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Po-Γ-Ideals in Po-Γ-Semigroups

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ABSTRACT **:** *In this paper the terms; a completely prime po-Γ-ideal, c-system,a prime po-Γ-ideal, m-system of a po-**Γ**-semigroup are introduced. It is proved that every po- **Γ**-subsemigroup of a po-**Γ**-semigroup is a c-system. It is also proved that a po-**Γ**-ideal P of a po-**Γ**-semigroup S is completely prime if and only if S\P is either a c-system or empty. It is proved that if P is a po-Γ-ideal of a po-Γ-semigroup S, then the conditions (1) if A, B are po-Γ-ideals of S and AΓB⊆P then either A⊆P or B⊆P, (2) if a, b ∈ S such that a*$\Gamma S^1\Gamma b \subseteq P$*, then either a ∈ P or b ∈ P, are equivalent. It is proved that every completely prime po-**Γ**-ideal of a po-**Γ**-semigroup S is a prime po-**Γ**-ideal of S. It is also proved that in a commutative po-**Γ**-semigroup S, a po-**Γ**-ideal P is a prime po-**Γ**-ideal if and only if P is a completely prime po-**Γ**-ideal. Further it is proved that a po-**Γ**-ideal P of a po-**Γ**-semigroup S is a prime po-**Γ**-ideal of S if and only if S\P is an m-system or empty. In a globally idempotent po-Γ-semigroup, it is proved that every maximal po- Γ-ideal is a prime po- Γ-ideal. It is also proved that a globally idempotent po-Γ-semigroup having a maximal po-Γ-ideal, contains semisimple elements. The terms completely semiprime po- Γ-ideal,a semiprime po-Γ-ideal, n-system, d-system are introduced. It is proved that (1) every completely semiprime po-Γ-ideal of a po-Γ-semigroup is a semiprime po-Γ-ideal, (2) every completely prime po- Γ-ideal of a po- Γ-semigroup is a completely semiprime po-Γ-ideal. It is also proved that the nonempty intersection of any family of (1) completely prime po-Γ-ideals of a po-Γ-semigroup is a completely semiprime po-Γ-ideal, (2) prime po- Γ-ideals of a po-Γ-semigroup is a semiprime po-Γ-ideal. It is also proved that a po- Γ-ideal Q of a po-Γ-semigroup S is a semiprime iff S\Q is either an n-system or empty. Further it is proved that if N is an n-system in a po-Γ-semigroup S and a ∈ N, then there exists an m-system M of S such that a ∈ M and M ⊆ N. Mathematics Subject Clasification (2010) : 06F05, 06F99, 20M10, 20M99*

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I . Introduction

Γ-semigroup was introduced by Sen and Saha [16] as a generalization of semigroup. Anjaneyulu. A [1], [2] and [3] initiated the study of ideals and radicals in semigroups. Many classical notions of semigroups have been extended to Γ-semigroups by Madhusudhana Rao, Anjaneyulu and Gangadhara Rao [11]. The concept of po-Γ-semigroup was introduced by Y. I. Kwon and S. K. Lee [10] in 1996, and it has been studied by several authors. In this paper we introduce the notions of a po-Γ-semigroups and characterize po-Γ-semigroups.

II . PRELIMINARIES

DEFINITION 2.1 : Let S and Γ be two non-empty sets. Then S is called a ***Γ-semigroup*** if there exist a mapping from $S \times \Gamma \times S$ to S which maps $(a, \alpha, b) \rightarrow a\alpha b$ satisfying the condition : $(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a, b, c \in S$ and $\gamma, \mu \in \Gamma$.

NOTE 2.2 : Let S be a Γ-semigroup. If A and B are two subsets of S, we shall denote the set $\{ a\gamma b : a \in A, b \in B \text{ and } \gamma \in \Gamma \}$ by AΓB.

DEFINITION 2.3: A Γ-semigroup S is said to a po-Γ-semigroup if S is a po- set such that $a \leq b \Rightarrow a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$ $\forall a,b,c \in S$ and $\gamma \in \Gamma$.

NOTE 2.4: A partially ordered Γ-semigroup simply called a po-Γ-semigroup or ordered Γ-semigroup.

DEFINITION 2.5 : An element ***a*** of a po- Γ-semigroup S is said to be a ***left identity*** of S provided $a\alpha s = s$ and $s \leq a$ for all $s \in S$ and $\alpha \in \Gamma$.

DEFINITION 2.6 : An element ***a*** of a po-Γ-semigroup S is said to be a ***right identity*** of S provided $s\alpha a = s$ and $s \leq a$ for all $s \in S$ and $\alpha \in \Gamma$.

DEFINITION 2.7 : An element '***a***' of a po-Γ-semigroup S is said to be a ***two sided identity*** or an ***identity*** provided it is both a left identity and a right identity of S.

NOTE 2.8 : An element '***a***' of a po-Γ-semigroup S is said to be a ***two sided identity*** or an ***identity*** provided $s\alpha a = a\alpha s = s$ and $s \leq a$ for all $s \in S$ and $\alpha \in \Gamma$.

THEOREM 2.9 : Any po-Γ-semigroup S has at most one identity.

NOTE 2.10 : The identity (if exists) of a po-Γ-semigroup is usually denoted by 1 or e.

DEFINITION 2.11 : An element a of a po-Γ-semigroup S is said to be a ***left zero*** of S provided $a\alpha s = a$ and $a \leq s$ for all $s \in$ S and $\alpha \in \Gamma$.

DEFINITION 2.12 : An element a of a po-Γ-semigroup S is said to be a ***right zero*** of S provided $s\alpha a = a$ and $a \leq s$ for all $s \in$ S and $\alpha \in \Gamma$.

DEFINITION 2.13 : An element a of a po-Γ-semigroup S is said to be a ***two sided zero*** or ***zero*** provided it is both a left zero and a right zero of S.

NOTE 2.14 : An element a of a po-Γ-semigroup S is said to be a ***two sided zero*** or ***zero*** provided $a\alpha s = s\alpha a = a$ and $a \leq s$ for all $s \in$ S and $\alpha \in \Gamma$.

DEFINITION 2.15 : A po-Γ-semigroup in which every element is a left zero is called a ***left zero po-Γ-semigroup***.

DEFINITION 2.16 : A po-Γ-semigroup in which every element is a right zero is called a ***right zero po-Γ-semigroup***.

DEFINITION 2.17 : A po- Γ-semigroup with 0 in which the product of any two elements equals to 0 is called a ***zero po-Γ-semigroup*** or a ***null po-Γ-semigroup***.

NOTATION 2.18 : Let S be a po-Γ-semigroup and T is a nonempty subset of S. If H is a nonempty subset of T, we denote the set $\{t \in$ T : $t \leq h$ for some $h \in$ H$\}$ by $(H]_T$. The $\{t \in$ T : $h \leq t$ for some $h \in$ H$\}$ by $[H)_T$. Also $(H]_s$ and $[H)_s$ are simply denoted by (H] and [H) respectively.

DEFINITION 2.19 : Let S be a po-Γ-semigroup. A nonempty subset T of S is said to be a po-**Γ-*subsemigroup*** of S if $a\gamma b \in$ T, for all a, $b \in$ T and $\gamma \in \Gamma$ and $t \in$ T, $s \in$ S, $s \leq t \Rightarrow s \in$ T.

THEOREM 2.20 : A nonempty subset T of a po-Γ-semigroup S is a po-Γ-subsemigroup of S iff (1) TΓT ⊆ T, (2) (T] ⊆ T.

THEOREM 2.21 : Let S be a po- Γ-semigroup and A is a subset of S. Then for all A, B ⊆ S (i) A ⊆ (A], (ii) ((A]] = (A], (iii) (A]Γ(B] ⊆ (AΓB] and (iv) A ⊆ (B] for A ⊆ B, (v) (A] ⊆ (B] for A ⊆ B.

THEOREM 2.22 : The nonempty intersection of two po-Γ-subsemigroups of a po-Γ-semigroup S is a po-Γ-subsemigroup of S.

THEOREM 2.23 : The nonempty intersection of any family of po-Γ-subsemigroups of a po-Γ-semigroup S is a po-Γ-subsemigroup of S.

III. PO-Γ-IDEALS

We now introduce the term, a left po-Γ-ideal in a po- Γ-semigroup.

DEFINITION 3.1 : A nonempty subset A of a po-Γ-semigroup S is said to be a ***left po-Γ-ideal*** of S if

(1) $s \in S, a \in A, \alpha \in \Gamma$ implies $s\alpha a \in A$.

(2) $s \in S, a \in A$, $s \leq a \Rightarrow s \in$ A.

NOTE 3.2 : A nonempty subset A of a po-Γ-semigroup S is a left po-Γ-ideal of S iff (1) SΓA⊆A, and (2) (A] ⊆ A.

NOTE 3.3 : Let S be a po-Γ-semigroup. Then the set

(SΓa] = $\{t \in$ S / $t \leq x\alpha a$ for some $x \in$ S and $\alpha \in \Gamma\}$

THEOREM 3.4: Let S be a po-Γ-semigroup. Then (SΓa] is a left po-Γ-ideal of S for all $a \in$ S.

***Proof* :** Since (SΓa] Γ(SΓa] ⊆ (SΓaΓSΓa] = (SΓSΓa] = (SΓa].

Therefore (SΓa] is the nonempty subset of S. Let $t \in$ (SΓa], $s \in$ S, $\gamma \in \Gamma$.

$t \in$ (SΓa] $\Rightarrow t \leq s_1\alpha a$ where $s_1 \in$ S and $\alpha \in \Gamma$.

Now $s\gamma t \leq s\gamma(s_1\alpha a) = (s\gamma s_1)\alpha a \in$ (SΓa]

Therefore $t \in$ (SΓa], $s \in$ S, $\gamma \in \Gamma \Rightarrow s\gamma t \in$ (SΓa] and hence (SΓa] is a left po-Γ-ideal of S.

THEOREM 3.5 : The nonempty intersection of any two left po-Γ-ideals of a po-Γ-semigroup S is a left po-Γ-ideal of S.

THEOREM 3.6 : The nonempty intersection of any family of po- left Γ-ideals of a po- Γ-semigroup S is a left po-Γ-ideal of S.

THEOREM 3.7 : The union of any two left po-Γ-ideals of a po-Γ-semigroup S is a left po-Γ-ideal of S.

THEOREM 3.8 : The union of any family of left po-Γ-ideals of a po-Γ-semigroup S is a left po-Γ-ideal of S.

We now introduce the notion of a right po-Γ-ideal in a po-Γ-semigroup.

DEFINITION 3.9 : A nonempty subset A of a po- Γ-semigroup S is said to be a ***right po- Γ-ideal*** of S if

(1) $s \in S, a \in A, \alpha \in \Gamma$ implies $a\alpha s \in A$.

(2) $s \in S, a \in A$, $s \leq a \Rightarrow s \in$ A.

NOTE 3.10 : A nonempty subset A of a Γ-semigroup S is a po- right Γ- ideal of S iff (1) AΓS⊆A and (2) (A] ⊆ A.

NOTE 3.11 : Let S be a po-Γ-semigroup. Then the set
(aΓS] = {$t \in S$ / $t \leq a\alpha x$ for some $x \in S$ and $\alpha \in \Gamma$}

THEOREM 3.12: Let S be a po-Γ-semigroup. Then (aΓS] is a po- right Γ-ideal of S for all *a* ∈ S.

***Proof* :** Since (aΓS] Γ(aΓS] ⊆ (aΓSΓaΓS] = (aΓaΓS] = (aΓS].
Therefore (aΓS] is the nonempty subset of S. Let $t \in$ (aΓS], $s \in S$, $\gamma \in \Gamma$.
$t \in$ (aΓS] ⇒ $t \leq a\alpha s_1$ where $s_1 \in S$ and α ∈ Γ.
Now $t\gamma s \leq (a\alpha s_1)\gamma s = a\alpha(s\gamma s_1) \in$ aΓS ⇒ $t\gamma s \in$ (aΓS]
Therefore $t \in$ (aΓS], $s \in S$, $\gamma \in \Gamma$ ⇒ $t\gamma s \in$ (aΓS] and hence (aΓS] is a right po-Γ-ideal of S.

THEOREM 3.13 : The nonempty intersection of any two right po-Γ-ideals of a po-Γ-semigroup S is a right po-Γ-ideal of S.

THEOREM 3.14 : The nonempty intersection of any family of right po-Γ-ideals of a po- Γ-semigroup S is a right Γ-ideal of S.

THEOREM 3.15 : The union of any two right po-Γ-ideals of a po-Γ-semigroup S is a right po-Γ-ideal of S.

THEOREM 3.16 : The union of any family of right po-Γ-ideals of a po-Γ-semigroup S is a right po-Γ-ideal of S.

We now introduce the notion of a po-Γ-ideal of a po- Γ-semigroup.

DEFINITION 3.17 : A nonempty subset A of a po-Γ-semigroup S is said to be a ***two sided po-Γ- ideal*** or simply a ***po-Γ- ideal*** of S if
(1) $s \in S$, $a \in A$, $\alpha \in \Gamma$ imply $s\alpha a \in A$, $a\alpha s \in A$.
(2) $s \in S, a \in A, s \leq a \Rightarrow s \in A$.

NOTE 3.18 : A nonempty subset A of a po-Γ-semigroup S is a two sided po-Γ-ideal iff it is both a left po-Γ-ideal and a right po-Γ- ideal of S.

The following examples are due to MANOJ SIRIPITUKDET AND AIYARED IAMPAN [13]

EXAMPLE 3.19 : Let $M = \{a, b, c, d\}$ and $\Gamma = \{\gamma\}$ with the multiplication and the relation ≤ on M defined by

$$x\gamma y = \begin{cases} b & \text{if } x, y \in \{a, b\} \\ c & \text{otherwise} \end{cases}$$

and ≤:= { (a, a), (b, b), (c, c), (d, d), (b, c), (b, d), (c, d)}. Then M is a po- Γ-semigroup and {b, c} is a po- Γ-ideal of M.

EXAMPLE 3.20 : Let S = { a, b, c, d} be then a po-Γ-semigroup defined by the following multiplication and relation ≤ on S as follows:

*	a	b	c	d
a	**b**	**b**	**d**	**d**
b	**b**	**b**	**d**	**d**
c	**d**	**d**	**c**	**d**
d	**d**	**d**	**d**	**d**

≤ := { (a, a), (b,b), (c,c), (d,d), (a,b), (d,b), (d,c)}.
Let M = S and Γ = {*}. Then M is a po-Γ-semigroup and {d} is a po-Γ-ideal of M.

THEOREM 3.21 : Let S be a po-Γ-semigroup. Then (SΓ*a*ΓS] is a right po-Γ-ideal of S for all *a* ∈ S.

***Proof* :** Since (SΓaΓS]Γ(SΓaΓS] ⊆ (SΓaΓSΓ SΓaΓS] = (SΓSΓaΓS] = (SΓaΓS]
Therefore (SΓaΓS] is a nonempty subset of S. Let $x \in$ (SΓaΓS], $s \in S$.
$x \in$ (SΓaΓS] ⇒ $x \leq t\alpha a\beta u$ for some $t, u \in S$ and $\alpha, \beta \in \Gamma$.
$x \leq t\alpha a\beta u \Rightarrow s\gamma x \leq s\gamma t\alpha a\beta u \Rightarrow s\gamma x \in$ (SΓSΓaΓS] = (SΓaΓS]
and $x\gamma s \leq t\alpha a\beta u\gamma s \Rightarrow x\gamma s \in$ (SΓaΓSΓS] = (SΓaΓS]
and ((SΓaΓS]] ⊆ (SΓaΓS] and hence (SΓaΓS] is a po-Γ-ideal of S.

THEOREM 3.22 : The nonempty intersection of any two po-Γ-ideals of a po-Γ-semigroup S is a po- Γ-ideal of S.

THEOREM 3.23 : The nonempty intersection of any family of po-Γ-ideals of a po-Γ-semigroup S is a po-Γ-ideal of S.

THEOREM 3.24 : The union of any two po-Γ-ideals of a po-Γ-semigroup S is a po-Γ-ideal of S.

THEOREM 3.25 : The union of any family of po-Γ-ideals of a po-Γ-semigroup S is a po-Γ-ideal of S.

We now introduce a proper po- Γ-ideal, trivial po- Γ-ideal, maximal left po-Γ-ideal, maximal right po-Γ-ideal, maximal po-Γ-ideal and globally idempotent po-Γ-ideal of a po-Γ-semigroup.

DEFINITION 3.26 : A po- Γ-ideal A of a po-Γ-semigroup S is said to be an ***proper po- Γ-ideal*** of S if A is different from S.

DEFINITION 3.27 : A Γ-ideal A of a po-Γ-semigroup S is said to be a ***trivial po-Γ-ideal*** provided S\A is singleton.

DEFINITION 3.28 : A Γ-ideal A of a po-Γ-semigroup S is said to be a ***maximal left po-Γ-ideal*** provided A is a proper left po-Γ-ideal of S and is not properly contained in any proper left po-Γ-ideal of S.

DEFINITION 3.29 : A Γ-ideal A of a po-Γ-semigroup S is said to be a ***maximal right po-Γ-ideal*** provided A is a proper right Γ-ideal of S and is not properly contained in any proper right po-Γ-ideal of S.

DEFINITION 3.30 : A Γ-ideal A of a po-Γ-semigroup S is said to be a ***maximal po-Γ-ideal*** provided A is a proper Γ-ideal of S and is not properly contained in any proper po-Γ-ideal of S.

DEFINITION 3.31 : A po-Γ-ideal A of a po-Γ-semigroup S is said to be ***globally idempotent*** if (AΓA] = A.

THEOREM 3.32 : If A is a po-Γ-ideal of a po- Γ-semigroup S with unity 1 and 1 ∈ A then A = S.

***Proof* :** Clearly A ⊆ S. Let s ∈ S.

1 ∈ A, s ∈ S, A is a po-Γ-ideal of S ⇒ 1Γs ⊆ A and $s \le 1 \Rightarrow s$ ∈ A.

Thus S ⊆ A. A ⊆ S, S ⊆ A ⇒ S = A.

THEOREM 3.33 : If S is a po-Γ-semigroup with unity 1 then the union of all proper po- Γ-ideals of S is the unique maximal po- Γ-ideal of S.

***Proof* :** Let M be the union of all proper po-Γ-ideals of S. Since 1 is not an element of any proper po- Γ-ideal of S, 1 ∉ M. Therefore M is a proper subset of S. By theorem 3.24, M is a po- Γ-ideal of S. Thus M is a proper po-Γ-ideal of S. Since M contains all proper po- Γ-ideals of S, M is a maximal po-Γ-ideal of S. If M_1 is any maximal po-Γ-ideal of S, then $M_1 \subseteq M \subset S$ and hence $M_1 = M$. Therefore M is the unique maximal po-Γ-ideal of S.

We now introducing left po-Γ-ideal generated by a subset, a right po-Γ-ideal generated by a subset, po-Γ-ideal generated by a subset of a po-Γ-semigroup.

DEFINITION 3.34 : Let S be a po- Γ-semigroup and A be a nonempty subset of S. The smallest po- left Γ-ideal of S containing A is called ***left po-Γ-ideal of* S *generated by A*** and it is denoted by L(A).

THEOREM 3.35 : Let S be a po-Γ-semigroup and A is a nonempty subset of S, then L(A) = (A ∪ SΓA].

***Proof* :** Let s ∈ S, r ∈ (A ∪ SΓA] and γ ∈ Γ.

r ∈ (A ∪ SΓA] ⇒ r ∈ (A] or r ∈(SΓA] ⇒ $r \le a$ or $r \le t\alpha a$ for some a ∈ A, t ∈ S, α ∈ Γ.

If $r \le a$ then $s\gamma r \le s\gamma a \Rightarrow s\gamma r$ ∈ (SΓA] ⊆ (A ∪ SΓA].

If $r \le t\alpha a$ then $s\gamma r \le s\gamma(t\alpha a) = (s\gamma t)\alpha a$ ∈ SΓa ⇒ $s\gamma r$ ∈ (SΓA] ⊆ (A ∪ SΓA].

Therefore $s\gamma a$ ∈ (A ∪ SΓA] and hence (A ∪ SΓA] is a po- left Γ-ideal of S.

Let L be a left po-Γ-ideal of S containing A.

Let r ∈ (A ∪ SΓA]. Then $r \le a$ or $r \le t\alpha a$ for some a ∈ A, t ∈ S, α ∈ Γ.

If $r \le a$ then $r \le a$ ∈ L. If $r \le t\alpha a$ then $r \le t\alpha a$ ∈ L.

Therefore (A ∪ SΓA]⊆ L and hence (A ∪ SΓA] is the smallest left po-Γ-ideal containing A.

Therefore L(A) = (A ∪ SΓA].

THEOREM 3.36 : The left po-Γ-ideal of a po-Γ-semigroup S generated by a nonempty subset A is the intersection of all left po-Γ-ideals of S containing A.

***Proof* :** Let Δ be the set of all left po-Γ-ideals of S containing A.

Since S itself is a left po-Γ-ideal of S containing A, S ∈ Δ. So Δ ≠ ∅.

Let $T^* = \bigcap_{T \in \Delta} T$. Since $A \subseteq T$ for all T ∈ Δ, $A \subseteq T^*$.

By theorem 3.6, T^* is a left po-Γ-ideal of S.

Let K is a left po-Γ-ideal of S containing A.

Clearly A ⊆ K and K is a left po-Γ-ideal of S.

Therefore K ∈ Δ ⇒T^* ⊆ K. Therefore T^* is the left po-Γ-ideal of S generated by A.

DEFINITION 3.37 : Let S be a po-Γ-semigroup and A be a nonempty subset of S. The smallest po- right Γ-ideal of S containing A is called ***right po-Γ-ideal of* S *generated by A*** and it is denoted by R(A).

THEOREM 3.38 : Let S be a po-Γ-semigroup and A is a nonempty subset of S, then R(A) = (A ∪ AΓS].

***Proof* :** Let s ∈ S, r ∈ (A ∪ AΓS] and γ ∈ Γ.

r ∈ (A ∪ AΓS] ⇒ r ∈ (A] or r ∈(AΓS] ⇒ $r \le a$ or $r \le a\alpha t$ for some a ∈ A, t ∈ S, α ∈ Γ.

If $r \le a$ then $r\gamma s \le a\gamma s \Rightarrow r\gamma s$ ∈ (AΓS] ⊆ (A ∪ AΓS].

If $r \le a\alpha t$ then $r\gamma s \le (a\alpha t)\gamma s = a\alpha(t\gamma s)$ ∈ AΓS ⇒ $r\gamma s$ ∈ (AΓS] ⊆ (A ∪ AΓS].

Therefore $r\gamma s$ ∈ (A ∪ AΓS] and hence (A ∪ AΓS] is a right po-Γ-ideal of S.

Let R be a right po-Γ-ideal of S containing A.

Let $r \in (A \cup A\Gamma S]$. Then $r \le a$ or $r \le a\alpha t$ for some $a \in A$, $t \in S$, $\alpha \in \Gamma$.
If $r \le a$ then $r \le a \in R$. If $r \le a\alpha t$ then $r \le a\alpha t \in R$.
Therefore $(A \cup A\Gamma S] \subseteq R$ and hence $(A \cup A\Gamma S]$ is the smallest right po-Γ-ideal containing A.
Therefore $R(A) = (A \cup A\Gamma S]$.

THEOREM 3.39 : The right po-Γ-ideal of a po-Γ-semigroup S generated by a nonempty subset A is the intersection of all right po-Γ-ideals of S containing A.

***Proof* :** Let Δ be the set of all right po-Γ-ideals of S containing A.
Since S itself is a right po-Γ-ideal of S containing A, $S \in \Delta$. So $\Delta \neq \emptyset$.
Let $T^* = \bigcap_{T \in \Delta} T$. Since $A \subseteq T$ for all $T \in \Delta$, $A \subseteq T^*$.
By theorem 3.14, T^* is a right po-Γ-ideal of S.
Let K is a right po-Γ-ideal of S containing A.
Clearly $A \subseteq K$ and K is a right po-Γ-ideal of S.
Therefore $K \in \Delta \Rightarrow T^* \subseteq K$. Therefore T^* is the right po-Γ-ideal of S generated by A.

DEFINITION 3.40 : Let S be a po-Γ-semigroup and A be a nonempty subset of S. The smallest po-Γ-ideal of S containing A is called ***po-Γ-ideal of* S *generated by A*** and it is denoted by J(A).

THEOREM 3.41 : If S is a po-Γ-semigroup and A ⊆ S then
$J(A) = (A \cup A\Gamma S \cup S\Gamma A \cup S\Gamma A\Gamma S]$.

***Proof*:** Let $s \in S$, $r \in (A \cup A\Gamma S \cup S\Gamma A \cup S\Gamma A\Gamma S]$ and $\gamma \in \Gamma$.
$r \in (A \cup A\Gamma S \cup S\Gamma A \cup S\Gamma A\Gamma S] \Rightarrow r \le a$ or $r \le a\alpha t$ or $r \le t\alpha a$ or $r \le t\alpha a\beta u$ for some $a \in A$ $t, u \in S$ and $\alpha, \beta \in \Gamma$.
If $r \le a$ then $r\gamma s \le a\gamma s \in A\Gamma S \Rightarrow r\gamma s \in (A\Gamma S]$ and $s\gamma r \le s\gamma a \in S\Gamma A \Rightarrow s\gamma r \in (S\Gamma A]$.
If $r \le a\alpha t$ then $r\gamma s \le (a\alpha t)\gamma s = a\alpha(t\gamma s) \in A\Gamma S \Rightarrow r\gamma s \in (A\Gamma S]$
and $s\gamma r \le s\gamma(a\alpha t) = s\gamma a\alpha t \in S\Gamma A\Gamma S \Rightarrow r\gamma s \in (S\Gamma A\Gamma S]$.
If $r \le t\alpha a$ then $r\gamma s \le (t\alpha a)\gamma s = t\alpha a\gamma s \in S\Gamma A\Gamma S \Rightarrow r\gamma s \in (S\Gamma A\Gamma S]$
or $s\gamma r \le s\gamma(t\alpha a) = (s\gamma t)\alpha a \in S\Gamma A \Rightarrow s\gamma r \in (S\Gamma A]$.
If $r \le t\alpha a\beta u$ then $r\gamma s \le (t\alpha a\beta u)\gamma s = t\alpha a\beta(u\gamma s) \in S\Gamma A\Gamma S \Rightarrow r\gamma s \in (S\Gamma A\Gamma S]$
and $s\gamma r \le s\gamma(t\alpha a\beta u) = (s\gamma t)\alpha a\beta u \in S\Gamma A\Gamma S \Rightarrow s\gamma r \in (S\Gamma A\Gamma S]$.
But $(A\Gamma S]$, $(S\Gamma A]$, $(S\Gamma A\Gamma S]$ are all subsets of $(A\Gamma S \cup S\Gamma A \cup S\Gamma A\Gamma S]$.
Therefore $r\gamma s, s\gamma r \in (A \cup A\Gamma S \cup S\Gamma A \cup S\Gamma A\Gamma S]$ and hence $(A \cup A\Gamma S \cup S\Gamma A \cup S\Gamma A\Gamma S]$ is a po-Γ-ideal of S.
Let J be a Γ-ideal of S containing A. Let $r \in (A \cup A\Gamma S \cup S\Gamma A \cup S\Gamma A\Gamma S]$.
Then $r \le a$ or $r \le a\alpha t$ or $r \le t\alpha a$ or $r \le t\alpha a\beta u$ for some $a \in A$ $t, u \in S$ and $\alpha, \beta \in \Gamma$.
If $r \le a$ then $r \le a \Rightarrow r \in J$. If $r \le a\alpha t$ then $r \le a\alpha t \Rightarrow r \in J$.
If $r \le t\alpha a$ then $r \le t\alpha a \Rightarrow r \in J$. If $r \le t\alpha a\beta u$ then $r \le t\alpha a\beta u \Rightarrow r \in J$.
Therefore $(A \cup A\Gamma S \cup S\Gamma A \cup S\Gamma A\Gamma S] \subseteq J$.
Hence $(A \cup A\Gamma S \cup S\Gamma A \cup S\Gamma A\Gamma S]$is the smallest po-Γ-ideal of S containing a.
Therefore $J(A) = (A \cup A\Gamma S \cup S\Gamma A \cup S\Gamma A\Gamma S]$.

THEOREM 3.42 : The po-Γ-ideal of a Γ-semigroup S generated by a nonempty subset A is the intersection of all po-Γ-ideals of S containing A.

***Proof* :** Let Δ be the set of all po- Γ-ideals of S containing A.
Since S itself is a po-Γ-ideal of S containing A, $S \in \Delta$. So $\Delta \neq \emptyset$.
Let $T^* = \bigcap_{T \in \Delta} T$. Since $A \subseteq T$ for all $T \in \Delta$, $A \subseteq T^*$.
By theorem 3.23, T^* is a po-Γ-ideal of S.
Let K is a po-Γ-ideal of S containing A.
Clearly $A \subseteq K$ and K is a po-Γ-ideal of S. Therefore $K \in \Delta \Rightarrow T^* \subseteq K$.
Therefore T^* is the po-Γ-ideal of S generated by A.

We now introduce a principal left po-Γ-ideal of a po-Γ-semigroup and characterize principal left po-Γ-ideal.

DEFINITION 3.43 : A left po-Γ-ideal A of a po-Γ-semigroup S is said to be the ***principal left po-Γ-ideal generated by a,*** if A is a po- left Γ-ideal generated by $\{a\}$ for some $a \in S$. It is denoted by $L(a)$.

THEOREM 3.44 : If S is a po-Γ-semigroup and $a \in S$ then $L(a) = (a \cup S\Gamma a]$.

***Proof*:** In the theorem 3.35., for $A = \{a\}$ we have $L(a) = (a \cup S\Gamma a]$.

NOTE 3.45: If S is a po-Γ-semigroup and $a \in S$ then $L(a) = \{ t \in S / t \le a \text{ or } t \le x\gamma a \text{ for some } x \in S, \gamma \in \Gamma\}$.

NOTE 3.46 : If S is a po-Γ-semigroup and $a \in S$ then $L(a) = (S^1\Gamma a]$.

We now introduce principal right po-Γ-ideal of a po-Γ-semigroup and characterize principal right po-Γ-ideal.

DEFINITION 3.47 : A right po-Γ-ideal A of a po-Γ-semigroup S is said to be the ***principal right po-Γ-ideal generated by a*** if A is a right po-Γ-ideal generated by $\{a\}$ for some $a \in S$. It is denoted $R(a)$.

THEOREM 3.48 : If S is a po-Γ-semigroup and $a \in$ S then R(a) = ($a \cup a$ΓS].

***Proof*:** In the theorem 3.38., for A = {a} we have R(a) = ($a \cup a$ΓS].

NOTE 3.49 : If S is a po-Γ-semigroup and $a \in$ S then R(a) = { $t \in$ S / $t \leq a$ or $t \leq a\gamma x$ for some $x \in$ S, $\gamma \in \Gamma$}.

NOTE 3.50 : If S is a po-Γ-semigroup and $a \in$ S then R (a) = ($a\Gamma S^1$].

We now introduce a principal po-Γ-ideal of a po-Γ-semigroup and characterize principal po-Γ-ideal.

DEFINITION 1.3.51 : A po- Γ-ideal A of a po-Γ-semigroup S is said to be a ***principal po-Γ-ideal*** provided A is a po-Γ-ideal generated by {a} for some $a \in$ S. It is denoted by J[a] or <a>.

THEOREM 3.52 : If S is a po-Γ-semigroup and $a \in$ S then J(a) = ($a \cup a$ΓS ∪ SΓa ∪ SΓaΓS].

***Proof*:** In the theorem 3.41., for A = {a} we have J(a) = ($a \cup$ SΓa ∪ aΓS ∪ SΓaΓS].

NOTE 3.53:If S is a po-Γ-semigroup and $a \in S$, then

<a> = { $t \in$ S / $t \leq a$ or $t \leq a\gamma x$ or $t \leq x\gamma a$ or $t \leq x\gamma a\delta y$ for some $x, y \in$ S and $\gamma, \delta \in \Gamma$}

NOTE 3.54 : If S is a po-Γ-semigroup and $a \in S$, then

<a> = ($a \cup a$ΓS ∪ SΓa ∪ SΓaΓS]= ($S^1\Gamma a\, \Gamma S^1$].

DEFINITION 3.55 : A partial order ≤ on a set S is linear if for any $a, b \in$ S, either $a \leq b$ or $b \leq a$.

DEFINITION 3.56 : Let ≤ is a partial order on a set S is linear. Then S is called a ***Chain*.**

THEOREM 3.57 : In any po-Γ-semigroup S, the following are equivalent.

1. **Principal po-Γ-ideals of S form a chain.**
2. **Po-Γ-ideals of S form a chain.**

***Proof* :** (1) ⇒ (2) : Suppose that principal po-Γ-ideals of S form a chain.

Let A, B be two po-Γ-ideals of S. Suppose if possible A ⊄ B, B ⊄ A.

Then there exists $a \in$ A\B and b ∈ B\A.

$a \in$ A ⇒ < a > ⊆ A and $b \in$ B ⇒ < b > ⊆ B.

Since principal po-Γ-ideals form a chain, either < a > ⊆ < b > or < b > ⊆ < a >.

If < a > ⊆ < b >, then $a \in$ < b > ⊆ B. It is a contradiction.

If < b > ⊆ < a >, then $b \in$ < a > ⊆ A. It is also a contradiction.

Therefore either A ⊆ B or B ⊆ A and hence po-Γ-ideals from a chain.

(2) ⇒ (1) : Suppose that po-Γ-ideals of S form a chain.

Then clearly principal po-Γ-ideal of S form a chain.

We now introduce a left simple po-Γ-semigroup and characterize left simple po-Γ-semigroups.

DEFINITION 3.58 : A po-Γ-semigroup S is said to be a ***left simple po-Γ-semigroup*** if for every $a, b \in$ S, $\alpha, \beta \in \Gamma$, there exist $x, y \in$ S such that $b \leq x\alpha a$ and $a \leq y\beta b$.

NOTE 3.59 : A po- Γ-semigroup S is said to be a left simple po-Γ-semigroup if S is its only left po-Γ-ideal.

THEOREM 3.60 : A po-Γ-semigroup S is a left simple po-Γ-semigroup if and only if (SΓa] = S for all $a \in$ S.

***Proof*:** Suppose that S is a left simple po-Γ-semigroup and $a \in$ S.

Let $t \in$ (SΓa], $s \in$ S, $\gamma \in \Gamma$.

$t \in$ (SΓa] ⇒ $t \leq s_1\alpha a$ where $s_1 \in S$ and α ∈ Γ.

Now $s\gamma t \leq s\gamma(s_1\alpha a) = (s\gamma s_1)\alpha a \in S\Gamma a \Rightarrow s\gamma t \in$ (SΓa]. Therefore (SΓa] is a left po-Γ-ideal of S.

Since S is a left simple po-Γ-semigroup, (SΓa] = S.

Therefore (SΓa] = S for all $a \in$ S.

Conversely suppose that (SΓa] = S for all $a \in$ S. Let L be a left Γ-ideal of S.

Let $l \in$ L. Then $l \in$ S. By assumption, (SΓl] = S.

Let $s \in$ S. Then $s \in$ (SΓl] ⇒ $s \leq t\alpha l$ for some $t \in$ S, α ∈ Γ.

$l \in$ L, $t \in$ S, α ∈ Γ and L is a left po-Γ-ideal ⇒ $t\alpha l \in$ L ⇒ $s \in$ L. Therefore S ⊆ L.

Clearly L ⊆ S and hence S = L. Therefore S is the only left po-Γ-ideal of S.

Hence S is left simple po-Γ-semigroup.

We now introduce a right simple po-Γ-semigroup and characterize right simple po-Γ-semigroups.

DEFINITION 3.61 : A po-Γ-semigroup S is said to be a ***right simple po-Γ-semigroup*** if for every $a, b \in$ S, $\alpha, \beta \in \Gamma$, there exist $x, y \in$ S such that $b \leq a\alpha x$ and $a \leq b\beta y$.

NOTE 3.62 : A po-Γ-semigroup S is said to be a right simple po- Γ-semigroup if S is its only right Γ-ideal.

THEOREM 3.63: A po-Γ-semigroup S is a right simple po-Γ-semigroup if and only if (aΓS]=S for all $a \in$ S.

***Proof* :** Suppose that S is a right simple po-Γ-semigroup and $a \in$ S.

Let $t \in$ (aΓS], $s \in$ S, $\gamma \in \Gamma$.

$t \in$ (aΓS] ⇒ $t \leq a\alpha s_1$ where $s_1 \in S$ and α ∈ Γ.

Now $t\gamma s \leq (a\alpha s_1)\gamma s = a\alpha(s_1\gamma s) \in a\Gamma S \Rightarrow t\gamma s \in$ (aΓS]. Therefore (aΓS] is a right po-Γ-ideal of S.

Since S is a right simple po- Γ-semigroup, (aΓS] = S. Therefore (aΓS] = S for all $a \in$ S.

Conversely suppose that $(a\Gamma S] = S$ for all $a \in S$.
Let R be a right po-Γ-ideal of a po- Γ-semigroup S.
Let $r \in R$. Then $r \in S$. By assumption $(r\Gamma S] = S$.
Let $s \in S$. Then $s \in (r\Gamma S] \Rightarrow s \leq r\alpha t$ for some $t \in S$, $\alpha \in \Gamma$.
$r \in R$, $t \in S$, $\alpha \in \Gamma$ and R is a right po-Γ-ideal $\Rightarrow r\alpha t \in R \Rightarrow s \in R$.
Therefore $S \subseteq R$. Clearly $R \subseteq S$ and hence $S = R$.
Therefore S is the only right po-Γ-ideal of S. Hence S is right simple po-Γ-semigroup.

We now introduce a simple po-Γ-semigroup and characterize simple po-Γ-semigroups.

DEFINITION 3.64 : A po-Γ-semigroup S is said to be a ***simple po-Γ-semigroup*** if for every $a, b \in S$, $\alpha, \beta \in \Gamma$, there exist $x, y \in S$ such that $a \leq x\alpha b\beta y$.

NOTE 3.65 : A po-Γ-semigroup S is said to be simple po-Γ-semigroup if S is its only two-sided po-Γ-ideal.

THEOREM 3.66 : If S is a left simple po-Γ-semigroup or a right simple po-Γ-semigroup then S is a simple po- Γ-semigroup.

***Proof* :** Suppose that S is a left simple po-Γ-semigroup. Then S is the only left po-Γ-ideal of S. If A is a po-Γ-ideal of S, then A is a left po-Γ-ideal of S and hence A = S.
Therefore S itself is the only po-Γ-ideal of S and hence S is a simple po-Γ-semigroup.
Suppose that S is a right simple po-Γ-semigroup. Then S is the only right po-Γ-ideal of S. If A is a po-Γ-ideal of S, then A is a right po-Γ-ideal of S and hence A = S.
Therefore S itself is the only po-Γ-ideal of S and hence S is a simple po-Γ-semigroup.

THEOREM 3.67 : A po-Γ-semigroup S is simple po-Γ-semigroup if and only if $(S\Gamma a\Gamma S] = S$ for all $a \in S$.

***Proof*:** Suppose that S is a simple po-Γ-semigroup and $a \in S$.
Let $t \in (S\Gamma a\Gamma S]$, $s \in S$ and $\gamma \in \Gamma$.
$t \in (S\Gamma a\Gamma S] \Rightarrow t \leq s_1\alpha a\beta s_2$ where $s_1, s_2 \in S$ and $\alpha, \beta \in \Gamma$.
Now $t\gamma s \leq (s_1\alpha a\beta s_2)\gamma s = s_1\alpha a\beta(s_2\gamma s) \in S\Gamma a\Gamma S \Rightarrow t\gamma s \in (S\Gamma a\Gamma S]$
and $s\gamma t \leq s\gamma(s_1\alpha a\beta s_2) = (s\gamma s_1)\alpha a\beta s_2 \in S\Gamma a\Gamma S \Rightarrow s\gamma t \in (S\Gamma a\Gamma S]$.
Therefore $(S\Gamma a\Gamma S]$ is a po-Γ-ideal of S.
Since S is a simple po-Γ-semigroup, S itself is the only po-Γ-ideal of S and hence $(S\Gamma a\Gamma S] = S$.
Conversely suppose that $(S\Gamma a\Gamma S] = S$ for all $a \in S$. Let I be a Γ-ideal of S.
Let $a \in I$. Then $a \in S$. So $(S\Gamma a\Gamma S] = S$.
Let $s \in S$. Then $s \in (S\Gamma a\Gamma S] \Rightarrow s \leq t_1\alpha a\beta t_2$ for some $t_1, t_2 \in S$, $\alpha, \beta \in \Gamma$.
$a \in I$, $t_1, t_2 \in S$, $\alpha, \beta \in \Gamma$, I is a Γ-ideal of S $\Rightarrow t_1\alpha a\beta t_2 \in I \Rightarrow s \in I$.
Therefore $S \subseteq I$. Clearly $I \subseteq S$ and hence $S = I$.
Therefore S is the only Γ-ideal of S. Hence S is a simple po-Γ-semigroup.

We now introduce a regular po-Γ-ideal of a po-Γ-semigroup.

DEFINITION 3.68 : A po-Γ-ideal A of a po-Γ-semigroup S is said to be ***regular*** if every element of A is regular in A.

NOTE 3.69 : A po-Γ-ideal A of a po-Γ-semigroup S is said to be regular if $A \subseteq (A\Gamma S\Gamma A]$

THEOREM 3.70 : Every po-Γ-ideal of a regular po-Γ-semigroup S is a regular po-Γ-ideal of S.

Proof : Let A be a po-Γ-ideal of S and $a \in A$. Then $a \in S$ and hence a is regular in S. Therefore $a \leq a\alpha b\beta a$ where $b \in S$ and $\alpha, \beta \in \Gamma$.
Hence $a \leq a\alpha b\beta a \leq (a\alpha b\beta)(a\alpha b\beta a) \leq a\alpha[(b\beta a)\alpha b]\beta a$.
Let $b_1 = (b\beta a)\alpha b \in S\Gamma A\Gamma S$. Now $a \leq a\alpha b_1\beta a$.
Therefore a is regular in A and hence A is a regular po-Γ-ideal.

IV. Completely Prime Po-Γ-Ideals And Prime Po-Γ-Ideals

DEFINITION 4.1 : A po-Γ-ideal P of a po-Γ-semigroup S is said to be a ***completely prime po-Γ- ideal*** provided $x, y \in S$ and $x\Gamma y \subseteq P$ implies either $x \in P$ or $y \in P$.

We now introduce the notion of a c-system of a po-Γ-semigroup.

DEFINITION 4.2 : Let S be a po-Γ-semigroup. A nonempty subset A of S is said to be a ***po-c-system*** of S if for each $a, b \in A$ and $\alpha \in \Gamma$ there exists an element $c \in A$ such that $c \leq a\alpha b$.

NOTE 4.3 : A nonempty subset A of a po-Γ-semigroup S is said to be a po-c-system of S if for each $a, b \in A$ there exists an element $c \in A$ such that $c \in (a\Gamma b]$.

THEOREM 4.4 : Every po-Γ-subsemigroup of a po-Γ-semigroup is a *po-c-system.*

***Proof* :** Let T be a po-Γ-subsemigroup of S and $a, b \in T$, $\alpha \in \Gamma$.
Since T is a po-Γ-subsemigroup of S, $a\alpha b \in T$.

Let $c = a\alpha b \Rightarrow c \leq a\alpha b$. Therefore there exist an element $c \in$ T such that $c \leq a\alpha b$. Therefore T is a po-c-system.

THEOREM 4.5: Let S be a po-Γ-semigroup and P is a po-Γ-ideal of S. Then (aΓb]⊆P if and only if aΓb⊆ P.

Proof **:** Suppose that $(a\Gamma b] \subseteq$ P. By theorem 2.21, $a\Gamma b \subseteq (a\Gamma b] \subseteq$ P and hence $a\Gamma b \subseteq$ P. Conversely suppose that $a\Gamma b \subseteq$ P. Let $x \in (a\Gamma b] \Rightarrow x \leq a\alpha b$ where $a\alpha b \in a\Gamma b \Rightarrow x \leq a\alpha b \in a\Gamma b \subseteq$ P $\Rightarrow x \in$ P. Therefore $(a\Gamma b] \subseteq$ P.

We now prove a necessary and sufficient condition for a po-**Γ**-ideal to be a completely prime po-**Γ**-ideal in a po-Γ-semigroup.

THEOREM 4.6 : A po-Γ-ideal P of a po-Γ-semigroup S is completely prime if and only if S\P is either a *c*-system of S or empty.

Proof **:** Suppose that P is a completely prime po-Γ-ideal of S and S\P ≠ ∅.
Let a, $b \in$ S\P. Then $a \notin$ P, $b \notin$ P. Suppose if possible $c \notin (a\Gamma b]$ for every $c \in$ S\P.
Then $(a\Gamma b] \subseteq$ P $\Rightarrow a\Gamma b \subseteq$ P. Since P is completely prime, either $a \in$ P or $b \in$ P.
It is a contradiction. Therefore $c \in (a\Gamma b]$ for some $c \in$ S\P. Hence there exists an element $c \in$ S\P such that $c \leq a\alpha b$ for $\alpha \in \Gamma$ and hence S\P is a c-system.
Conversely suppose that S\P is a c-system of S or S\P is empty.
If S\P is empty then P = S and hence P is a completely prime.
Assume that S\P is a c-system of S. Let a, $b \in$ S and $a\Gamma b \subseteq$ P.
Suppose if possible $a \notin$ P and $b \notin$ P. Then $a \in$ S\P and $b \in$ S\P.
Since S\P is a c-system, there exists $c \in$ S\P such that $c \leq a\alpha b$ for some $\alpha \in \Gamma$.
$c \leq a\alpha b \in a\Gamma b \subseteq$ P. Thus $c \in$ P.
It is a contradiction. Hence either $a \in$ P or $b \in$ P.
Therefore P is a completely prime po-Γ-ideal of S.

We now introduce the notion of a prime po-Γ-ideal of a po-Γ-semigroup.

DEFINITION 4.7 : A po-Γ-ideal P of a po-Γ-semigroup S is said to be a ***prime po-Γ-ideal*** provided A, B are two po- Γ-ideals of S and AΓB ⊆ P ⇒ either A ⊆ P or B ⊆ P.

THEOREM 4.8 : If P is a po-Γ-ideal of a po-Γ-semigroup S, then the following conditions are equivalent.

(1) If *A, B* are po- Γ- ideals of S and AΓB ⊆ P then either A⊆P or B⊆P.

(2) If a, $b \in$ S such that $a\Gamma S^1\Gamma b \subseteq$ P, then either $a \in$ P or $b \in$ P.

COROLLARY 4.9 : A po-Γ-ideal P of a po-Γ-semigroup S is a prime po-Γ- ideal iff a, $b \in$ S such that $a\Gamma S^1\Gamma b \subseteq$ P, then either $a \in$ P or $b \in$ P.

THEOREM 4.10 : Let P be a po-Γ-ideal of a po-Γ-semigroup S. Then $(a\Gamma S^1\Gamma b] \subseteq$ P if and only if $a\Gamma S^1\Gamma b \subseteq$ P.

Proof **:** Suppose that $(a\Gamma S^1\Gamma b] \subseteq$ P. By theorem 2.21, $a\Gamma S^1\Gamma b \subseteq (a\Gamma S^1\Gamma b] \subseteq$ P and hence $a\Gamma S^1\Gamma b \subseteq$ P.
Conversely suppose that $a\Gamma S^1\Gamma b \subseteq$ P. Let $x \in (a\Gamma S^1\Gamma b]$
$\Rightarrow x \leq a\alpha s\beta b$ for some $a\alpha s\beta b \in a\Gamma S^1\Gamma b \Rightarrow x \leq a\alpha s\beta b \in a\Gamma S^1\Gamma b \subseteq$ P $\Rightarrow x \in$ P.
Therefore $(a\Gamma S^1\Gamma b] \subseteq$ P.

DEFINITION 4.11 : A po-Γ-ideal A is said to be ***exceptional prime po-Γ-ideal*** if A is a po-prime Γ-ideal which is not completely prime po-Γ-ideal.

THEOREM 4.12 : Every completely prime po-Γ-ideal of a po-Γ-semigroup S is a prime po-Γ-ideal of S.

Proof: Suppose that A is a completely prime po-**Γ**-ideal of a po-**Γ**-semigroup S. Let a, $b \in$ S and $a\Gamma S^1\Gamma b \subseteq$ A.
Now $a\Gamma b \subseteq a\Gamma S^1\Gamma b \subseteq$ A.
Since A is a completely prime, either $a \in$ A or $b \in$ A.
Therefore A is a prime po-**Γ**-ideal of S.

THEOREM 4.13 : Let S be a commutative po-Γ-semigroup. A po-Γ-ideal P of S is a prime po-Γ-ideal if and only if P is a completely prime po-Γ-ideal.

Proof **:** Suppose that P is a prime po-Γ-ideal of po-Γ-semigroup S.
Let x, $y \in$ S and $x\Gamma y \subseteq$ P. Now $x\Gamma y \subseteq$ P, P is a po-Γ-ideal $\Rightarrow x\Gamma y\Gamma S^1 \subseteq$ P.
Since S is commutative, $x\Gamma S^1\Gamma y = x\Gamma y\, \Gamma\, S^1 \subseteq$ P.
By corollary 4.18, either $x \in$ P or $y \in$ P. Hence P is a completely prime po-Γ-ideal.
Conversely suppose that P is a completely prime po-Γ-ideal of S.
By theorem 4.19, P is a prime po-Γ-ideal of S.

We now introduce the notion of an m-system of a po-Γ-semigroup.

DEFINITION 4.14 : A nonempty subset A of a po-Γ-semigroup S is said to be an ***po-m-system*** provided for any a,$b \in$ A and $\alpha, \beta \in \Gamma$ there exists an $c \in$ A and $x \in$ S such that $c \leq a\alpha x\beta b$.

NOTE 4.15 : A nonempty subset A of a po-Γ-semigroup S is said to be an ***m-system*** provided for any a,$b \in$ A there exists an $c \in$ A and $x \in$ S such that $c \in (a\Gamma S\Gamma b]$.

THEOREM 4.16 : A nonempty set A is an *m*-system of Γ-semigroup (S, Γ, .) if and only if A is an *m*-system of po-Γ-semigroup (S, Γ, ., ≤).

Proof **:** Suppose that a nonempty set A is an *m*-system of Γ-semigroup S. Then for each a, $b \in$ A and α, $\beta \in \Gamma$ there exist an $x \in$ S such that $a\alpha x\beta b \in$ A. $a\alpha x\beta b \in$ A. Let $c = a\alpha x\beta b$. Then⇒ $c \leq a\alpha x\beta b$ for $a\alpha x\beta b \in$ A ⇒ $c \in$ A and hence there exists an element $c \in$ A such that $c \leq a\alpha x\beta b$. Therefore A is an *m*-system of S.

Conversely suppose that A is a po-*m*-system of a po-Γ-semigroup S. Then for each $a,b \in$ A and $\alpha, \beta \in \Gamma$ there exists an element $c \in$ A and $x \in$ S such that $c \leq a\alpha x\beta b$.

$c \leq a\alpha x\beta b \Rightarrow c \leq a\alpha x\beta b \in a\Gamma S\Gamma b \subseteq A \Rightarrow a\alpha x\beta b \in A$ and hence A is an *m*-system of Γ-semigroup S.

We now prove a necessary and sufficient condition for a po-Γ-ideal to be a prime po-Γ-ideal in a po-Γ-semigroup.

THEOREM 4.17 : A po-Γ-ideal P of a po-Γ-semigroup S is a prime po-Γ-ideal of S if and only if S\P is an *m*-system of S or empty.

Proof **:** Suppose that P is a prime po-Γ-ideal of a po-Γ-semigroup S and S\P ≠ ∅.

Let a, $b \in$ S\P. Then $a \notin$ P, $b \notin$ P.

Suppose if possible $c \notin (a\Gamma S^1\Gamma b]$ for every $c \in$ S\P.

Then $(a\Gamma S^1\Gamma b] \subseteq P \Rightarrow a\Gamma S^1\Gamma b \subseteq P$. Since P is prime, either $a \in$ P or $b \in$ P. It is a contradiction.

Therefore there exist an element $c \in (a\Gamma S^1\Gamma b]$ for some $c \in$ S\P. Hence there exists $c \in$ S\P such that $c \leq a\alpha x\beta b$ for some $a\alpha x\beta b \in a\Gamma S^1\Gamma b$. Hence S\P is an *m*-system.

Conversely suppose that S\P is either an *m*-system of S or S\P = ∅.

If S\P is empty then P = S and hence P is a prime po-Γ-ideal.

Assume that S\P is an *m*-system of S.

Let a, $b \in$ S and $a\Gamma S^1\Gamma b \subseteq$ P. Suppose if possible $a \notin$ P, $b \notin$ P. Then a, $b \in$ S\P.

Since S\P is an *m*-system, there exists $c \in$ S\P such that $c \leq a\alpha x\beta b$ for $x \in$ S, $\alpha, \beta \in \Gamma$.

$c \leq a\alpha x\beta b \in a\Gamma S^1\Gamma b \subseteq$ P. Thus $c \in$ P.

It is a contradiction. Therefore either $a \in$ P or $b \in$ P.

Hence P is a prime po-Γ-ideal of S.

We now introduce the notion of a globally idempotent po-Γ-semigroup.

DEFINITION 4.18 : A po-Γ-semigroup S is said to be a ***globally idempotent po-Γ-semigroup*** if (SΓS] = S.

THEOREM 4.19 : If S is a globally idempotent po-Γ-semigroup then every maximal po-Γ-ideal of S is a prime po-Γ-ideal of S.

Proof **:** Let M be a maximal po-Γ-ideal of S.

Let A, B be two po-Γ-ideals of S such that AΓB ⊆ M.

Suppose if possible A ⊈ M, B ⊈ M.

Now A ⊈ M ⇒ M ∪ A is a po-Γ-ideal of S and M ⊂ M ∪ A ⊆ S.

Since M is maximal, M ∪ A = S. Similarly B ⊈ M ⇒ M ∪ B = S.

Now S = (SΓS] = ((M ∪ A)Γ(M ∪ B)] = ((MΓM) ∪ (MΓB) ∪ (AΓM) ∪ (AΓB)] ⊆ (M] ⇒ S ⊆ M. Thus M = S. It is a contradiction. Therefore either A ⊆ M or B ⊆ M. Hence M is a prime.

THEOREM 4.20 : If S is a globally idempotent po-Γ-semigroup having maximal po-Γ-ideals then S contains semisimple elements.

Proof **:** Suppose that S is a globally idempotent po-Γ-semigroup having maximal po-Γ-ideals. Let M be a maximal po-Γ-ideal of S. Then by theorem 4.30, M is a prime po-Γ-ideal of S.

Now if $a \in$ S\M then $< a >\Gamma< a > \not\subseteq M \Rightarrow (< a >\Gamma< a >] \not\subseteq M$

and hence S = M ∪ $(< a >]$ = M ∪ $(< a >\Gamma< a >]$. Therefore $a \in (< a >\Gamma< a >]$.

Thus a is semisimple. Therefore S contains semisimple elements.

V . Completely Semiprime Po-Γ-Ideals And Semiprime Po-Γ-Ideals

We now introduce the notion of a completely semiprime po-Γ-ideal and a semiprime po-Γ-ideal in a po-Γ-semigroup.

DEFINITION 5.1 : A po-Γ-ideal A of a po- Γ-semigroup S is said to be a ***completely semiprime po-Γ- ideal*** provided $x\Gamma x \subseteq A$; $x \in$S implies $x \in$A.

THEOREM 5.2 : Every completely prime po-Γ-ideal of a po-Γ-semigroup S is a completely semiprime po-Γ-ideal of S.

Proof: Let A be a po- completely prime **Γ**-ideal of a po-Γ-semigroup S.

Suppose that $x \in S$ and $x\Gamma x \subseteq A$. Since A is a completely prime po-**Γ**-ideal of S, $x \in$A.

Therefore *S* is a completely semiprime po-**Γ**-ideal.

THEOREM 5.3 : The nonempty intersection of any family of a completely prime po- Γ-ideals of a po-Γ-semigroup S is a completely semiprime po-Γ-ideal of S.

Proof **:** Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a family of a completely prime po-Γ-ideals of S such that $\bigcap_{\alpha\in\Delta} A_\alpha \neq \emptyset$. By theorem 3.26, $\bigcap_{\alpha\in\Delta} A_\alpha$ is a po-Γ-ideal.

Let $a \in S$, $a\Gamma a \subseteq \bigcap_{\alpha\in\Delta} A_\alpha$. Then $a\Gamma a \subseteq A_\alpha$ for all $\alpha \in \Delta$.

Since A_α is a completely prime, $a \in A_\alpha$ for all $\alpha \in \Delta$ and hence $a \in \bigcap_{\alpha\in\Delta} A_\alpha$.

Therefore $\bigcap_{\alpha\in\Delta} A_\alpha$ is a completely semiprime po-Γ-ideal of S.

We now introduce the notion of a *d*-system of a po-Γ-semigroup.

DEFINITION 5.4 : Let S be a po-Γ-semigroup. A nonempty subset A of S is said to be a ***po-d-system*** of S if for each $a \in A$ and $\alpha \in \Gamma$, there exists an element $c \in A$ such that $c \leq a\alpha a$.

NOTE 5.5 : A nonempty subset A of a po-Γ-semigroup S is said to be a po-*d*-system of S if for each $a \in A$, there exists $c \in A$ such that $c \in (a\Gamma a]$.

We now prove a necessary and sufficient condition for a po-Γ-ideal to be a completely semiprime po-Γ-ideal in a po-Γ-semigroup.

THEOREM 5.6 : A po-Γ-ideal P of a po-Γ-semigroup S is a completely semiprime iff S\P is a po-*d*-system of S or empty.

Proof **:** Suppose that P is a completely semiprime po-Γ-ideal of S and $S\backslash P \neq \emptyset$.
Let $a \in S\backslash P$. Then $a \notin P$. Suppose if possible $c \notin (a\Gamma a]$ for every $c \in S\backslash P$.
Then $(a\Gamma a] \subseteq P \Rightarrow a\Gamma a \subseteq P$. Since P is a completely semiprime, $a \in P$.
It is a contradiction. Therefore there exists an element $c \in S\backslash P$ such that $c \leq a\alpha a$.
Therefore S\P is a po-*d*-system of S.
Conversely suppose that S\P is a *d*-system of S or S\P is empty.
If S\P is empty then P = S and hence P is completely semiprime.
Assume that S\P is a po-*d*-system of S.
Let $a \in S$ and $a\Gamma a \subseteq P$. Suppose if possible $a \notin P$. Then $a \in S\backslash P$.
Since S\P is a *d*-system, there exists an element $c \in S\backslash P$ such that $c \leq a\alpha a$ for $\alpha \in \Gamma$.
$c \leq a\alpha a \in a\Gamma a \subseteq P$. Therefore $c \in P$. It is a contradiction. Hence $a \in P$.
Thus P is a completely semiprime po-Γ-ideal of S.

We now introduce the notion of a semiprime po-Γ-ideal of a po-Γ-semigroup.

DEFINITION 5.7 : A po-Γ- ideal A of a po-Γ-semigroup S is said to be a ***semiprime po-Γ-ideal*** provided $x \in S$, $x\Gamma S^1\Gamma x \subseteq A$ implies $x \in A$.

THEOREM 5.8 : Every completely semiprime po-Γ-ideal of a po-Γ-semigroup S is a semiprime po-Γ-ideal of S.

Proof: Suppose that A is a completely semiprime po-**Γ**-ideal of a po-**Γ**-semigroup S.
Let $a \in S$ and $a\Gamma S^1\Gamma a \subseteq A$.
Now $a\Gamma a \subseteq a\Gamma S^1\Gamma a \subseteq A$. Since A is a completely semiprime, $a \in A$.
Therefore A is a semiprime po-**Γ**-ideal of S.

THEOREM 5.9 : Let S be a commutative po-Γ-semigroup. A po-Γ-ideal A of S is completely semiprime iff it is semiprime.

Proof **:**Suppose that A is a completely semiprime po-Γ-ideal of S.
By theorem 5.8, A is a semiprime po-Γ-ideal of S.
Conversely suppose that A is a semiprime po-Γ-ideal of S. Let $x \in S$ and $x\Gamma x \subseteq A$.
Now $x\Gamma x \subseteq A \Rightarrow s\Gamma x\Gamma x \subseteq A$ for all $s \in S \Rightarrow x\Gamma s\Gamma x \subseteq A$ for all $s \in S \Rightarrow x\Gamma S\Gamma x \subseteq A$
$\Rightarrow x \in A$, since A is a semiprime.
Therefore A is a completely semiprime po-Γ-ideal of S.

THEOREM 5.10 : Every prime po-Γ-ideal of a po-Γ-semigroup S is a semiprime po-Γ-ideal of S.

Proof: Suppose that A is a prime po-**Γ**-ideal of a po-**Γ**-semigroup S.
Let $a \in S$ and $a\Gamma S^1\Gamma a \subseteq A$. By corollary 4.19, $a \in A$.
Therefore A is a semiprime po-**Γ**-ideal of S.

THEOREM 5.11 : The nonempty intersection of any family of prime po-Γ-ideals of a po-Γ-semigroup S is a semiprime po-Γ-ideal of S.

Proof **:** Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a family of a prime po-Γ-ideals of S such that $\bigcap_{\alpha\in\Delta} A_\alpha \neq \emptyset$.

By theorem 3.26, $\bigcap_{\alpha\in\Delta} A_\alpha$ is a po-Γ-ideal.

Let $a \in S$, $a\Gamma S\Gamma a \subseteq \bigcap_{\alpha\in\Delta} A_\alpha$. Then $a\Gamma S\Gamma a \subseteq A_\alpha$ for all $\alpha \in \Delta$.

Since A_α is prime, $a \in A_\alpha$ for all $\alpha \in \Delta$ and hence $a \in \bigcap_{\alpha\in\Delta} A_\alpha$. Therefore $\bigcap_{\alpha\in\Delta} A_\alpha$ is a semiprime po-Γ-ideal of S.

We now introduce the notion of an *n*-system of a po-Γ-semigroup.

DEFINITION 5.12 : A nonempty subset A of a po-Γ-semigroup S is said to be an ***po-n-system*** provided for any $a \in A$ and some $\alpha, \beta \in \Gamma$ there exists an element $c \in A$, $x \in S$ such that $c \leq a\alpha x\beta a$.

NOTE 5.13 : A nonempty subset A of a po-Γ-semigroup S is said to be an **po-*n*-*system*** provided for any $a \in A$, $x \in S$ there exists an element $c \in A$ such that $c \in (a\Gamma S\Gamma a]$.

THEOREM 5.14 : Every po-*m*-system in a po-Γ-semigroup S is an po-*n*-system.

Proof **:** Let A be po-*m*-system of a po-Γ-semigroup S. Let $a \in A$. Since A is a po-*m-system.* $a,a \in A$ and $\alpha, \beta \in \Gamma$ there exists an $c \in A$ and $x \in S$ such that $c \leq a\alpha x\beta b \Rightarrow c \leq a\alpha x\beta a$ and hence A is an po-*n*-system of S.

THEOREM 5.15 : A nonempty set A is an *n*-system of Γ-semigroup (S, Γ, .) if and only if A is an *n*-system of a po- Γ-semigroup (S, Γ, ., ≤).

We now prove a necessary and sufficient condition for a po-Γ-ideal to be a semiprime po-Γ-ideal in a po-Γ-semigroup.

THEOREM 5.16 : A po-Γ- ideal Q of a po-Γ-semigroup S is a semiprime po-Γ-ideal iff S\Q is a po-*n*-system of S or empty.

Proof **:** Suppose that Q is a semiprime po-Γ-ideal of a po-Γ-semigroup S and $S\backslash Q \neq \emptyset$.

Let $a \in S\backslash Q$. Then $a \notin Q$. Suppose if possible $c \notin (a\Gamma S^1\Gamma a]$ for every $c \in S\backslash Q$.

Then $(a\Gamma S^1\Gamma a] \subseteq Q \Rightarrow a\Gamma S^1\Gamma a \subseteq Q$. Since Q is a semiprime, $a \in Q$. It is a contradiction.

Therefore there exist an element $c \in S\backslash Q$ such that $c \leq a\alpha x\beta a$ for some $a\alpha x\beta a \in a\Gamma S^1\Gamma a$.

Hence S\Q is an *n*-system.

Conversely suppose that S\Q is either an *n*-system of S or $S\backslash Q = \emptyset$.

If S\Q is empty then Q = S and hence Q is a semiprime.

Assume that S\Q is an *n*-system of S. Let $a \in S$ and $a\Gamma S^1\Gamma a \subseteq Q$.

Suppose if possible $a \notin Q$. Then $a \in S\backslash Q$. Since S\Q is a po-*n*-system.

There exists $c \in S\backslash Q$ such that $c \leq a\alpha x\beta a$ for some $x \in S$, $\alpha, \beta \in \Gamma$.

$c \leq a\alpha x\beta a \in a\Gamma S^1\Gamma a \subseteq Q$. Thus $c \in Q$.

It is a contradiction. Therefore $a \in Q$. Hence Q is a semiprime po-Γ-ideal of S.

THEOREM 5.17 : If N is an *n*-system in a po-Γ-semigroup S and *a* ∈ N, then there exists an *m*-system M in S such that *a* ∈ M and M ⊆N.

Proof: We construct a subset M of N as follows. Define $a_1 = a$.

Since $a_1 \in N$ and N is an *n*-system, there exists $c_1 \in N$ such that $c_1 \leq a_1\alpha x\beta a_1$ for some $x \in S$, $\alpha, \beta \in \Gamma$. Then $c_1 \in (a_1\Gamma S\Gamma a_1]$. Thus $(a_1\Gamma S\Gamma a_1] \cap N \neq \emptyset$. Let $a_2 \in (a_1\Gamma S\Gamma a_1] \cap N$.

Since $a_2 \in N$ and N is an *n*-system, there exists $c_2 \in N$ such that $c_2 \leq a_2\alpha x\beta a_2$ for some $x \in S$, $\alpha, \beta \in \Gamma$. Then $c_2 \in (a_2\Gamma S\Gamma a_2]$. Thus $(a_2\Gamma S\Gamma a_2] \cap N \neq \emptyset$ and so on.

In general, if a_i has been defined with $a_i \in N$, choose a_{i+1} as an element of $(a_i\Gamma S\Gamma a_i] \cap N$ there exists $c_{i+1} \in N$ such that $c_{i+1} \leq a_{i+1}\alpha x\beta a_{i+1}$ for some $x \in S$, $\alpha, \beta \in \Gamma$.

Then $c_{i+1} \in (a_{i+1}\Gamma S\Gamma a_{i+1}]$. Thus $(a_{i+1}\Gamma S\Gamma a_{i+1}] \cap N \neq \emptyset$.

Let $M = \{ a_1, a_2, \ldots\ldots, a_i, a_{i+1} \ldots\ldots\}$. Now $a \in M$ and $M \subseteq N$.

We now show that M is an *m*-system.

Let $a_i, a_j \in M$. If $i = j$ then, for the element $a_{j+1} \in S$, We have $a_{i+1} \in (a_i\Gamma S\Gamma a_i] \subseteq (a_i\Gamma S\Gamma a_j]$

$\Rightarrow a_{i+1} \leq a_i\alpha x\beta a_j$, $x \in S$, $\alpha, \beta \in \Gamma$.

If $i < j$ then, for the element $a_{j+1} \in S$,

We have $a_{j+1} \in (a_i\Gamma S\Gamma a_i] \subseteq ((a_{j-1}\Gamma S\Gamma a_{j-1}]Sa_j] \subseteq (a_{j-1}\Gamma S\Gamma a_j] \subseteq \ldots \subseteq (a_i\Gamma S\Gamma a_j]$.

Hence $a_{j+1} \leq a_i\alpha x\beta a_j \in S$, for $x \in S$ $\alpha, \beta \in \Gamma$.

If $j < i$ then, for the element $a_{i+1} \in S$.

We have $a_{i+1} \in (a_i\Gamma S\Gamma a_i] \subseteq (a_i\Gamma S\Gamma(a_{i-1}Sa_{i-1})] \subseteq (a_i\Gamma S\Gamma a_{i-1}] \subseteq \ldots \subseteq (a_i\Gamma S\Gamma a_j]$.

Therefore M is an *m*-system.

VI . Conclusion

It is proved that (1) every completely semiprime po-Γ-ideal of a po-Γ-semigroup is a semiprime po-Γ-ideal, (2) every po- completely prime Γ-ideal of a po- Γ-semigroup is a po-completely semiprime Γ-ideal. It is also proved that the nonempty intersection of any family of (1)a po- completely prime Γ-ideals of a po-Γ-semigroup is a po- completely semiprime Γ-ideal, (2)a po- prime Γ-ideals of a po-Γ-semigroup is a semiprime po-Γ-ideal. It is also proved that a po- Γ-ideal Q of a po-Γ-semigroup S is a semiprime iff S\Q is either an n-system or empty. Further it is proved that if N is an n-system in a po-Γ-semigroup S and a ∈ N, then there exists an m-system M of S such that a ∈ M and M ⊆ N. The study of ideals in semigroups, Γ-semigroups creates a platform for the ideals in po- Γ-semigroups.

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