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On Generalized Stancu's Polynomials

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***Abstract**: We have tested the convergence of the Generalized Stancu's Polynomial $R_n^{\alpha}(f,x)$ and have also tested the degree of approximation of Lebesgue integrable functions by $R_n^{\alpha}(f,x)$*

I. Introduction & Results

If f(x) is a function defined on [0,1], the Bernstein Polynomial $B_n^f(x)$ of f(x) as

$$B_n^f(x) = \sum_{k=0}^{n} w_{n,k}(x,\alpha)\, f(k/n) \binom{n}{k} x^k (1-x)^{n-k} \qquad \text{------------- (1.1)}$$

Bernstein [2] proved that if f(x) is continuous in the closed interval, then

$$B_n^f(x) \longrightarrow f(x) \qquad \text{---------------- (1.2)}$$

uniformly as n→∞. this yields a simple constructive proof of weierstrass's approximation theorem
A more precise version of this result due to Popoviciu[5] states that

$$|B_n^f(x) \longrightarrow f(x)| \le \frac{5}{4} w_f(n^{-\frac{1}{2}}) \qquad \text{---------------- (1.3)}$$

where w_f is the uniform modulus of continuity of f defined by

$$w_f(h) = \max\{|f(x) - f(y)| : x, y \in [0,1], |x-y| \le h\}$$

A small modification of Bernstein polynomial due to Kantorovic[3] makes it possible to approximate lebesgue integrable functions in L_1-norm by the modified polynomials

$$P_n^f(x) = (n+1) \sum_{k=0}^{n} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt \right) \binom{n}{k} x^k (1-x)^{n-k} \qquad \text{--------------- (1.4)}$$

Stancu [6] defined a polynomial in a closed interval [0, 1] as

$$P_n^{[\alpha]}(f,x) = \sum_{k=0}^{n} w_{n,k}(x,\alpha)\, f(k/n), \qquad \text{-------------------- (1.5)}$$

where

$$w_{n,k}(x;\alpha) = \binom{n}{k} \frac{\prod_{v=0}^{k-1}(x+v\alpha) \prod_{\mu=0}^{n-k-1}(1-x+\mu\alpha)}{(1+\alpha)(1+2\alpha).....(1+(n-1)\alpha)}. \qquad \text{---------------------- (1.6)}$$

We now define a Kantorovic type polynomial with the help of (1.6) for lebesgue integrable function on [0,1] in L_1-norm as:

$$R_n^{\alpha}(f,x) = (n+1) \sum_{k=0}^{n} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt \right) w_{n,k}(x;\alpha), \qquad \text{-------------------- (1.7)}$$

where $w_{n,k}(x;\alpha)$ is same as (1.6)
such that

$$\sum_{k=0}^{n} w_{n,k}(x;\alpha) = 1, \qquad \text{-------------- (1.8)}$$

$$\sum_{k=0}^{n} k\ w_{n,k}(x;\alpha) = nx, \qquad \text{----------------- (1.9)}$$

and

$$\sum_{k=0}^{n} k^2\ w_{n,k}(x;\alpha) = \frac{nx(1-x)+n^2x(x+\alpha)}{(1+\alpha)}; \qquad \text{----------------- (1.10)}$$

In this paper, we shall prove the corresponding results of approximation due to Bernstein and Popoviciu for lebegue integrable functions in L_1-norm by our newly defined Generalized Stancu's polynomial in terms of L_1-modulus of continuity

$$w_f(h)_{L_1} = \sup_{|t| \le h} \int_0^1 |f(x+t) - f(x)| dx$$

Infact our results are as follows

Theorem 1: let $f(x)$ be a continuous lebesgue integrable function on [0,1] and $\alpha=\alpha n\rightarrow 0$, as $n\rightarrow\infty$,then

$$\left|R_n^{\alpha}(f,x)-f(x)\right|<\in$$

uniformly on [0, 1]

Theorem 2: Let $f(x)$ be a continuous lebesgue integrable function on [0,1] and w(δ) is modulus of continuity of $f(x)$, for $\alpha > 0$, we have

$$\left|f(x)-R_n^{\alpha}(f,x)\right|\leq\frac{5}{4}w\left(\sqrt{\frac{1+\alpha n}{n+\alpha n}}\right).$$

II. Lemma

In order to prove our results we need the following lemma:

Lemma: For all values of x□[0,1] , we have

$$(n+1)\sum_{k=0}^{n}\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^2dt\right)w_{n,k}(x;\alpha)$$

$$=x(1-x)\left(\frac{1+\alpha n}{n+\alpha n}\right).$$

Proof : we have

$$(n+1)\sum_{k=0}^{n}\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^2dt\right)w_{n,k}(x;\alpha)$$

$$=(n+1)\sum_{k=0}^{n}\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(x^2-2xt+t^2)dt\right)w_{n,k}(x;\alpha)$$

$$=\sum_{k=0}^{n}\left[x^2-x\frac{2k+1}{(n+1)}+\frac{k^2+k}{(n+1)^2}+\frac{1}{3(n+1)^2}\right]w_{n,k}(x;\alpha)$$

$$=x^2-\frac{2nx+1}{(n+1)}x+\frac{nx(1-x)+n^2x(x+\alpha)}{(1+\alpha)(n+1)^2}+\frac{nx}{(n+1)^2}+\frac{1}{3(n+1)^2}$$

$$=\frac{nx^2+x^2-2nx^2-x}{(n+1)}+\frac{nx(1-x)+n^2x(x+\alpha)+nx(1+\alpha)}{(1+\alpha)(n+1)^2}+\frac{1}{3(n+1)^2}$$

$$=\frac{-x(1-x)\alpha-x(1-x)+n^2x(1-x)\alpha+nx(1-x)}{(n+1)^2(1+\alpha)}+\frac{1}{3(n+1)^2}$$

$$\leq\frac{-x(1-x)\alpha-x(1-x)+n^2x(1-x)\alpha+nx(1-x)}{n^2(1+\alpha)}+\frac{1}{3n^2}$$

$$=-\frac{x(1-x)\alpha}{n^2(1+\alpha)}-\frac{x(1-x)}{n^2(1+\alpha)}+\frac{x(1-x)\alpha}{(1+\alpha)}+\frac{x(1-x)}{n(1+\alpha)}+\frac{1}{3n^2}$$

$$\leq\frac{x(1-x)\alpha}{(1+\alpha)}+\frac{x(1-x)}{n(1+\alpha)}\ ,\qquad\dots\dots\dots\dots\text{for large } n$$

$$=x(1-x)\left\{\frac{1+n\alpha}{n+n\alpha}\right\}.$$

which completes the proof of Lemma

III. Proof of Theorems

Proof of Theorem 1:

$$\left|f(x)-R_n^{\alpha}(f,x)\right| = (n+1)\sum_{k=0}^{n}\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(x)-f(t)|dt\right) w_{n,k}(x;\alpha)$$

$$= \sum_{|t-x|<\delta} (n+1)w_{n,k}(x;\alpha)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(x)-f(t)|dt$$

$$+ \sum_{|t-x|\geq\delta} (n+1)w_{n,k}(x;\alpha)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(x)-f(t)|dt$$

$$= I_1+I_2 \quad \text{-- (3.1)}$$

$$I_1= \sum_{|t-x|<\delta}(n+1)w_{n,k}(x;\alpha)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}|f(x)-f(t)|dt$$

$$\leq \frac{\epsilon}{2}\sum_{|t-x|<\delta}(n+1)w_{n,k}(x;\alpha)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}dt \quad \text{by uniform continuity of f}$$

$$\leq \frac{\epsilon}{2}(n+1)\sum_{k=0}^{n}\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right) w_{n,k}(x;\alpha)$$

$$= \frac{\epsilon}{2} \quad \text{---------------------------------- (3.2)}$$

$$I_2= \sum_{|t-x|\geq\delta}(n+1)w_{n,k}(x;\alpha)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}|f(x)-f(t)|dt$$

$$\leq 2M\sum_{|t-x|\geq\delta}(n+1)w_{n,k}(x;\alpha)(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}dt \quad \text{by boundedness of f}$$

$$\leq 2M(n+1)\delta^{-2}\sum_{k=0}^{n}\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^2dt\right)w_{n,k}(x;\alpha)$$

$$\leq 2M\delta^{-2}\frac{1}{4n}\left(\frac{1+n\alpha}{1+\alpha}\right) \quad \text{by lemma and the fact x(1-x)≤1/4 on [0,1]}$$

$$= \frac{M}{2n\delta^2(1+\alpha)}+\frac{M\alpha}{2\delta^2(1+\alpha)} \quad \text{-------------------------- (3.3)}$$

and hence by (3.1) ,(3.2) & (3.3) we have

$$\left|f(x)-R_n^{\alpha}(f,x)\right| \leq I_1+I_2 \leq \frac{\epsilon}{2}+\frac{M}{2n\delta^2(1+\alpha)}+\frac{M\alpha}{2\delta^2(1+\alpha)}$$

for sufficiently large value of n & $\alpha=\alpha_n=o\left(\frac{1}{n}\right)$, $\frac{M}{2n\delta^2(1+\alpha)}+\frac{M\alpha}{2\delta^2(1+\alpha)} <\epsilon/2$ (independence of x)

and consequently

$$\left|f(x)-R_n^{\alpha}(f,x)\right| <\epsilon$$

which completes the proof of theorem 1

Proof of Theorem 2: For arbitrary x_1 , x_2 in [0,1] and δ>0,we denote $\lambda=\lambda(x_1,x_2;\delta)$ the integers $[|x_1-x_2|\delta^{-1}]$; the difference $\{f(x_1)-f(x_2)\}$ is then a sum of (λ+1) differences of f(x) on intervals of length < δ thus it follows

$$\left|f(x)-R_n^{\alpha}(f,x)\right|$$

$$\leq (n+1)\sum_{k=0}^{n}\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}|f(x)-f(t)|dt\right)w_{n,k}(x;\alpha)$$

$$\leq (n+1)w(\delta)_{L_1}\sum_{k=0}^{n}\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}[1+\lambda(x,t;\delta)]dt\right)w_{n,k}(x;\alpha)$$

(by hypothesis together with modulus of the continuity)

$$= (n+1)w(\delta)_{L_1}[\sum_{k=0}^{n}\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right) w_{n,k}(x;\alpha) + \sum_{\lambda\geq 1}\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \lambda(x,t;\delta)dt\right) w_{n,k}(x;\alpha)$$

$$= w(\delta)_{L_1}\left[1 + (n+1)\delta^{-1}\sum_{\lambda\geq 1}\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |x-t|dt\right) w_{n,k}(x;\alpha)\right]$$

$$\leq w(\delta)_{L_1}\left[1 + (n+1)\delta^{-2}\sum_{\lambda\geq 1}\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (x-t)^2 dt\right) w_{n,k}(x;\alpha)\right]$$

since $x(1-x) \leq \frac{1}{4}$ $on[0,1]$ and so by the given lemma we have

$$\left|f(x) - R_n^{\alpha}(f,x)\right| \leq w(\delta)_{L_1}\left[1 + \delta^{-2}\frac{1}{4n}\left(\frac{1+n\alpha}{1+\alpha}\right)\right]$$

For $\delta = (\frac{1+n\alpha}{1+\alpha})^{1/2}$ we get our required result

$$\left|f(x) - R_n^{\alpha}(f,x)\right| \leq \frac{5}{4} w\left(\sqrt{\frac{1+\alpha n}{n+\alpha n}}\right)$$

IV. Conclusion

Results of Berstein & Popoviciu have been extended by our newly defined Generalized Stancu's polynomials.

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