On The Integrated Formulation of the Tau Method Involving At Most Two Tau Parameters for Ivps in Odes

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Abstract: An error estimation of the integrated variant of the tau method for ordinary differential equations is hereby considered for the class of equations characterized by \( m+s \leq 2 \) where \( m \) and \( s \) are, respectively, the order and the number of overdetermination of the differential equation. Some general results are obtained and applied to test problems. Numerical evidences show that the estimate adequately captures the order of the tau approximation.

Key Words: Tau Method, Overdetermination, perturbation, class, Error estimation, Variant

I. Introduction

The tau method was originally developed by Lanzos (1938) for the solution of the \( m \)-th order problem

\[
Ly(x) \equiv \sum_{r=0}^{m} P_r(x) y^{(r)}(x) = 0
\]

With the conditions

\[
L \ast y(x_{rk}) \equiv \sum_{r=0}^{m} a_{rk}(x) y^{(r)}(x_{rk}) a_k \quad k \quad m
\]

Where \( y^{(r)}(x) \) stands for the derivatives of order \( y \), \( f(x) \) and \( P_r(x) \), \( r = 0, 1, \ldots, m \), are polynomials (or polynomial approximations immediately derivable by using the tau method) of given function; where \( a_{rk} \), \( x_{rk} \), and \( a_k \) are given real numbers.

The method solves problem (1) by seeking an approximation:

\[
y_n(x) = \sum_{r=0}^{m} a_r x^r
\]

This is the exact solution of the perturbed form:

\[
Ly_n(x) \equiv \sum_{r=0}^{m} P_r(x) y_n^{(r)}(x) f(x) H_n(x)
\]

With the conditions stated by Eq.(2)

\[
P_r(x) \sum_k P_{rk} x^k
\]

\( H_n(x) \) is a linear combination of Chebyshev polynomials valid in the interval \([a, b]\) and it may be of the form

\[
H_n(x) \sum_{r=0}^{m} T_{m-s-r} x
\]

The parameters in Eq. (6) is the number of over-determinations of Eq.(4). \( T_r \)'s are the tau parameters to be determined. By inserting (3) into Eq. (4) and then applying the conditions (2), we get the system of linear equation in \((n+m+s+1)\) unknown constants \( a_r \), \( r=0(1)n \), \( T_r \) \( T_0 \ldots, T_m \). This system is then solved to obtain the \((n+m+s+1)\) unknown constants which are to be substituted into Eq. (2) in order to get our approximate solution of Eq. (1).

Lanzos introduced the use of the canonical polynomials \( Q_r(x) \), \( r \geq 0 \),

\[
LQ_r(x) = x^r
\]

Where \( L \) is the linear operator

\[
L \sum_{r=0}^{m} P_r(x) \frac{d^r}{dx^r}
\]

The expression of the approximate solution \( y_n(x) \) in terms of a canonical polynomials offers several advantages because it does not depend on the boundary condition of the problems which we want to solve nor
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The interval in which the solution is sought, allowing for every segmentation of the domain. They are permanent in the sense that if an approximation of higher degree is required, the computation does not need to be repeated from the beginning. Furthermore, the tau method does not require a stage of discretisation of the given differential operator; as discrete variable method do.

A recursive generation of polynomial was introduced by Ortiz (1969) to give some flexibility in the computation of the conical polynomials.

An approach developed for an improved accuracy of the approximation \( y_n \), is the integrated function, whereby we first integrate through Eq. (1), to have

\[
I_L(\tau_n(x)) \int \int m \int \sum_{r} P_r(x) y^{(r)}(x) \, dx \, dx \, dx
\]

Thus,

\[
I_L(\tau_n(x)) \int \int m \int \sum_{r} f_r x^r \sum_{r} T_{m+n-r} T_{n-m-r} x \, dx \, dx \, dx H_{n,m} x
\]

the higher order of the perturbation in Eq. (10) account for the improvement in accuracy of \( y(x) \) in contrast of the differential and recursive formulations.

1.1 Definition of Terms

Definition 1.1.1
A differential equation (or a system of differential equation) together with its associated given conditions will be referred to a Differential system.

Definition 1.1.2
The differential system

\[
L y_n(x) \equiv \sum_{r} P_r(x) y^{(r)}(x) f(x) H_n(x)
\]

will be called the Tau problem corresponding to the differential system

\[
L y(x) \sum_{r} P_r(x) y^{(r)}(x) f(x)
\]

We call the n-th degree polynomial, \( y_n(x) \), which satisfies the Tau problem (12), the tau approximant of (13) and the tau solution of Eq. (12) resulting in the process of solution of (13) will be referred to as tau system of problem (12).

Definition 1.1.3
The system of equation \( A \tau = B \) where \( \tau = (a, a, a, \ldots, a_n, \tau, \tau, \ldots, \tau_m) \).

Resulting the process of solution tau of (13) will be referred to as Tau System of (12).

Definition 1.1.4
The number of over-determination of the DE (1) is defined by

\[
S = \max \{N_r - r : 0 \leq r \leq m\} \geq f or \ N_r \geq r
\]

II. Derivation of Tau Approximant

We consider here the derivation of tau approximants of varying orders and degrees, for the class of problem:

\[
L y(x) \sum_{r} P_r(x) y^{(r)}(x) f(x) a \leq x \leq b
\]

\[
L y(x) : \sum_{r} a_{r+} y^{(r)}(x) \alpha_k \quad k \quad (m-)
\]

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where,

\[ P_r(x) \sum_{k=0}^{N_r} p_{rk} x^k \]

The amount of work as well as the size of the space this will involve is enormous; we shall illustrate the procedure for a fifth degree approximant and then provide only the results for approximants of other degrees. In the work, we shall derive a fifth degree approximant for

\[ L y_n(x) \equiv \sum_{r=0}^{m} P_r(x) y^{(r)}(x) \sum_{r}^{n} f_r x^r H_{mn}(x) \]

\textbf{The case} \( m = 1, s = 0 \)

From (1) above, the general case for \( m=1 \) and \( s=0 \) is given by;

\[ L y(x) \equiv (P x y(x) P y(x) \sum_{i}^{n} f_i x^r \tau T_n(x) \]

And from (11) we have

\[ I_{2} y(x) = \int_{-\infty}^{\infty} (P x t) dt \int_{-\infty}^{\infty} y(t) dt \int_{-\infty}^{\infty} \sum_{r}^{n} f_r t^r dt \tau T_n(x) \]

where,

\[ T_n(x) = \sum_{r}^{n} C_r x^r \]

This leads to

\[ P [y_n(x) - \alpha_] \sum \sum_{r}^{n} a_r x^r \]

We seek an approximant solution of the form

\[ y_n(x) = \sum_{r}^{n} a_r x^r \]

With (23), (22) now becomes,

\[ P \sum_{r}^{n} a_r x^r - P \sum_{r}^{n} a_r x^r \]

\[ \sum_{r}^{n} f_r x^r - \tau \sum_{r}^{n} C_r x^r \]

This gives

\[ P \sum_{r}^{n} a_r x^r \sum_{r}^{n} \left[ \frac{P}{r} r^P \right] a_r x^r - \tau \sum_{r}^{n} C_r x^r \sum_{r}^{n} f_r x^r \]

Thus, for example, when \( n = 5 \), we have:

\[ P \sum_{r}^{n} a_r x^r \sum_{r}^{n} \left[ \frac{P}{r} r^P \right] a_r x^r - \tau \sum_{r}^{n} C_r x^r \sum_{r}^{n} f_r x^r \]

Equating corresponding coefficients power of \( x \), we obtain the tau system

\[ \left( \begin{array}{cccccccc}
\beta_{25} & 0 & 0 & 0 & 0 & 0 & -c_{25} & 0 \\
\beta_{26} & \beta_{27} & \beta_{28} & \beta_{29} & 0 & 0 & 0 & -c_{26} \\
0 & 0 & 0 & 0 & 0 & 0 & -c_{27} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -c_{28} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -c_{29} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right) \left( \begin{array}{c}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5 \\
a_6 \\
a_7 \\
\end{array} \right) = \left( \begin{array}{c}
\delta_{25} \\
\delta_{26} \\
\delta_{27} \\
\delta_{28} \\
\delta_{29} \\
\delta_{30} \\
\delta_{31} \\
\delta_{32} \\
\end{array} \right) \]

Continuing with the process, using \( m = 1 \) and \( s = 1 \). By expanding (27), we obtain the following tau system

For \( m=1 \), \( s=1 \), we have,
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where

\[ V_{32} = \frac{p}{p}, \quad V_{32} = \frac{p}{p}, \quad V_{43} = \frac{p}{p}, \quad V_{53} = \frac{p}{p}, \quad V_{54} = \frac{p}{p}, \quad V_{65} = \frac{p}{p}, \quad V_{75} = \frac{p}{p}, \quad V_{76} = \frac{p}{p}, \quad V_{86} = \frac{p}{p} \]

The case \( m = 2, s = 0 \)

From (1) the general case for \( m = 2 \) and \( s = 0 \) is given by

\[
Ly(x) = (P \ P x \ P x) \frac{y(x)}{y(x)} \quad (P \ P x) y(x) \quad P \ y(x) \quad \sum_r f_r x^r \ H_{n,m}(x) \\
y(0) = \alpha, \quad y'(0) = \alpha, \quad \alpha \leq x \leq b
\]

where,

\[
H_{n,m}(x) = \int_0^x \int_0^x (P \ P t \ P t) y_n(t) dtdu \int_0^x \int_0^x (P \ P t \ y_n(t) dtdu
\]

\[
\int_0^x \sum_r f_r x^r \ \tau T_n \ x \ \tau T_n \ x
\]

We integrate the terms in (31) to have,

\[
-P \ \alpha - P \ \alpha \ P \ \alpha \ x \ P \ \sum_r a_r x^r \ P \ \sum_r a_r x^r - P \ \sum_r a_r x^r - P \ \sum_r a_r x^r - P \ \sum_r a_r x^r - P \ \sum_r a_r x^r - P \ \sum_r a_r x^r
\]

\[
\Sigma_r f_r \ x^r \tau
\]

This gives,

\[
\Sigma_r P \ a_r x^r \ \sum_r \left[ \frac{(r-1)p}{r} \right] \ a_r x^r \ \Sigma_r \left[ \frac{r(p \ r)}{r} \right] \ a_r x^r - \tau \ \sum_r C_r x^r - \tau \ \sum_i C_i x^r \ P \ \alpha \ \left[ (P - P) \ \alpha \ P \ \alpha \right] x
\]

Equating the corresponding coefficients of powers of \( x \) in (33) when \( n = 5 \) we have the tau system

\[
D = \begin{pmatrix}
\frac{p}{p} & -C^{(1)} & -C \\
\frac{p}{p} & -C^{(1)} & -C \\
R & R & P \\
\end{pmatrix}, \quad E = \begin{pmatrix}
\left( \begin{array}{c}
a \\
a \\
a \\
a \\
a \\
\end{array} \right)
\end{pmatrix}, \quad F = \begin{pmatrix}
P \ \alpha \\
f / f \\
f / f \\
f / f \\
f / f \\
\end{pmatrix}
\]

(34)

Then, for \( m = 2, s = 0 \) and \( n = 5 \) we have

\[
DE = F
\]

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\[ R_{42} = p_{42}, \quad R_{43} = p_{43}, \quad R_{44} = p_{44}, \quad R_{45} = p_{45}, \quad R_{55} = p_{55}, \quad R_{66} = p_{66}. \]

and,

\[ a_{4k} = P_{4m} \forall k = 1(1)(n+1) \forall m. \]  \hspace{1cm} \text{(36)}

\[ a_{k\pi} \]

\[ a_{k\pi} = \begin{cases} 
C_{k_\pi} & \text{k = 1(1)(n+3)} \\
C_{k_\pi} & \text{k = 1(1)(n+2)} 
\end{cases} \]

\[ b \frac{p_{m}}{(m-1)} \left\{ \sum_r \frac{r}{r} - \frac{r r}{r} P_r r \right\} \quad \forall m \]

\[ b_i \]

\[ b_i \left\{ \begin{array}{ll}
\phi_{i-1} & \forall m \forall i (n) \\
\phi_{i-1} & \forall m \forall i (n) 
\end{array} \right. \]

III. Procedure For Error Estimation

The integrated formulation of the tau method often leads to better accuracy of the tau solution. (See Fox(1962) and Ortiz(1993)). To this end, let \( \int \ldots \int g(x)dx \) denote the indefinite integration i times applied to the function \( g(x) \) and let

\[ I_{L} \int \ldots \int L(\ )dx \]

The integration form of

\[ L I_{n}(x) = -H_{n}(x)dx \]

is therefore

\[ I_{L} \left( I_{n}(x) \right) = - \int \ldots \int H_{n} x \ dx \]

We considered the perturbed form of (42) i.e. the perturbed integrated error equation

\[ \left( I_{n}(x) \right)_{n} = - \int \ldots \int H_{n} x \ dx \bar{H}_{m n} x \]

which is equivalent to

\[ \left( I_{n}(x) \right)_{n} = - \int \ldots \int \int \sum_{r} \sum_{s} T_{n} x \ dx \bar{H}_{m n} x \]

and which is satisfied by \( \left( e_{n}(x) \right)_{n} \), given by

\[ \left( e_{n}(x) \right)_{n} = \frac{\mu_{m}(x) \Phi_{n} T_{n-m} x}{C_{n-m}} \]

With \( \Phi_{n} \) replaced by \( \Phi_{n} \) where

\[ \bar{H}_{m n} (x) \] 

\[ \bar{H}_{m n} (x) \]

We insert (48) in (47) and then equate the coefficient of \( x^{m+n} \), \( x^{m+n} \), \ldots, \( x^{n-m} \)
for the determination of the parameter \( \bar{\theta}_n \) of \( \{ e_n(x) \} \). We then have
\[
e^{-\frac{\bar{\theta}_n}{n-m}}
\]
as an estimation of \( \varepsilon \).
We shall carry out these steps for obtaining \( \bar{\theta}_n \) with various values of \( m \) and \( s \) and then generalize the result to obtain a recursive formula for \( \bar{\theta}_n \).

IV. Error Estimation For The Integrated Formulation

The case \( m = 1, s = 0 \)

From (46) we have for the problem:
\[
Ly(x) = (P P x)y' + P x y + \sum_{r=0}^{P} f_r x^r, \quad a \leq x \leq b
\]

The equation:
\[
Ly(x) := \int_{a}^{x} (P P u) e_n'(u) du - \int_{a}^{x} P e_n(u) du
\]

where
\[
\left( e_n(x) \right)_n = \frac{xT_n x \bar{\theta}_n}{C_n^n}
\]

that is,
\[
\left( e_n(x) \right)_n = \frac{\bar{\theta}_n}{k} K x^n K x^n K x^n - \ldots
\]

Now,
\[
\int_{a}^{x} \left( e_n(x) \right)_n \bar{\theta}_n \left[ \frac{k}{n} x^n \frac{k}{n} x^n \frac{k}{n} x^n \right]
\]

Inserting (54) and (55) into (52) gives
\[
\frac{\bar{\theta}_n}{k} \left[ \lambda x^n \lambda x^n \ldots \right] \bar{\pi} C_n^n x^n \left[ \lambda C_n^n - \frac{\bar{\pi} C_n^n}{n} \right] x^n \ldots
\]

where,
\[
\lambda = \left[ P \left( \frac{(n-P)}{n} \right) P \right] k \lambda P k \left[ P \left( \frac{(n-P)}{n} \right) P \right] k
\]

Equating coefficient of corresponding powers of \( x \) from both sides of (52), gives
\[
\bar{\pi} C_n^n = \frac{\bar{\theta}_n}{k}
\]
\[
\bar{\pi} C_n^n - \frac{\tau k}{n} \lambda \frac{\bar{\theta}_n}{k}
\]

From (57) we have
\[
\bar{\pi} \frac{\bar{\theta}_n}{k} C_n^n
\]

Inserting this into (59) gives
\[
\bar{\theta}_n = \frac{\tau k}{R}
\]

where,
\[
R = \lambda - \frac{\lambda C_n^n}{C_n^n}
\]

Let \( R_1 = \lambda \), then \( R_2 \) can be put in the following recursive form:
\[
R_i = \lambda
\]

(63)
Thus, we have the following recursive form

Thus, Inserting \((e_n(x))_n\) and its integral into (65) gives

where \(\tau\) is defined by (48).

Equating coefficients of corresponding powers of from both sides of (66), we obtain the following system of equations

From (68) we have

Inserting (71) into (69) and solving for \(\tau\) we have

Inserting () and () into () and solving for we obtain

where

Thus, we have the following recursive form,
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\[ R \lambda - \frac{e^n}{C_a^n} R \]

We repeat the analysis discussed above with \( m = 2 \) and thus, we obtain the following results:

**For** \( m = 2, s = 0 \)

\[ \phi_n = -k r m \frac{1}{n^m} \]

Thus, we obtain general expression for \( \phi_n \) as:

\[ \phi_n \left( \frac{m}{s} r - R_m s \right) \forall m s \]

Where \( k = C_n m = \frac{m-s}{m} \)

\[ \phi_n \left( \frac{m}{s} r - R_m s \right) \forall m s \]

Where \( R_{m+1} \) is given recursively in terms of \( R_1, R_2, R_3, \ldots, R_{m+1} \) as follows:

\[ R_1 = \lambda \]

\[ R_u \lambda_u = \sum_{i=1}^{n} \left( \frac{C_n u-i}{C_n u-i} R_i \right) u \ldots m s \]

and

\[ \lambda_u \sum_{i=1}^{n} \left( \frac{\sum_{j=m}^{n} \left( \prod_{r=0}^{m} \left( n-r \right) \right) \left( \begin{pmatrix} n \ n \ s \ m \ -u-r \end{pmatrix} k_i u \ldots m s \right)}{\prod_{r=0}^{m} \left( n-r \right) \right) k_i u \ldots m s \right) \]

Provided \( i \geq u-m+1 \)

Thus, from (50), we have the following expression for \( \varepsilon^* \):

\[ \varepsilon^* \left( \frac{\prod_{r=0}^{m} \left( n-r \right) \right) \left( \begin{pmatrix} n \ n \ s \ m \ -u-r \end{pmatrix} k_i u \ldots m s \right)}{\prod_{r=0}^{m} \left( n-r \right) \right) \left( \begin{pmatrix} n \ n \ s \ m \ -u-r \end{pmatrix} k_i u \ldots m s \right)} \forall m s \]

\[ \varepsilon^* \left( \frac{\prod_{r=0}^{m} \left( n-r \right) \right) \left( \begin{pmatrix} n \ n \ s \ m \ -u-r \end{pmatrix} k_i u \ldots m s \right)}{\prod_{r=0}^{m} \left( n-r \right) \right) \left( \begin{pmatrix} n \ n \ s \ m \ -u-r \end{pmatrix} k_i u \ldots m s \right)} \forall m s \]

\[ \varepsilon^* \left( \frac{\prod_{r=0}^{m} \left( n-r \right) \right) \left( \begin{pmatrix} n \ n \ s \ m \ -u-r \end{pmatrix} k_i u \ldots m s \right)}{\prod_{r=0}^{m} \left( n-r \right) \right) \left( \begin{pmatrix} n \ n \ s \ m \ -u-r \end{pmatrix} k_i u \ldots m s \right)} \forall m s \]

**V. Numerical Examples**

We consider here some selected examples for experimentation with our results of the preceding section for \( m+s = 1 \) and \( m+s = 2 \), the exact error is defined as

\[ \xi = \max_{x \in [0,1]} \left| y(x) - e^x \right| \]

\[ 0 \leq x \leq 1 \]

The numerical results are presented in the tables below the examples

**Problem 4.1**

\[ y'(x) - y(x) = 0 \]

\[ y(1) = 1, \ y'(1) = 1, \ 0 \leq x \leq 1 \]

**Table 4.1**

<table>
<thead>
<tr>
<th>Error</th>
<th>n</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimates</td>
<td>4.61 x 10^{-3}</td>
<td>7.50 x 10^{-3}</td>
<td>9.46 x 10^{-3}</td>
<td>5.34 x 10^{-3}</td>
<td></td>
</tr>
<tr>
<td>Exact</td>
<td>3.67 x 10^{-3}</td>
<td>7.60 x 10^{-3}</td>
<td>9.53 x 10^{-3}</td>
<td>6.36 x 10^{-3}</td>
<td></td>
</tr>
</tbody>
</table>

**Problem 4.2**

\[ y'(x) - xy(x) = 1 - x \]

\[ y(0) = 0 \]

\[ 0 \leq x \leq 1 \]

**Table 4.2**

<table>
<thead>
<tr>
<th>Error</th>
<th>n</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimates</td>
<td>6.01 x 10^{-3}</td>
<td>6.7 x 10^{-3}</td>
<td>3.24 x 10^{-3}</td>
<td>4.6 x 10^{-3}</td>
<td></td>
</tr>
<tr>
<td>Exact</td>
<td>9.34 x 10^{-3}</td>
<td>7.11 x 10^{-3}</td>
<td>4.89 x 10^{-3}</td>
<td>8.53 x 10^{-3}</td>
<td></td>
</tr>
</tbody>
</table>
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Problem 4.3
\[ y(x) = \begin{cases} \sin x, & 0 \leq x \leq \pi/2 \\ \cos x, & \pi/2 < x \leq \pi \end{cases} \]

Table 4.3

<table>
<thead>
<tr>
<th>Error</th>
<th>n</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimates</td>
<td>2</td>
<td>3.51 x 10^1</td>
<td>4.60 x 10^2</td>
<td>5.65 x 10^3</td>
<td>9.02 x 10^4</td>
</tr>
<tr>
<td>Exact</td>
<td>2</td>
<td>1.34 x 10^1</td>
<td>2.37 x 10^2</td>
<td>3.1 x 10^3</td>
<td>4.67 x 10^4</td>
</tr>
</tbody>
</table>

Problem 4.4
\[ y''(x) + 2y'(x) + 3y(x) = 0, \quad y(0) = 3, \quad y'(0) = 1 \]

Table 4.4

<table>
<thead>
<tr>
<th>Error</th>
<th>n</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimates</td>
<td>2</td>
<td>6.46 x 10^1</td>
<td>4.23 x 10^2</td>
<td>7.0 x 10^3</td>
<td>3.85 x 10^4</td>
</tr>
<tr>
<td>Exact</td>
<td>2</td>
<td>8.84 x 10^1</td>
<td>1.06 x 10^2</td>
<td>2.86 x 10^3</td>
<td>2.14 x 10^4</td>
</tr>
</tbody>
</table>

VI. Conclusion

The integrated formulation of the tau method and its error estimation have been generalized for those ODEs, whose perturbed form involves a maximum of two tau parameters and consequently a maximum overdetermination number, s, and to one.

The tau system for the determination of the tau approximation \( y_n, x \) was first constructed. The error estimation which followed immediately provides the estimate of the error in \( y_n, x \). Numerical evidences, obtained for some selected problems, revealed that the estimate accurately captures the order of the exact error.

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