{\tilde{\mu}^u}(\delta)\right]$ for $\tilde{\mu}^l,\tilde{\mu}^u\in\tilde{\mathcal{F}}^*_{\neg N}$. The set of all GCNFNs is denoted by $[\tilde{\mathcal{F}}^*_{\neg N}]$. We call $\lambda([\tilde{Z}],\delta+i\,\Delta):\mathbb{C}\to\left\{[\gamma^-,\gamma^+]:\gamma^-,\gamma^+\in I^{1[}_{[0}:=[0,\alpha_Z^{max}];\ \gamma^-\leq\gamma^+\right\}$ is a GRVBCCCNFNs if $\lambda([\tilde{Z}],\delta+i\,\Delta)=\left[\lambda(\underline{\tilde{Z}},\delta+i\,\Delta),\lambda\left(\overline{\tilde{Z}},\delta+i\,\Delta\right)\right]$ for BCCCNFNs $\underline{\tilde{Z}}$ and $\overline{\tilde{Z}}$. Sometimes, we write $[\tilde{Z}]$ to be $[\tilde{Z}]=\left[\underline{\tilde{Z}},\overline{\tilde{Z}}\right]=[\tilde{\mu}]\,[+]\,i\,[\tilde{\lambda}]=[\tilde{\mu}^\ell\boxplus i\tilde{\lambda}^\ell,\tilde{\mu}^u\boxplus i\tilde{\lambda}^u]$.

Definition 12. Let $[*]\in\{[+],[-],[\cdot],[\div],[\vee],[\wedge]\}$. For GRVBCCCNFNs $\left[\underline{\tilde{Z}},\overline{\tilde{Z}}\right]$, $\left[\underline{\tilde{W}},\overline{\tilde{W}}\right]$, we define $\left[\underline{\tilde{Z}},\overline{\tilde{Z}}\right][*]\left[\underline{\tilde{W}},\overline{\tilde{W}}\right]$ by $\lambda\left(\left[\underline{\tilde{Z}},\overline{\tilde{Z}}\right][*]\left[\underline{\tilde{W}},\overline{\tilde{W}}\right],\gamma+i\,\beta\right)=\wedge_{\gamma+i\,\beta=(\delta+i\,\Delta)*(\mu+i\,\nu)}\left(\lambda\left(\left[\underline{\tilde{Z}},\overline{\tilde{Z}}\right],\delta+i\,\Delta\right)\vee\lambda\left(\left[\underline{\tilde{W}},\overline{\tilde{W}}\right],\mu+i\,\nu\right)\right)$.

Theorem 13. Let $[\tilde{Z}],[\tilde{W}]\in[\tilde{\mathcal{F}}^{**}_{\neg N}]$, then $[\tilde{Z}]\,[*]\,[\tilde{W}]\in[\tilde{\mathcal{F}}^{**}_{\neg N}]$.

Proof: $\lambda([\tilde{Z}]\,[*]\,[\tilde{W}],\gamma+i\beta)=\wedge_{\gamma+i\beta=(\delta+i\,\Delta)*(\mu+i\,\nu)}\left(\lambda([\tilde{Z}],\delta+i\,\Delta)\vee\lambda([\tilde{W}],\mu+i\,\nu)\right)$
$$=\wedge_{\gamma+i\beta=(\delta+i\,\Delta)*(\mu+i\,\nu)}\left(\left[\lambda(\underline{\tilde{Z}},\delta+i\,\Delta),\ \lambda\left(\overline{\tilde{Z}},\delta+i\,\Delta\right)\right]\vee\left[\lambda(\underline{\tilde{W}},\mu+i\,\nu),\ \lambda\left(\overline{\tilde{W}},\mu+i\,\nu\right)\right]\right)$$
$$=\wedge_{\gamma+i\beta=(\delta+i\,\Delta)*(\mu+i\,\nu)}\left[\lambda(\underline{\tilde{Z}},\delta+i\,\Delta)\vee\lambda(\underline{\tilde{W}},\mu+i\,\nu),\lambda\left(\overline{\tilde{Z}},\delta+i\,\Delta\right)\vee\lambda\left(\overline{\tilde{W}},\mu+i\,\nu\right)\right]$$
$$=[\wedge_{\gamma+i\beta=(\delta+i\,\Delta)*(\mu+i\,\nu)}\left(\lambda(\underline{\tilde{Z}},\delta+i\,\Delta)\vee\lambda(\underline{\tilde{W}},\mu+i\,\nu)\right),\ \wedge_{\gamma+i\beta=(\delta+i\,\Delta)*(\mu+i\,\nu)}\left(\lambda\left(\overline{\tilde{Z}},\delta+i\,\Delta\right)\vee\lambda\left(\overline{\tilde{W}},\mu+i\,\nu\right)\right)]=\left[\lambda(\underline{\tilde{Z}}\boxed{*}\underline{\tilde{W}},\gamma+i\beta),\lambda\left(\overline{\tilde{Z}}\boxed{*}\overline{\tilde{W}},\gamma+i\beta\right)\right].$$

On the other hand

$\lambda([\tilde{Z}]\,[*]\,[\tilde{W}],\gamma+i\beta)=\left[\lambda\left(\underline{(\tilde{Z}\boxed{*}\tilde{W})},\gamma+i\beta\right),\lambda\left(\overline{(\tilde{Z}\boxed{*}\tilde{W})},\gamma+i\beta\right)\right]$. So that

$\lambda(\underline{\tilde{Z}}\boxed{*}\underline{\tilde{W}},\gamma+i\beta)=\lambda\left(\underline{(\tilde{Z}\boxed{*}\tilde{W})},\gamma+i\beta\right)$ and $\lambda\left(\overline{\tilde{Z}}\boxed{*}\overline{\tilde{W}},\gamma+i\beta\right)=\lambda\left(\overline{(\tilde{Z}\boxed{*}\tilde{W})},\gamma+i\beta\right)$.

Since $\underline{\tilde{Z}}$, $\underline{\tilde{W}}$, $\overline{\tilde{Z}}$, $\overline{\tilde{W}}$ are BCCCNFNs, it follows that $\underline{\tilde{Z}}\boxed{*}\underline{\tilde{W}}$ and $\overline{\tilde{Z}}\boxed{*}\overline{\tilde{W}}$ are BCCCNFNs, and so $\underline{(\tilde{Z}\boxed{*}\tilde{W})}$, $\overline{(\tilde{Z}\boxed{*}\tilde{W})}$ are BCCCNFNs. Hence $[\tilde{Z}]\,[*]\,[\tilde{W}]$ is GRVBCCCNFNs.

Theorem 14. Let $[\tilde{Z}],[\tilde{W}],[\tilde{Y}]\in[\tilde{\mathcal{F}}^{**}_{\neg N}]$, $[*']\in\{[+],[\cdot]\}$, $[*'']\in\{[\wedge],[\vee]\}$ then

1. $\left[\underline{\tilde{Z}},\overline{\tilde{Z}}\right][+]\left(\left[\underline{\tilde{W}},\overline{\tilde{W}}\right][*'']\left[\underline{\tilde{Y}},\overline{\tilde{Y}}\right]\right)=\left(\left[\underline{\tilde{Z}},\overline{\tilde{Z}}\right][+]\left[\underline{\tilde{W}},\overline{\tilde{W}}\right]\right)[*'']\left(\left[\underline{\tilde{Z}},\overline{\tilde{Z}}\right][+]\left[\underline{\tilde{Y}},\overline{\tilde{Y}}\right]\right)$
2. $\left[\underline{\tilde{Z}},\overline{\tilde{Z}}\right][-]\left(\left[\underline{\tilde{W}},\overline{\tilde{W}}\right][\vee]\left[\underline{\tilde{Y}},\overline{\tilde{Y}}\right]\right)=\left(\left[\underline{\tilde{Z}},\overline{\tilde{Z}}\right][-]\left[\underline{\tilde{W}},\overline{\tilde{W}}\right]\right)[\wedge]\left(\left[\underline{\tilde{Z}},\overline{\tilde{Z}}\right][-]\left[\underline{\tilde{Y}},\overline{\tilde{Y}}\right]\right)$

3. $\left[\underline{\tilde{Z}},\overline{\tilde{Z}}\right][-]\left(\left[\underline{\widetilde{W}},\overline{\widetilde{W}}\right][\wedge]\left[\underline{\tilde{Y}},\overline{\tilde{Y}}\right]\right)=\left(\left[\underline{\tilde{Z}},\overline{\tilde{Z}}\right][-]\left[\underline{\widetilde{W}},\overline{\widetilde{W}}\right]\right)[\vee]\left(\left[\underline{\tilde{Z}},\overline{\tilde{Z}}\right][-]\left[\underline{\tilde{Y}},\overline{\tilde{Y}}\right]\right)$
4. $\left[\underline{\tilde{Z}},\overline{\tilde{Z}}\right][*']\left[\underline{\widetilde{W}},\overline{\widetilde{W}}\right]=\left[\underline{\widetilde{W}},\overline{\widetilde{W}}\right][*']\left[\underline{\tilde{Z}},\overline{\tilde{Z}}\right]$
5. $\left(\left[\underline{\tilde{Z}},\overline{\tilde{Z}}\right][*']\left[\underline{\widetilde{W}},\overline{\widetilde{W}}\right]\right)[*']\left[\underline{\tilde{Y}},\overline{\tilde{Y}}\right]=\left[\underline{\tilde{Z}},\overline{\tilde{Z}}\right][*']\left(\left[\underline{\widetilde{W}},\overline{\widetilde{W}}\right][*']\left[\underline{\tilde{Y}},\overline{\tilde{Y}}\right]\right)$

Proof: The proofs of (2), (3), (4), and (5) are similar to (1), so we only prove (1).

$$
\begin{aligned}
\left[\underline{\tilde{Z}},\overline{\tilde{Z}}\right][+]\left(\left[\underline{\widetilde{W}},\overline{\widetilde{W}}\right][*'']\left[\underline{\tilde{Y}},\overline{\tilde{Y}}\right]\right) &= \left[\underline{\left(\tilde{Z}\boxplus(\widetilde{W}\boxed{*''}\tilde{Y})\right)},\overline{\left(\tilde{Z}\boxplus(\widetilde{W}\boxed{*''}\tilde{Y})\right)}\right]\\
&= \left[\underline{\tilde{Z}}\boxplus\underline{(\widetilde{W}\boxed{*''}\tilde{Y})},\overline{\tilde{Z}}\boxplus\overline{(\widetilde{W}\boxed{*''}\tilde{Y})}\right]\\
&= \left[\underline{\tilde{Z}}\boxplus(\underline{\widetilde{W}}\boxed{*''}\underline{\tilde{Y}}),\overline{\tilde{Z}}\boxplus\left(\overline{\widetilde{W}}\boxed{*''}\overline{\tilde{Y}}\right)\right]\\
&= \left[(\underline{\tilde{Z}}\boxplus\underline{\widetilde{W}})\boxed{*''}(\underline{\tilde{Z}}\boxplus\underline{\tilde{Y}}),\left(\overline{\tilde{Z}}\boxplus\overline{\widetilde{W}}\right)\boxed{*''}\left(\overline{\tilde{Z}}\boxplus\overline{\tilde{Y}}\right)\right]\\
&= \left[\underline{(\tilde{Z}\boxplus\widetilde{W})}\boxed{*''}\underline{(\tilde{Z}\boxplus\tilde{Y})},\overline{(\tilde{Z}\boxplus\widetilde{W})}\boxed{*''}\overline{(\tilde{Z}\boxplus\tilde{Y})}\right]\\
&= \left[\underline{\left((\tilde{Z}\boxplus\widetilde{W})\boxed{*''}(\tilde{Z}\boxplus\tilde{Y})\right)},\overline{\left((\tilde{Z}\boxplus\widetilde{W})\boxed{*''}(\tilde{Z}\boxplus\tilde{Y})\right)}\right]\\
&= \left(\left[\underline{\tilde{Z}},\overline{\tilde{Z}}\right][+]\left[\underline{\widetilde{W}},\overline{\widetilde{W}}\right]\right)[*'']\left(\left[\underline{\tilde{Z}},\overline{\tilde{Z}}\right][+]\left[\underline{\tilde{Y}},\overline{\tilde{Y}}\right]\right)
\end{aligned}
$$

Definition 15. Let $[\tilde{Z}]\in[\tilde{\mathcal{F}}^{**}_{\neg N}]$ and $(\gamma^-,\gamma^+)\in I^{1[}_{[0}\times I^{1[}_{[0}$. We define $[\gamma^-,\gamma^+]$-level of $[\tilde{Z}]$ as ${}^{[\gamma^-,\gamma^+]+}[\tilde{Z}]={}^{\gamma^-+}\underline{\tilde{Z}}\cap{}^{\gamma^++}\overline{\tilde{Z}}$.

Theorem 16. For GRVBCCCNFNs $[\tilde{Z}]$ and $[\widetilde{W}]$ and $[\circ]\in\{[+],[-],[\cdot],[\div]\}$ we have ${}^{[\gamma^-,\gamma^+]+}([\tilde{Z}][\circ][\widetilde{W}])={}^{[\gamma^-,\gamma^+]+}[\tilde{Z}]\circ{}^{[\gamma^-,\gamma^+]+}[\widetilde{W}]$.

Proof: $\lambda([\tilde{Z}][\circ][\widetilde{W}],\gamma+i\beta)=\left[\lambda(\underline{\tilde{Z}}\boxed{\circ}\underline{\widetilde{W}},\gamma+i\beta),\lambda\left(\overline{\tilde{Z}}\boxed{\circ}\overline{\widetilde{W}},\gamma+i\beta\right)\right]$. Hence

$$
\begin{aligned}
{}^{[\gamma^-,\gamma^+]+}([\tilde{Z}][\circ][\widetilde{W}]) &= {}^{\gamma^-+}(\underline{\tilde{Z}}\boxed{\circ}\underline{\widetilde{W}})\cap{}^{\gamma^++}\left(\overline{\tilde{Z}}\boxed{\circ}\overline{\widetilde{W}}\right)\\
&= \left({}^{\gamma^-+}\underline{\tilde{Z}}\circ{}^{\gamma^-+}\underline{\widetilde{W}}\right)\cap\left({}^{\gamma^++}\overline{\tilde{Z}}\circ{}^{\gamma^++}\overline{\widetilde{W}}\right)\\
&= \left({}^{\gamma^-+}\underline{\tilde{Z}}\cap{}^{\gamma^++}\overline{\tilde{Z}}\right)\circ\left({}^{\gamma^-+}\underline{\widetilde{W}}\cap{}^{\gamma^++}\overline{\widetilde{W}}\right)\\
&= {}^{[\gamma^-,\gamma^+]+}[\tilde{Z}]\circ{}^{[\gamma^-,\gamma^+]+}[\widetilde{W}]
\end{aligned}
$$

IV. Fuzzy Complex Derivatives

In this section, we will use the "dot" notation for partial derivatives with respect to z. Otherwise, we employ the "prime" notation for the derivative of a complex function of one variable. Furthermore, we use the standard notations and results of Yang and Yi in [16].

The complex fuzzy valued function $\tilde{f}:\mathbb{C}\to\tilde{\mathcal{F}}^{**}_{\neg N}$ is fuzzy differentiable in its domain if the derivative of ${}^{\gamma+}(\tilde{f}(z))={}^{z,\gamma+}\tilde{f}$ denoted by ${}^{z,\gamma+}\dot{\tilde{f}}$ exists for all $\gamma\in I^{1[}_{[0}$. We call $\tilde{f}$ is fuzzy meromorphic if ${}^{z,\gamma+}\tilde{f}$ is meromorphic for any $\gamma\in I^{1[}_{[0}$. We say that $\tilde{f}$ has a pole (resp. zero) if ${}^{z,\gamma+}\tilde{f}$ has a poles (resp. zeros) for any $\gamma\in I^{1[}_{[0}$.

Theorem 1. Let $\tilde{Z},\widetilde{W}\in\tilde{\mathcal{F}}^{**}_{\neg N}$ and $\tilde{f}$ be a fuzzy meromorphic function. If $\tilde{f}(z)=\widetilde{W}$ and $\tilde{f}'(z)=\widetilde{W}$ have the same zeros, $\tilde{f}(z)=\tilde{Z}$ and $\tilde{f}'(z)=\tilde{Z}$ have the same zeros with the same order, and $\overline{N}\left(r,\frac{1}{{}^{z,\gamma+}\tilde{f}}\right)+\overline{N}\left(r,{}^{z,\gamma+}\tilde{f}\right)=o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)$ then $\tilde{f}'=\tilde{f}$.

Proof: Assume that ${}^{z,\gamma+}\dot{\tilde{f}}\neq{}^{z,\gamma+}\tilde{f}$.

$$
\begin{aligned}
2T\left(r,{}^{z,\gamma+}\tilde{f}\right) &\le \overline{N}\left(r,{}^{z,\gamma+}\tilde{f}\right)+\overline{N}\left(r,\frac{1}{{}^{z,\gamma+}\tilde{f}-{}^{\gamma+}\tilde{Z}}\right)+\overline{N}\left(r,\frac{1}{{}^{z,\gamma+}\tilde{f}-{}^{\gamma+}\widetilde{W}}\right)+\overline{N}\left(r,\frac{1}{{}^{z,\gamma+}\tilde{f}}\right)+o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)\\
&= \overline{N}\left(r,\frac{1}{{}^{z,\gamma+}\tilde{f}-{}^{\gamma+}\tilde{Z}}\right)+\overline{N}\left(r,\frac{1}{{}^{z,\gamma+}\tilde{f}-{}^{\gamma+}\widetilde{W}}\right)+o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)\\
&= \overline{N}\left(r,\frac{1}{{}^{z,\gamma+}\dot{\tilde{f}}-{}^{\gamma+}\tilde{Z}}\right)+\overline{N}\left(r,\frac{1}{{}^{z,\gamma+}\dot{\tilde{f}}-{}^{\gamma+}\widetilde{W}}\right)+o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)\\
&\le N\left(r,\frac{1}{\frac{{}^{z,\gamma+}\dot{\tilde{f}}}{{}^{z,\gamma+}\tilde{f}}-1}\right)+o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)
\end{aligned}
$$

$$\leq T\left(r,\frac{{}^{z,\gamma+}\dot{\tilde{f}}}{{}^{z,\gamma+}\tilde{f}}\right)+o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)$$
$$\leq \overline{N}\left(r,{}^{z,\gamma+}\tilde{f}\right)+\overline{N}\left(r,\frac{1}{{}^{z,\gamma+}\tilde{f}}\right)+o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)$$
$$=o\left(T(r,{}^{z,\gamma+}\tilde{f})\right).$$

Theorem 2. Let $\tilde{Z},\widetilde{W}\in\tilde{\mathcal{F}}^{**}_{\neg N}$ and $\tilde{f}$ be a fuzzy meromorphic function. If $\tilde{f}(z)=\widetilde{W}$ and $\tilde{f}'(z)=\widetilde{W}$ have the same zeros, $\tilde{f}(z)=\tilde{Z}$ and $\tilde{f}'(z)=\tilde{Z}$ have the same zeros with the same order, and $\overline{N}\left(r,\frac{1}{{}^{z,\gamma+}\tilde{f}}\right)=o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)$ then $\tilde{f}'=\tilde{f}$.

Proof: Assume that ${}^{z,\gamma+}\dot{\tilde{f}}\neq{}^{z,\gamma+}\tilde{f}$. Then

$$2T\left(r,{}^{z,\gamma+}\tilde{f}\right)\leq\overline{N}\left(r,{}^{z,\gamma+}\tilde{f}\right)+\overline{N}\left(r,\frac{1}{{}^{z,\gamma+}\tilde{f}-{}^{\gamma+}\tilde{Z}}\right)+\overline{N}\left(r,\frac{1}{{}^{z,\gamma+}\tilde{f}-{}^{\gamma+}\widetilde{W}}\right)+\overline{N}\left(r,\frac{1}{{}^{z,\gamma+}\tilde{f}}\right)+o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)$$
$$=\overline{N}\left(r,{}^{z,\gamma+}\tilde{f}\right)+\overline{N}\left(r,\frac{1}{{}^{z,\gamma+}\dot{\tilde{f}}-{}^{\gamma+}\tilde{Z}}\right)+\overline{N}\left(r,\frac{1}{{}^{z,\gamma+}\dot{\tilde{f}}-{}^{\gamma+}\widetilde{W}}\right)+o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)$$
$$\leq\overline{N}\left(r,{}^{z,\gamma+}\tilde{f}\right)+N\left(r,\frac{1}{\frac{{}^{z,\gamma+}\dot{\tilde{f}}}{{}^{z,\gamma+}\tilde{f}}-1}\right)+o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)$$
$$=\overline{N}\left(r,{}^{z,\gamma+}\tilde{f}\right)+N\left(r,\frac{{}^{z,\gamma+}\dot{\tilde{f}}}{{}^{z,\gamma+}\tilde{f}}\right)+o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)$$
$$=\overline{N}\left(r,{}^{z,\gamma+}\tilde{f}\right)+\overline{N}\left(r,{}^{z,\gamma+}\tilde{f}\right)+\overline{N}\left(r,\frac{1}{{}^{z,\gamma+}\tilde{f}}\right)+o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)$$
$$=2\overline{N}\left(r,{}^{z,\gamma+}\tilde{f}\right)+o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)$$
$$\leq 2T\left(r,{}^{z,\gamma+}\tilde{f}\right)+o\left(T(r,{}^{z,\gamma+}\tilde{f})\right).$$

It follows that

$$\overline{N}\left(r,\frac{1}{{}^{z,\gamma+}\tilde{f}-{}^{\gamma+}\tilde{Z}}\right)+\overline{N}\left(r,\frac{1}{{}^{z,\gamma+}\tilde{f}-{}^{\gamma+}\widetilde{W}}\right)+\overline{N}\left(r,\frac{1}{{}^{z,\gamma+}\dot{\tilde{f}}-{}^{\gamma+}\tilde{Z}}\right)+\overline{N}\left(r,\frac{1}{{}^{z,\gamma+}\dot{\tilde{f}}-{}^{\gamma+}\widetilde{W}}\right)$$
$$=\overline{N}\left(r,{}^{z,\gamma+}\tilde{f}\right)+\overline{N}\left(r,{}^{z,\gamma+}\tilde{f}\right)=2\overline{N}\left(r,{}^{z,\gamma+}\tilde{f}\right)$$
$$=2T\left(r,{}^{z,\gamma+}\tilde{f}\right)+o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)\tag{1}$$

and

$$m\left(r,\frac{1}{{}^{z,\gamma+}\dot{\tilde{f}}}\right)+m\left(r,\frac{1}{{}^{z,\gamma+}\dot{\tilde{f}}-{}^{\gamma+}\tilde{Z}}\right)+m\left(r,\frac{1}{{}^{z,\gamma+}\dot{\tilde{f}}-{}^{\gamma+}\widetilde{W}}\right)+m\left(r,\frac{1}{{}^{z,\gamma+}\tilde{f}-{}^{\gamma+}\tilde{Z}}\right)+m\left(r,\frac{1}{{}^{z,\gamma+}\tilde{f}-{}^{\gamma+}\widetilde{W}}\right)$$
$$\leq m\left(r,\frac{1}{{}^{z,\gamma+}\ddot{\tilde{f}}}\right)+m\left(r,\frac{1}{{}^{z,\gamma+}\dot{\tilde{f}}}\right)+o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)\tag{2}$$

From (1) and (2) we have

$$T\left(r,\frac{1}{{}^{z,\gamma+}\tilde{f}-{}^{\gamma+}\tilde{Z}}\right)+T\left(r,\frac{1}{{}^{z,\gamma+}\tilde{f}-{}^{\gamma+}\widetilde{W}}\right)+T\left(r,\frac{1}{{}^{z,\gamma+}\dot{\tilde{f}}-{}^{\gamma+}\tilde{Z}}\right)+T(r,\frac{1}{{}^{z,\gamma+}\dot{\tilde{f}}-{}^{\gamma+}\widetilde{W}})$$
$$\leq m\left(r,\frac{1}{{}^{z,\gamma+}\ddot{\tilde{f}}}\right)+2\overline{N}\left(r,{}^{z,\gamma+}\tilde{f}\right)+o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)$$
$$\leq T\left(r,\frac{1}{{}^{z,\gamma+}\ddot{\tilde{f}}}\right)+2\overline{N}\left(r,{}^{z,\gamma+}\tilde{f}\right)+o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)$$
$$\leq T\left(r,{}^{z,\gamma+}\ddot{\tilde{f}}\right)+2\overline{N}\left(r,{}^{z,\gamma+}\tilde{f}\right)+o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)$$
$$\leq T\left(r,{}^{z,\gamma+}\dot{\tilde{f}}\right)+\overline{N}(r,{}^{z,\gamma+}\dot{\tilde{f}})+2\overline{N}\left(r,{}^{z,\gamma+}\tilde{f}\right)+o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)$$
$$\leq T\left(r,{}^{z,\gamma+}\dot{\tilde{f}}\right)+T(r,{}^{z,\gamma+}\dot{\tilde{f}})+\overline{N}\left(r,{}^{z,\gamma+}\tilde{f}\right)+o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)$$
$$\leq 2T\left(r,{}^{z,\gamma+}\dot{\tilde{f}}\right)+T(r,{}^{z,\gamma+}\tilde{f})+o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)$$

Hence

$2T\left(r,{}^{z,\gamma+}\dot{\tilde{f}}\right)+2T\left(r,{}^{z,\gamma+}\tilde{f}\right)\leq 2T\left(r,{}^{z,\gamma+}\dot{\tilde{f}}\right)+T\left(r,{}^{z,\gamma+}\tilde{f}\right)+o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)$. This implies that $T(r,{}^{z,\gamma+}\tilde{f})\leq o\left(T(r,{}^{z,\gamma+}\tilde{f})\right)$.

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