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Legendre-coefficients Comparison Methods for the Numerical Solution of a Class of Ordinary Differential Equations

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Abstract : *We explore the use of Legendre polynomials of the first kind in solving constant coefficients, non-homogenous differential equations. To achieve this, trial solution is formulated with the use of Legendre polynomials as basis functions. We thereafter apply direct and indirect comparison techniques to reduce the entire problem whether initial or boundary value problems into a system of algebraic equations. Numerical examples are given to illustrate the efficiency and good performance of these methods.*
Keywords: *Algebraic equations, Direct comparison, Indirect comparison, Legendre polynomials.*

I. INTRODUCTION

Special attention has over the years been given to applications of orthogonal functions, such as Chebyshev polynomials[1], Laguerre polynomials[2], Legendre polynomials[3], Fourier series[4], block-pulse functions to mention a few. These functions and polynomial series have recieved considerable attention in dealing with various problems in Engineering and Scientific applications [5]. The main characteristic that cuts across these applications is that they reduce these problems to those of solving a system of algebraic equations, thus greatly simplifying the problem[1].

The focus of this paper however, involves solving the n^{th} order differential equation of the form;

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^{n-1} y}{dx^{n-1}}\right) \tag{1}$$

with sufficient conditions attached to the physical boundaries of the problem.
The problem of this sort finds relevant applications in a good number of scientific studies such as in population dynamics, decomposition of radioactive substances and motion of a vibrating body e.t.c. (see[6])

The approach in this paper basically entails substituting into equation (1) a trial solution of the form:

$$\bar{y}_N(x,a) = \sum_{i=0}^{N} a_i \phi_i \tag{2}$$

where N is the order of the trial solution, a_i are specialised cordinates called Degree of Freedom (DOF), $\phi_i(x)$ are orthogonal functions.

As means of obtaining the numerical values of the approximants a_i, coefficients comparison techniques are applied, first in terms of coefficients of independent variable x (Direct comparison) and secondly, in a bid to enhance results produced by this first technique, comparison is likewise carried out in terms of coefficients of Legendre polynomials $P_r(x)$ (Indirect comparison).

II. LEGENDRE POLYNOMIAL

For easy reference, the definitions and certain properties of the Legendre polynomials of the first kind are presented. These polynomials, which are special cases of Legendre functions have the first kind $P_r(x)$ defined as;

$$P_{r+1}(x) = \frac{(2r+1)xP_r(x) - rP_{r-1}(x)}{r+1} \tag{3}$$

with initial conditions $P_0(x) = 1$ and $P_1(x) = x$
from (3), the following are the first few Legendre polynomials;

$P_0(x) = 1$

$P_1(x) = x$

$$P_2(x) = \frac{3x^2 - 1}{2}$$

$$P_3(x) = \frac{5x^3 - 3x}{2}$$

$$P_4(x) = \frac{35x^4 - 30x^2 + 3}{8}$$

One of the techniques established in this paper demands that a differential equation be written purely in Legendre form. To achieve this, a recurrence relation of the form;

$$P'_n(x) = \frac{n(P_{n-1}(x) - xP_n(x))}{(1-x^2)} \tag{4}$$

for $r \geq 1$ is applied for the problem of order 1 (see [7]).

For higher order equations, a recurrence relation for obtaining n^{th} order derivatives of Legendre polynomials is derived as;

$$P_r^{(n)}(x) = \sum_{k=0}^{r} (-1)^{(r+k)} \frac{(r+k)!\, x^{k-n}}{(r-k)!(k!)^2} \tag{5}$$

For the sake of problems that exists in intervals other than the natural interval $-1 \leq x \leq 1$, a shifted version is obtained via the use of;

$$t = \frac{b-a}{2} x + \frac{b+a}{2} \tag{6}$$

which transforms the interval [a,b] into the interval [-1,1]. (see[1] and [8])

Owing to orthogonality property of these polynomials (Legendre) , they play a very important role in the numerical solution of differential equations[9].

III. CONSTRUCTION OF TRIAL SOLUTION

The construction of a trial solution consists of constructing expressions for each of the trial functions $\phi_i(x)$ in equation (2). In choosing expression for these functions, an important practical consideration is the use of functions that are algebraically as simple as possible and easy to work with [10]. Owing to these reasons and its orthogonality properties, Legendre polynomials illustrated in section 2 of this paper is used for the construction of trial functions.

Therefore the applied trial solution takes the form;

$$\bar{y}_N(x,a) = \sum_{r=0}^{N} a_r P_r(x) \tag{7}$$

where $P_r(x)$ are Legendre polynomials of degree r .

IV. SOLUTION TECHNIQUES

The approach in this paper involves substituting (7) into equation (1), to yield residual equation of the form;

$$R(x,a) \Rightarrow \frac{C_n d^n \bar{y}}{dx^n} + \frac{C_{n-1} d^{n-1} \bar{y}}{dx^{n-1}} + \frac{C_{n-2} d^{n-2} \bar{y}}{dx^{n-2}} + \cdots + C_0 \bar{y} = f(x) \; . \tag{8}$$

The subsequent steps demands generating system of equations from which coefficients of expansions a_i could be derived. In achieving this, several researchers have applied a number of techniques such as collocation method , least squares method, method of moments to mention a few. (see[10] and [11])

In this paper, we apply comparison techniques and this is carried out in two forms namely Direct and Indirect comparison techniques.

A. Direct Comparison Technique.

This technique entails comparing the coefficients of independent variable x in equation (8), such that coefficients of variable x of different degrees on the right hand side (RHS) are equated correspondingly to coefficients of x on the left hand side (LHS) of the same residual equation. Through this approach, we generate a system of algebraic equations of the form;

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & \cdots & C_{1N} \\ C_{21} & C_{22} & C_{23} & \cdots & C_{2N} \\ C_{31} & C_{32} & C_{33} & \cdots & C_{3N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \\ C_{N1} & C_{N2} & C_{N3} & \cdots & C_{NN} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ \\ a_N \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ \\ f_N \end{pmatrix} \quad (9)$$

where C_{ij} and f_i are known constants and a_i are the DOFs.

The matrix in (9) is then solved alondside with equations derived from imposition of boundary/initial conditions on (7), thus producing numerical values of a_i. These are thereafter substituted back into trial solution (7) thereby yielding the numerical solution of equations (1).

B. Indirect Comparison Technique.

This technique entails writing the entire differential equation in linear combination of Legendre polynomials $P_r(x)$. This is acheived in such a way that after trial solution (7) is substituted into (1), there exists an inclusion of Legendre polynomials and its derivatives of different degrees n. These derivatives are expressed in terms of Legendre polynomials without derivatives by the use of equation (4) and (5).
At this point, equation (1) is being converted into linear combination of Legendre polynomials of the form;

$$\sum_{r=0}^{\infty} D_r P_r(x) = f(P_0(x), P_1(x), \cdots) \quad (10)$$

where D_r are the coefficients of corresponding Legendre polynomials

and $P_r(x)$ is the Legendre polynomial of order r. It should however be noted that the RHS of (10) is thus converted into Legendre polynomial form by the use of techniques discussed in [1]. The coefficients D_r on the LHS are then equated corresponding to the coefficients of $P_r(x)$ on the RHS. By this approach, a system of linear algebraic equations is generated. This system in conjunction with equations derived from the imposition of boundary conditions are solved to obtian the DOFs, which are thereafter substituted into trial solution (7) thus yielding the numerical solution of differential equation (1).

It is necessary to note that equations derived from the imposition of given conditions with selected equations from coefficients comparison are taylored to yield a system of $N+1$ equations, this is basically to guide against over-determined and under-determined cases (see [12]). In cases where RHS contained non-polynomial function, the function is first expanded through the use of taylor series expansion after which subsequent steps are applied.

V. NUMERICAL EXAMPLES

The described methods of direct and indirect comparisons are in this section applied to a number of examples typical of the class of considered problems (constant coefficients, non-homogenous differential equations) and cutting across problems of order one and higher orders. This in principle is to illustrate the applicability and efficiency of these techniques and also with the aim of comparing their performances. The entire solution technique are automated via the use of symbolic algebraic program – MATLAB.

Example 5.1 Solve the initial value problem;

$$\frac{dy}{dx} - y = x$$
$$y(0) = 1$$

the analytical solution is $2e^x - x - 1$

Example 5.2 Solve the initial value problem;

$$\frac{dy}{dx} - 2y = 3e^x$$
$$y(0) = 0$$

the analytical solution is $3(e^{2x} - e^{x})$

Example 5.3 Solve the boundary value problem;

$$\frac{d^2y}{dx^2} - y = x$$

$$y(0) = y(1) = 0$$

with analytical solution $\left(\frac{\sinh x}{\sinh 1}\right) - x$

Example 5.4 Find the numerical solution of the boundary value problem;

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} - y = e^x - 2x$$

$$y(0) = 1 \quad y''(0) = 1$$

$$y(1) = 2 + e$$

$$y''(1) = e$$

the analytical solution is $2x + e^x$

5.1 Estimation of Numerical errors.

In this subsection, we report some representative results in terms of error accrued from the solution of examples 5.1 – 5.4. The exact errors are computed by the use of the formula;

$$\text{Error} = \left| y(x) - \bar{y}_N(x) \right|$$

Where $y(x)$ is the exact solution and $\bar{y}_N(x)$ is the approximate solution of degree N .

Table 1: Table of errors for Example 5.1

X		N=4	N=6	N=8	N=10
0.0	Direct	0	0	0	0
	Indirect	0	0	0	0
0.1	Direct	1.6948e-007	4.0184e-011	5.5511e-015	0
	Indirect	7.7989e-004	6.1724e-006	2.6012e-008	0
0.2	Direct	5.5163e-006	5.2092e-009	2.8793e-012	8.8818e-016
	Indirect	1.4875e-003	1.0478e-005	3.7061e-008	8.8818e-016
0.3	Direct	4.2615e-005	9.0152e-008	1.1183e-010	9.1260e-014
	Indirect	1.9820e-003	1.1103e-005	2.5257e-008	9.1260e-014
0.4	Direct	1.8273e-004	6.8417e-007	1.5048e-009	2.1738e-012
	Indirect	2.1526e-003	7.6716e-006	1.2115e-009	2.1738e-012
0.5	Direct	5.6754e-004	3.3053e-006	1.1328e-008	2.5525e-011
	Indirect	1.9445e-003	1.6680e-006	2.1509e-008	2.5525e-011
0.6	Direct	1.4376e-003	1.2001e-005	5.9067e-008	1.9130e-010
	Indirect	1.3870e-003	3.7519e-006	1.6311e-008	1.9130e-010
0.7	Direct	3.1637e-003	3.5779e-005	2.3903e-007	1.0518e-009
	Indirect	6.2547e-004	4.8527e-006	1.2843e-008	1.0518e-009
0.8	Direct	6.2819e-003	9.2346e-005	8.0352e-007	4.6096e-009
	Indirect	4.4580e-005	3.7802e-007	3.3771e-008	4.6096e-009
0.9	Direct	1.1531e-002	2.1350e-004	2.3444e-006	1.6990e-008
	Indirect	1.3861e-004	8.3659e-006	1.0617e-008	1.6990e-008
1.0	Direct	1.9897e-002	4.5255e-004	6.1172e-006	5.4625e-008
	Indirect	1.0592e-003	4.1410e-006	1.0654e-008	5.4625e-008

Table 2: Table of erros for Example 5.2

X		N=4	N=6	N=8	N=10
0.0	Direct	0	0	0	0
	Indirect	0	0	0	0
0.1	Direct	8.0203e-006	1.1920e-008	4.3548e-009	4.3506e-009
	Indirect	4.6164e-002	1.3833e-003	2.2562e-005	4.3506e-009
0.2	Direct	2.6582e-004	1.2851e-006	2.9311e-007	2.9093e-007
	Indirect	9.3495e-002	2.5140e-003	3.5200e-005	2.9093e-007
0.3	Direct	2.0925e-003	2.0903e-005	3.5506e-006	3.4651e-006
	Indirect	1.3436e-001	2.9692e-003	3.2686e-005	3.4651e-006
0.4	Direct	9.1487e-003	1.5456e-004	2.1537e-005	2.0376e-005
	Indirect	1.6208e-001	2.5879e-003	3.0908e-005	2.0376e-005
0.5	Direct	2.8994e-002	7.3896e-004	9.0248e-005	8.1439e-005
	Indirect	1.7251e-001	1.5822e-003	7.6646e-005	8.1439e-005
0.6	Direct	7.4994e-002	2.6776e-003	3.0144e-004	2.5516e-004
	Indirect	1.6593e-001	5.1962e-004	2.5503e-004	2.5516e-004
0.7	Direct	1.6865e-001	8.0074e-003	8.6510e-004	6.7646e-004
	Indirect	1.4937e-001	1.0266e-004	7.0030e-004	6.7646e-004
0.8	Direct	3.4247e-001	2.0802e-002	2.2272e-003	1.5888e-003
	Indirect	1.3948e-001	6.4707e-004	1.6244e-003	1.5888e-003
0.9	Direct	6.4345e-001	4.8527e-002	5.2809e-003	3.4063e-003
	Indirect	1.6605e-001	1.1449e-003	3.3901e-003	3.4063e-003
1.0	Direct	1.1373e+000	1.0399e-001	1.1728e-002	6.8069e-003
	Indirect	2.7621e-001	2.2308e-003	6.6739e-003	6.8069e-003

Table 3: Table of errors for Example 5.3

X		N=4	N=6	N=8	N=10
0.0	Direct	0	0	0	0
	Indirect	0	0	0	0
0.1	Direct	6.2344e-004	1.4594e-005	2.0169e-007	1.8287e-009
	Indirect	7.0245e-004	5.4295e-006	2.2779e-008	1.7893e-009
0.2	Direct	1.2510e-003	2.9333e-005	4.0541e-007	3.6756e-009
	Indirect	1.3205e-003	9.2506e-006	3.3312e-008	3.5211e-009
0.3	Direct	1.8782e-003	4.4332e-005	6.1313e-007	5.5593e-009
	Indirect	1.7781e-003	1.0309e-005	2.5872e-008	5.0954e-009
0.4	Direct	2.4834e-003	5.9570e-005	8.2647e-007	7.4978e-009
	Indirect	2.0166e-003	8.2713e-006	4.8077e-009	6.3595e-009
0.5	Direct	3.0191e-003	7.4601e-005	1.0447e-006	9.5027e-009
	Indirect	2.0032e-003	3.8251e-006	1.6993e-008	7.1420e-009
0.6	Direct	3.4028e-003	8.8011e-005	1.2582e-006	1.1545e-008
	Indirect	1.7401e-003	1.3875e-006	2.5113e-008	7.3082e-009
0.7	Direct	3.5074e-003	9.6525e-005	1.4324e-006	1.3426e-008
	Indirect	1.2739e-003	5.1820e-006	1.3891e-008	6.7867e-009
0.8	Direct	3.1517e-003	9.3669e-005	1.4717e-006	1.4375e-008
	Indirect	7.0548e-004	5.6894e-006	5.7264e-009	5.5167e-009
0.9	Direct	2.0897e-003	6.7884e-005	1.1518e-006	1.2016e-008
	Indirect	2.0044e-004	2.7577e-006	1.0933e-008	3.3353e-009
1.0	Direct	0	0	0	1.2309e-017
	Indirect	0	0	1.2739e-017	1.4410e-017

Table 4: Table of errors for Example 5.4

X		N=4	N=6	N=8	N=10
0.0	Direct	0	0	0	0
	Indirect	0	0	0	0
0.1	Direct	1.7147e-003	1.0437e-004	2.7046e-006	3.7884e-008
	Indirect	2.1368e-003	9.7659e-005	4.6582e-007	1.6750e-009
0.2	Direct	3.3100e-003	2.0304e-004	5.2726e-006	7.3925e-008

	Indirect	4.1087e-003	1.8665e-004	8.6086e-007	3.0201e-009
0.3	Direct	4.6575e-003	2.8936e-004	7.5445e-006	1.0597e-007
	Indirect	5.7509e-003	2.5836e-004	1.1252e-006	3.8116e-009
0.4	Direct	5.6291e-003	3.5584e-004	9.3362e-006	1.3153e-007
	Indirect	6.9096e-003	3.0532e-004	1.2259e-006	4.0219e-009
0.5	Direct	6.1099e-003	3.9446e-004	1.0443e-005	1.4780e-007
	Indirect	7.4546e-003	3.2225e-004	1.1668e-006	3.8052e-009
0.6	Direct	6.0130e-003	3.9748e-004	1.0648e-005	1.5173e-007
	Indirect	7.2936e-003	3.0693e-004	9.8544e-007	3.3756e-009
0.7	Direct	5.2946e-003	3.5872e-004	9.7493e-006	1.4022e-007
	Indirect	6.3880e-003	2.6073e-004	7.3892e-007	2.8501e-009
0.8	Direct	3.9716e-003	2.7547e-004	7.6063e-006	1.1068e-007
	Indirect	4.7703e-003	1.8865e-004	4.8036e-007	2.1709e-009
0.9	Direct	2.1408e-003	1.5132e-004	4.2399e-006	6.2459e-008
	Indirect	2.5629e-003	9.8618e-005	2.3601e-007	1.2086e-009
1.0	Direct	0	0	0	0
	Indirect	0	0	0	0

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VI. Conclusion

In this paper, we gave a numerical treatment of a class of differential equation both in initial and boundary value problems through Legendre polynomials applied via the use of two kinds of comparison techniques namely direct and indirect. Considering the result produced, it is noticed that these two techniques are highly effective for the class of problem considered, that is, constant coefficient, non-homogeneous linear differential equations. Also from the achieved accuracies as depicted through the tabulated errors, it is further observed that as the degree of trial solution N increases, better accuracies were obtained thereby minimizing the errors.

Also observed, is the fact that indirect comparison technique facilitates an evenly distribution of errors across the interval of consideration, thereby serving as an improvement to direct comparison method especially on the note of an even distribution of obtained errors.

Through this paper, the two techniques of direct or indirect comparison have been demonstrated to yield result close enough to the exact solution as to be useful in application, and their computational cost are likewise minimal when compared to a good number of existing methods like Galekin weighted residual method, collocation method, methods of moment, finite difference method to mention a few.
We therefore suggest an extension to other classes of differential and integral equations.

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