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G-I-Lc* Sets and Decompositions of ∗ - Continuity

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Abstract: *In this paper, we introduce and investigate the notions of I_g^*-closed sets and Ig^*-continuous maps in ideal topological spaces. Also we introduce the notion of G-I-LC*-sets and G-I-LC*-continuous maps to obtain decompositions of ∗-continuity. Further, we introduce the notions of weakly GLC*-sets, rg*-closed sets and weakly GLC*-continuous maps, rg*-continuous maps in topological spaces to obtain decompositions of continuity.*
Keywords: *G-I-LC*-sets, weakly G-I-LC*-sets, I_g^*-closed sets, I_{rg}^*-closed sets.*

I. Introduction And Preliminaries

The concept of ideals in topological spaces is treated in the classic text by Kuratowski [1] and Vaidyanathaswamy [2]. The notion of I -open sets in topological spaces was introduced by Jankovic and Hamlett [3]. Dontchev et al. [4] introduced and studied the notion of I_g -closed sets. An ideal I on a topological space (X, τ) is a non-empty collection of subsets of X satisfying the following properties: (1) $A \in I$ and $B \subset A$ imply $B \in I$ (heredity), (2) $A \in I$ and $B \in I$ imply $A \cup B \in I$ (finite additivity). A topological space (X, τ) with an ideal I on X is called an ideal topological space and is denoted by (X, τ, I). For a subset $A \subset X$, $A^*(I) = \{x \in X : U \cap A \notin I \text{ for every } U \in \tau(x)\}$ is called the local function [1] of A with respect to I and τ. We simply write A^* in case there is no chance for confusion. A Kuratowski closure operator $cl^*(.)$ for a topology $\tau^*(I)$ called the ∗ -topology finer than τ is defined by $cl^*(A) = A \cup A^*$ [2]. Let (X, τ) denote a topological space on which no separation axioms are assumed unless explicitly stated. In a topological space (X, τ), the closure and the interior of any subset A of X will be denoted by cl(A) and int(A), respectively. A subset A of a space is said to be regular-open [5] if A = int(cl(A)).

Definition 1.1. A subset A of an ideal topological space (X, τ) is said to be
1. g-closed [6] if $cl(A) \subset U$ whenever $A \subset U$ and U is open.
2. rg-closed [7] if $cl(A) \subset U$ whenever $A \subset U$ and U is regular-open.
3. g^* -closed [8] if $cl(A) \subset U$ whenever $A \subset U$ and U is g-open.

The complements of the above sets are their respective open sets.

Definition 1.2. A subset A of an ideal topological space (X, τ) is said to be
1. locally closed [9] (briefly LC) if $A = U \cap V$ where U is open and V is closed.
2. GLC*-set [10] if $A = U \cap V$ where U is g-open and V is closed.

Definition 1.3. A subset A of an ideal topological space (X, τ, I) is ∗-closed [3] (resp. ∗-dense in itself [11]) if $A^* \subset A$ (resp. $A \subset A^*$).

Lemma 1.4. [12] Let (X, τ, I) be a topological space with an ideal I on X and A is a subset of X. If $A \subset A^*$, then $A^* = cl(A^*) = cl(A) = cl^*(A)$.

Definition 1.5. A function f: (X, τ) → (Y, σ) is said to be GLC*-continuous [10] (resp. g^*-continuous [8]) if $f^{-1}(A)$ is a GLC*-set (resp. g^*-closed set) in (X, τ) for every closed set A of (Y, σ).

Definition 1.6. [13] A function f: (X, τ, I) → (Y, σ) is said to be ∗-continuous if $f^{-1}(A)$ is ∗-closed in (X, τ, I) for every closed set A in (Y, σ).

II. I_g^*-closed sets

Definition 2.1. A subset A of an ideal topological space (X, τ, I) is said to be ***Ig****-closed if $A^* \subset U$ whenever $A \subset U$ and U is g-open in X.

Theorem 2.2. Let (X, τ, I) be an ideal topological space and $A \subset X$. Then A is $\{\phi\}_g^*$-closed if and only if A is g^*-closed.

Proof. Follows from the fact that $A^*(\{\phi\}) = cl(A)$.

Theorem 2.3. Let (X, τ, I) be an ideal topological space with $A \subset X$ and $B \subset X$. If A and B are I_g^*-closed sets, then their union $A \cup B$ is also I_g^*-closed.

Proof. Suppose $A \cup B \subset U$ and U is g-open in (X, τ, I). Then $A \subset U$ and $B \subset U$. Since A and B are I_g^*-closed sets, $A^* \subset U$ and $B^* \subset U$ and hence $(A \cup B)^* = A^* \cup B^* \subset U$. Thus $A \cup B$ is I_g^*-closed.

Theorem 2.4. If A is an I_g^*-closed set of (X, τ, I) such that $A \subset B \subset cl^*(A)$, then B is also an I_g^*-closed set.
Proof. Let U be any g-open set of (X, τ, I) such that $B \subset U$ and so $A \subset U$. Since A is I_g^*-closed, we have $A^* \subset U$ and $cl^*(A) = A^* \cup A \subset U$. Now $B^* \subset cl^*(B) \subset cl^*(A) \subset U$. Therefore, B is I_g^*-closed.

Theorem 2.5. A subset of an ideal topological space (X, τ, I) is I_g^*-closed if and only if $A^* - A$ does not contains any nonempty g-closed set.
Proof. Suppose that A is I_g^*-closed. Let U be a g-closed subset of $A^* - A$. Then $A \subset U^C$, the complement of U. Since A is I_g^*-closed, we have $A^* \cap U^C$. Consequently $U \subset (A^*)^C$. Hence $U \subset A^* \cap (A^*)^C = \phi$.

Conversely, suppose that $A^* - A$ contains no nonempty g-closed set. Let $A \subset U$ and U be g-open. If $A^* \not\subset U$, then $A^* \cap U^C \neq \phi$. A^* is closed and U^C is a g-closed set of X. Thus $A^* \cap U^C$ is a nonempty g-closed set. Therefore, $A^* \cap U^C \subset A^* - A$. This is a contradiction to the hypothesis. So, $A^* \subset U^C$. This implies that A is I_g^*-closed.

Theorem 2.6. Let (X, τ, I) be an ideal topological space and $A \subset X$, then the following are equivalent:

1. A is I_g^*-closed;
2. $cl^*(A) \subset U$ whenever $A \subset U$ and U is g-open in X;
3. For all $x \in cl^*(A)$, $cl(\{x\}) \cap A \neq \phi$;
4. $cl^*(A) - A$ contains no nonempty g-closed set;
5. $A^* - A$ contains no nonempty g-closed set.

Proof. (1) ⇒ (2): If A is I_g^*-closed, then $A^* \subset U$ whenever $A \subset U$ and U is g-open in X and so $cl^*(A) = A^* \cup A \subset U$ whenever $A \subset U$ and U is g-open in X. This proves (2).
(2) ⇒ (3): Suppose $x \in cl^*(A)$. If $cl(\{x\}) \cap A = \phi$, then $A \subset X - cl(\{x\})$. By (2) $cl^*(A) \subset X - cl(\{x\})$, a contradiction, since $x \in cl^*(A)$.
(3) ⇒ (4): Suppose $F \subset cl^*(A) - A$, F is g-closed and $x \in F$. Since $F \subset X - A$ and F is g-closed $cl(\{x\}) \cap A = \phi$. Also since $x \in cl^*(A)$ by (3), $cl(\{x\}) \cap A \neq \phi$, a contradiction.
(4) ⇒ (5): Follows from the fact that $cl^*(A) - A = A^* - A$.
(5) ⇒ (1): Follows from Theorem 2.5.
From Theorem 2.6 (2), it follows that every *-closed set is I_g^*-closed. The following Example 2.7 shows that the converse need not be true.

Example 2.7. Let X = {a, b, c, d}, τ = {ϕ, {b}, {b, c, d}, X} and I = {ϕ, {a}}. Then A = {a, b} is I_g^*-closed but not *-closed.

Theorem 2.8. If (X, τ, I) is an ideal topological space and A is a $*$-dense in itself, I_g^*-closed subset of X, then A is g^*-closed.
Proof. Suppose A is a $*$-dense in itself, I_g^*-closed subset of X. If U is any g-open set containing A, then by Theorem 2.6(2), $cl^*(A) \subset U$. Since A is $*$-dense in itself, by Lemma 1.4, $cl(A) \subset U$ and so A is g^*-closed.

III. G-I-LC* SETS

Definition 3.1. A subset A of an ideal topological space (X, τ, I) is said to be ***G***-I-LC*-set if $A = C \cap D$, where C is g-open and D is $*$-closed.

Definition 3.2. A subset A of an ideal topological space (X, τ, I) is said to be weakly G-I-LC*-set if $A = C \cap D$, where C is rg-open and D is $*$-closed.

Proposition 3.3. For a subset A of an ideal topological space (X, τ, I), the following hold:

1. If A is g-open, then A is a G-I-LC*-set;
2. If A is $*$-closed, then A is a G-I-LC*-set;
3. If A is a G-I-LC*-set, then A is a weakly G-I-LC*-set.

The converse of Proposition 3.3 need not be true as seen from the following examples.

Example 3.4. Let X = {a, b, c}, τ = {ϕ, {a}, {b}, {a,b}, X} and I = {ϕ, {c}}. Then,

1. A = {c} is a G-I-LC*-set but not a g-open set.

2. A = {a} is a G-I-LC*-set but not a $*$-closed set.

Example 3.5. Let X = {a, b, c, d, e}, τ = {ϕ, {a, b}, {c, d}, {a, b, c, d}, X} and I = {ϕ, {a}}. Then A = {b, e} is a weakly G-I-LC*-set but not a G-I-LC*-set.

Theorem 3.6. Let (X, τ, I) be an ideal topological space and A be a G-I-LC*-subset of X. Then the following hold:

1. If B is a $*$-closed set, then A ∩ B is a G-I-LC*-set;
2. If B is a g-open set, then A ∩ B is a G-I-LC*-set;
3. If B is a G-I-LC*-set, then A ∩ B is a G-I-LC*-set.

Proof.

1. Let B be a $*$-closed set. Then A ∩ B = (C ∩ D) ∩ B = C ∩ (D ∩ B), where D ∩ B is $*$-closed. Hence A ∩ B is a G-I-LC*-set.
2. Let B be a g-open set. Then A ∩ B = (C ∩ D) ∩ B = (C ∩ B) ∩ D, where C ∩ B is g-open. Hence A ∩ B is a G-I-LC*-set.
3. Let B be a G-I-LC*-set. Then A ∩ B = (C ∩ D) ∩ (U ∩ V) = (C ∩ U) ∩ (D ∩ V), where C ∩ U is g-open and D ∩ V is $*$-closed. Hence A ∩ B is a G-I-LC*-set.

Definition 3.7. A subset A of an ideal topological space (X, τ, I) is said to be I_{rg}^*-closed if $A^* \subset U$ whenever $A \subset U$ and U is rg-open in X.

Theorem 3.8. For a subset A of an ideal topological space (X, τ, I), the following hold:

1. If A is $*$-closed, then A is I_{rg}^*-closed;
2. If A is $*$-closed, then A is a weakly G-I-LC*-set;
3. If A is I_{rg}^*-closed, then A is I_g^*-closed.

The converse of Theorem 3.8 need not be true as seen from the following example.

Example 3.9. Let X = {a, b, c, d, e}, τ = {ϕ, {a, b}, {c, d}, {a, b, c, d}, X} and I = {ϕ, {a}}. Then,

1. A = {a, b, c, e} is an I_{rg}^*-closed set but not a $*$-closed set.
2. A = {b} is a weakly G-I-LC*-set but not a $*$-closed set.
3. A = {b, e} is an I_g^*-closed set but not an I_{rg}^*-closed set.

Theorem 3.10. A subset of an ideal topological space (X, τ, I) is $*$-closed if and only if it is a weakly G-I-LC*-set and an I_{rg}^*-closed set.

Proof. Necessity is trivial. We shall prove only sufficiency. Let A be a weakly G-I-LC*-set and an I_{rg}^*-closed set. Since A is a weakly G-I-LC*-set, A = C ∩ D where C is rg-open and D is $*$-closed. So we have A = C ∩ D ⊂ C. Since A is I_{rg}^*-closed, $A^* \subset C$. Also A = C ∩ D ⊂ D and D is $*$-closed implies $A^* \subset D$. Consequently, we have $A^* \subset C \cap D = A$ and hence A is $*$-closed.

Theorem 3.11. For a subset A of an ideal topological space (X, τ, I), the following are equivalent:

1. A is $*$-closed;
2. A is a G-I-LC*-set and an I_{rg}^*-closed set;
3. A is a G-I-LC*-set and an I_g^*-closed set.

Proof. (1) ⇒ (2): This is obvious.

(2) ⇒ (3): Follows from the fact that every I_{rg}^*-closed set is I_g^*-closed.

(3) ⇒ (1): Let A be a G-I-LC*-set and an I_g^*-closed set. Since A is a G-I-LC*-set, A = C ∩ D where C is g-open and D is $*$-closed. Now A ⊂ C and A is I_g^*-closed implies $A^* \subset C$. Also A ⊂ D and D is $*$-closed implies that $A^* \subset D$. Thus $A^* \subset C \cap D = A$. Hence A is $*$-closed.

Remark 3.12.

1. The notions of G-I-LC*-sets and Ig*-closed sets are independent.
2. The notions of weakly G-I-LC*-sets and I_g^*-closed sets are independent.
3. The notions of G-I-LC*-sets and Irg*-closed sets are independent.
4. The notions of weakly G-I-LC*-sets and I_{rg}^*-closed sets are independent.

Example 3.13. Let X = {a, b, c, d}, τ = {ϕ, {b}, {b, c, d}, X} and I = {ϕ, {a}}. Then,

1. A = {a, b} is an I_g^*-closed set but not a G-I-LC*-set.
2. A = {a, b} is a G-I-LC*-set but not an I_g^*-closed set.

Example 3.14. Let X = {a, b, c, d, e}, τ = {ϕ, {a, b}, {c, d}, {a, b, c, d}, X} and I = {ϕ, {a}}. Then,

1. A = {b} is a weakly G-I-LC*-set but not an I_g^*-closed set.
2. A = {b, c, d, e} is an I_g^*-closed set but not a weakly G-I-LC*-set.
3. A = {a, b, c, e} is an I_{rg}^*-closed set but not a G-I-LC*-set.
4. A = {a, b, c, d} is a G-I-LC*-set but not an I_{rg}^*-closed set.
5. A = {b} is a weakly G-I-LC*-set but not an I_{rg}^*-closed set.
6. A = {b, c, d, e} is an I_{rg}^*-closed set but not a weakly G-I-LC*-set.
7.

IV. Decompositions of $*$-continuity

Definition 4.1. A function f : (X, τ, I) → (Y, σ) is said to be I_g^*-continuous (resp. I_{rg}^*-continuous, G-I-LC*-continuous, weakly G-I-LC*- continuous) if f^{-1} (V) is I_g^*-closed (resp. I_{rg}^*-closed, G-I-LC*-set, weakly G-I-LC*-set) in (X, τ, I) for every closed set V of (Y, σ).

Theorem 4.2. A function f : (X, τ, I) → (Y, σ) is $*$-continuous if and only if it is weakly G-I-LC*- continuous and I_{rg}^*-continuous.

Proof. This is an immediate consequence of Theorem 3.10.

Definition 4.3. A subset A of a topological space (X, τ) is weakly GLC*-set if A= C ∩ D, where C is rg-open and D is closed.

Definition 4.4. A subset A of a topological space (X, τ) is rg*-closed if cl(A) ⊂ U whenever A ⊂ U and U is rg-open in X.

Definition 4.5. A function f : (X, τ) → (Y, σ) is said to be weakly GLC*- continuous (resp. rg*-continuous) if f^{-1} (V) is weakly GLC*- set (resp. rg*-closed) in (X, τ) for every closed set V of (Y, σ).

Corollary 4.6: Let (X, τ, I) be an ideal topological space and I = {ϕ}, then a function f : (X, τ, I) → (Y, σ) is continuous if and only if it is weakly GLC*- continuous and rg*-continuous.

Theorem 4.7. For a function f: (X, τ, I) → (Y, σ), the following are equivalent:

1. f is $*$-continuous;
2. f is G-I-LC*- continuous and I_g^*-continuous;
3. f is G-I-LC*- continuous and I_{rg}^*-continuous.

Proof. This is an immediate consequence of Theorem 3.11.

Corollary 4.8: Let (X, τ, I) be an ideal topological space and I = {ϕ}, then a function f : (X, τ, I) → (Y, σ), the following are equivalent:

1. f is continuous;
2. f is GLC*- continuous and g*-continuous;
3. f is GLC*- continuous and rg*-continuous.

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