Some Properties of semi-symmetric non-metric connection in LP-Sasakian manifold

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Abstract: S.K. Chaubey and R.H. Ojha [4] introduced a semi-symmetric non-metric connection in almost contact manifold and also studied the connection in Sasakian manifold. The present paper deals with some properties of semi-symmetric non-metric connection in LP-Sasakian manifold.

Key words: Lorentzian almost paracontact manifold, LP-Sasakian manifold and semi-symmetric non-metric connection.

I. Introduction

An n-dimensional differentiable manifold M, is called a Lorentzian almost paracontact manifold (briefly LAP- Sasakian manifold) [2],[3] if it admits (1, 1) tensor field \( \Phi \), a contravariant vector field \( \xi \), a 1-form \( \eta \) and a Lorentzian metric g which satisfy
\[
\Phi(\xi) = 0 , \hspace{1cm} \eta(\xi) = -1 \tag{1}
\]
\[
g(\Phi(X), \Phi(Y)) = g(X, Y) + \eta(X)\eta(Y) \tag{2}
\]
\[
g(X, \xi) = \eta(X) \tag{3}
\]
\[
\nabla_X \xi = \Phi(X) \tag{4}
\]
\[
(\nabla_X \Phi)(Y) = g(X, Y) + \eta(Y)X + 2\eta(X)\eta(Y) \tag{5}
\]
where \( \nabla \) denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

It can be easily seen that in an LAP- Sasakian manifold, the following relations holds

(a) \( \Box \xi = 0 \) , \hspace{1cm} (b) \( \eta(\Box X) = 0 \) \tag{6}

In an LAP manifold, if we put
\[
\Box(X, Y) = g(X, \Phi Y) \tag{7}
\]
for any vector fields X and Y, then the tensor field \( \Box(X, Y) \) is a symmetric (0, 2) tensor field [2], that is
\[
\Box(X, Y) = \Box(Y, X) \tag{8}
\]
An LAP- Sasakian manifold satisfying the relation [2]
\[
(\nabla_X \Box)(Y) = g(Y, Z)\eta(X) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z) \tag{9}
\]
is called a normal Lorentzian paracontact manifold or Lorentzian para-sasakian manifold (briefly LP-Sasakian) manifold.

Also, since the vector field \( \eta \) is closed in an LP- Sasakian manifold, we have [2], [5].
\[
(\nabla_X \eta)(Y) = -\Box(X, Y) \tag{10}
\]
for any vector field X and Y.

In an \( n \)-dimensional LP- Sasakian manifold, the following relations holds [1], [5]
\[
g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) + g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \tag{11}
\]
\[
R(\xi, X) Y = g(Y, \xi)X - \eta(Y)X \tag{12}
\]
\[
R(X, Y) \xi = \eta(Y)X - \eta(X)Y \tag{13}
\]
\[
R(\xi, X) \xi = X - \eta(X)\xi \tag{14}
\]
\[
S(X, \xi) = (n-1)\eta(X) \tag{15}
\]
\[
S(\Box X, \Box Y) = S(X, Y) + (n-1)\eta(X)\eta(Y) \tag{16}
\]
for any vector fields X, Y and Z , where R is the Riemannian curvature tensor and S is the Ricci tensor.
II. Semi-symmetric non-metric connection

A linear connection $B$ on $(M, g)$ defined as

$$B_X Y = \nabla_X Y - \eta(Y) X - g(X, Y)T$$

for arbitrary vector fields $X$ and $Y$, is said to be a semi-symmetric non-metric connection [4].

Now, if we put (19) as

$$B_X Y = \nabla_X Y + H(X, Y)$$

where $H(X, Y) = -\eta(Y) X - g(X, Y)T$

Let us define

(a) $S(X, Y, Z) = g(S(X, Y), Z)$
(b) $H(X, Y, Z) = g(H(X, Y), Z)$

then we can write

(a) $S(X, Y, Z) = \eta(X) g(Y, Z) - \eta(Y) g(X, Z)$
(b) $H(X, Y, Z) = -\eta(Y) g(X, Z) - \eta(Z) g(X, Y)$

**Theorem 1.** Let $B$ be a semi-symmetric non-metric connection in a LP-Sasakian manifold with a Riemannian connection $\nabla$, then we have

\(\text{(a)}\) \quad (B_X \Box)(Y, Z) = \eta(Y)([(\nabla_X \nabla)(Z)] + [\nabla_X \nabla](Z))

\(\text{(b)}\) \quad (B_X \Box)(Y, Z) = \eta(Z)((\nabla_X \nabla)(Y) + [\nabla_X \nabla](Y))

**Proof.** We have,

$$X(\Box(Y, Z)) = (\nabla_X \Box)(Y, Z) + \Box(\nabla_X Y, Z) + \Box(Y, \nabla_X Z)$$

$$= (B_X \Box)(Y, Z) + \Box(B_X Y, Z) + \Box(Y, B_X Z)$$

$$\quad (B_X \Box)(Y, Z) = (\nabla_X \Box)(Y, Z) + \Box(\nabla_X Y - B_X Y, Z) + \Box(Y, \nabla_X Z - B_X Z)$$

With the help of equation (20), the above equation becomes

\(\text{(B_X \Box)(Y, Z)} = (\nabla_X \Box)(Y, Z) - \Box(H(X, Y), Z) - \Box(Y, H(X, Z))$$

\(\text{(B_X \Box)(Y, Z)} = (\nabla_X \Box)(Y, Z) - \Box(H(X, Y), Z) - \Box(H(X, Z), Y)$$

Using equation (9) in this, we obtain

\(\text{(B_X \Box)(Y, Z)} = (\nabla_X \Box)(Y, Z) - g(H(X, Y), Z) + g(H(X, Z), Y)$$

From equation (23) (b) and equation (29), we obtain

\(\text{(B_X \Box)(Y, Z)} = (\nabla_X \Box)(Y, Z) - H(X, Y, Z)$$

Now from equation (25) above becomes

\(\text{(B_X \Box)(Y, Z)} = (\nabla_X \Box)(Y, Z) + \eta(Y)g(X, Z) + \eta(Z)g(X, Y)$$

\quad + \eta(Z)g(X, Y) + \eta(Y)g(X, Z)$$

\(\text{(B_X \Box)(Y, Z)} = (\nabla_X \Box)(Y, Z) + \eta(Y)g(X, Z) + \eta(Z)g(X, Y)$$

\quad + \eta(Z)g(X, Y) + \eta(Y)g(X, Z)$$

\quad \text{(B_X \Box)(Y, Z)} = (\nabla_X \Box)(Y, Z) + \eta(Y)g(X, Z) + \eta(Z)g(X, Y)$$

\quad + \eta(Z)g(X, Y) + \eta(Y)g(X, Z)$$

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\quad + \eta(Z)g(X, Y) + \eta(Y)g(X, Z)$$

\quad \text{(B_X \Box)(Y, Z)} = (\nabla_X \Box)(Y, Z) + \eta(Y)g(X, Z) + \eta(Z)g(X, Y)$$

\quad + \eta(Z)g(X, Y) + \eta(Y)g(X, Z)$$

Replace $Z$ by $\Box Z$ in equation (32) it becomes

\(\text{(B_X \Box)(Y, Z)} = g(Z, X)\eta(Y) + g(Y, X)\eta(Z) + 2\eta(X)\eta(Y)\eta(Z)$$

\quad + \eta(Y)\eta(Z)g(X, Z) + \eta(Z)\eta(Y)g(X, Y)$$

\quad \text{(B_X \Box)(Y, Z)} = g(Z, X)\eta(Y) + g(Y, X)\eta(Z)$$

\quad \text{(B_X \Box)(Y, Z)} = (\nabla_X \Box)(Z)\eta(Y) + \eta(Y)\eta(Z)\eta(X)$$

\quad \text{(B_X \Box)(Y, Z)} = (\eta(Y))((\nabla_X \Box)(Z) + (\nabla_X \Box)(Y))$$

Similarly replace $Y$ by $\Box Y$ in equation (32) we get equation (27).

**Theorem 2.** Let B be a semi-symmetric non-metric connection in LP-Sasakian manifold with a Riemannian connection $\nabla$, then we have

\(\text{(B_X \Box)(Y, Z)} - (B_Y \Box)(Z, X) - (B_Z \Box)(X, Y) = 0$$

**Proof:** Barring $X$ in equation (31), we get

\(\text{(B_X \Box)(Y, Z)} = (\nabla_X \Box)(Y, Z) + \eta(Y)(\nabla_X \Box)(Z) + \eta(Z)(\nabla_X \Box)(Y)$$

From equation (11), the above equation becomes

\(\text{(B_X \Box)(Y, Z)} = g(Z, X)\eta(Y) + g(Y, X)\eta(Z) + 2\eta(X)\eta(Y)\eta(Z)$$

\quad + \eta(Y)\eta(Z)g(X, Z) + \eta(Z)\eta(Y)g(X, Y)$$

\quad \text{(B_X \Box)(Y, Z)} = g(Z, X)\eta(Y) + g(Y, X)\eta(Z)$$

\quad \text{(B_X \Box)(Y, Z)} = (\nabla_X \Box)(Z)\eta(Y) + \eta(Y)\eta(Z)\eta(X)$$

\quad \text{(B_X \Box)(Y, Z)} = (\eta(Y))((\nabla_X \Box)(Z) + (\nabla_X \Box)(Y))$$
Theorem 3. Let B be a semi-symmetric non-metric connection in LP–Sasakian manifold with a Riemannian connection \( \nabla \), then we have

\[
(B \circ Y)(Z, X) = (B \circ Y)(X, Z) + \eta(Z)(\nabla_X \eta)(Y)
\]

From equations (34) and (36), we get the required result.

Theorem 4. Let B be a semi-symmetric non-metric connection in LP–Sasakian manifold with a Riemannian connection \( \nabla \), then we have

\[
(B \circ Y)(X, Z) = (B \circ Y)(Z, X) + \eta(Y)(\nabla_X \eta)(Z)
\]

From equations (38) and (39), we get the required result.

References