Asymptotic Behavior of a Generalized Polynomial

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Abstract: We have extended the corresponding result of Voronowskaja for Lebesgue integrable function in $L_1$-norm by our newly defined Generalized Polynomial.

$$A_n^\alpha(f,x) = (n+1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} f(t) \, dt \right\} p_{n,k}(x;\alpha)$$

where

$$p_{n,k}(x;\alpha) = \binom{n}{k} x^k (1-x)^{n-k}.$$

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I. Introduction and Results

If $f(x)$ is a function defined $[0,1]$, the Bernstein polynomial $B_n^f(x)$ of $f$ is given as

$$B_n^f(x) = \sum_{k=0}^{n} f(k/n) p_{n,k}(x) \quad \text{....(1.1)}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \quad \text{....(1.2)}$$

One question arises about the rapidity of convergence of $B_n^f(x)$ to $f(x)$. An answer to this question has been given in different directions. One direction is that in which $f(x)$ is supposed to be at least twice differentiable in a point $x$ of $[0,1]$.

Voronowskaja [6] proved that

$$\lim_{n \to \infty} n \left| f(x) - B_n^f(x) \right| = -\frac{1}{2} x(1-x)f''(x). \quad \text{....(1.3)}$$

In particular, if $f''(x) \neq 0$, difference $f(x) - B_n^f(x)$ is exactly of order $n^{-1}$.

A small modification of Bernstein polynomial due to Kantorovitch [4] makes it possible to approximate Lebesgue integrable function in $L_1$-norm by the modified polynomials

$$P_n^f(x) = (n+1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} f(t) \, dt \right\} p_{n,k}(x) \quad \text{....(1.4)}$$

where $p_{n,k}(x)$ is defined by (1.2).

By Abel’s formula (see Jensen [3])

$$(x+y+na)^n = \sum_{k=0}^{n} \binom{n}{k} x^k (x+k\alpha y + (n-k)\alpha)^{n-k} \quad \text{....(1.5)}$$

If we put $x = 1$ we obtain (see Cheney and Sharma [2])

$$(1+na)^n = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x+(n-k)\alpha)^{n-k}$$

Or

$$1 = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x+(n-k)\alpha)^{n-k} \quad \text{....(1.6)}$$

Thus defining

$$p_{n,k}(x;\alpha) = \binom{n}{k} x^k (1-x+(n-k)\alpha)^{n-k} \quad \text{....(1.7)}$$

we have

$$\sum_{k=0}^{n} p_{n,k}(x;\alpha) = 1 \quad \text{....(1.8)}$$

and (see Anwar Habib [1])

$$\sum_{k=0}^{n} k p_{n,k}(x;\alpha) \leq nx/(1 + \alpha) \quad \text{....(1.9)}$$

$$\sum_{k=0}^{n} k^2 p_{n,k}(x;\alpha) \leq \frac{nx(1-x)n^2}{1 + \alpha} \quad \text{....(1.10)}$$
we now define the Generalized Polynomial
\[ A^\alpha_n(f,x) = (n+1) \sum_{k=0}^{N} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} f(t) dt \right) p_{n,k}(x;\alpha) \] …(1.11)
where \( p_{n,k}(x;\alpha) \) is defined in (1.7) and moreover when \( \alpha = 0 \), (1.7) and (1.11) reduces to (1.2) and (1.4) respectively.

In this paper, we shall prove the corresponding results of approximation due to Voronowska[6] for Lebesgue integrable function in \( L_1 \)-norm by the our polynomial (1.11). In fact we state our result is as follows of Theorem: let \( f(x) \) be bounded Lebesgue integrable function with its first derivative in \([0,1]\) and suppose that the second derivative \( f''(x) \) exists at a certain point \( x \) of \([0,1]\), then for \( \alpha = \alpha_n = o(1/n) \)
\[ \lim_{n \to \infty} n [A^\alpha_n(f,x) - f(x)] = \frac{1}{2} [(1-2x)f''(x) - x(1-x)f''(x)] \]

II. Lemma

we first like to prove the lemma which would be useful for the proof of our theorem

**Lemma** - For all values of \( x \in [0,1] \) and for \( \alpha = \alpha_n = o(1/n) \), we have
\[ (n+1) \sum_{k=0}^{n} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 dt \right) p_{n,k}(x;\alpha) \leq x(1-x)/n \]

**Proof of Lemma:**
\[
\begin{align*}
(n+1) \sum_{k=0}^{n} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 dt \right) p_{n,k}(x;\alpha) &= \sum_{k=0}^{n} \left[ x^2 - \frac{2kx + x}{n+1} + \frac{k^2 + k}{(n+1)^2} + \frac{1}{3(n+1)^2} \right] p_{n,k}(x;\alpha) \\
&\leq x^2 - \frac{1}{n+1} \left[ \frac{2n^2}{2} + x \right] + \frac{1}{n+1} \left[ x(1+2\alpha) \right] \frac{n(n-1)}{(1+2\alpha)^2} \\
&\quad + \frac{(n-2)\alpha^2}{1+\alpha} + \frac{1}{3(n+1)^2} (by 1.9 & 1.10) \\
\end{align*}
\]
\[ \leq \frac{1}{n(1-\alpha)(1+2\alpha)^2(3+3\alpha)^2} \left[ x(1-x) + ax(1-x)(2n+9) + x \right. \\
+ \alpha^2 x(1-x)(17n+23) + 9x \]
\[ + \alpha^3 x(1-x)(57n+13) + 7nx^2 + x(5n^2+35) \]
\[ + \alpha^4 x(1-x)(96n-144) + 86nx^2 + x(65n^2+12n) \]
\[ + \alpha^5 x(1-x)(54n-216) + x(4n-12n+46) + 162nx^2 \]
\[ + \alpha^6 108x(1-x) + 108nx^2 + 1/3n^2 \]
\[ \leq \frac{x(1-x)}{n} \text{ for } \alpha = \alpha_n = o\left( \frac{1}{n} \right) \text{ and for large } n \]

which completes the proof of Lemma.

III. Proof of the Theorem

**Proof of Theorem**

We write(in view of Taylor’s Theorem)
\[ f(t) = f(x) + (t-x)f'(x) + (t-x)^2 \left[ \frac{1}{2} f''(x) + \eta(t-x) \right] \] …(2.1)
where \( \eta(h) \) is bounded \( |\eta(h)| \leq H \) for all \( h \) and converges to zero with \( h \). Multiplying eqn. (2.1) by \( (n+1)p_{n,k}(x;\alpha) \) and integrating it from \( k/(n+1) \) to \( (k+1)/(n+1) \), then on summing ,we get
\[
(n+1) \sum_{k=0}^{n} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} f(t) dt \right) p_{n,k}(x;\alpha) 
\]
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\[
\begin{align*}
= (n + 1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} f(x) \, dx \right\} p_{n,k}(x; \alpha) \\
+ (n + 1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} f(t-x)f'(x) \, dx \right\} p_{n,k}(x; \alpha) \\
+ \frac{1}{2}(n + 1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2f''(x) \, dx \right\} p_{n,k}(x; \alpha) \\
+ (n + 1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 \eta(t-x) \, dx \right\} p_{n,k}(x; \alpha)
\end{align*}
\]

\[= I_1 + I_2 + I_3 + I_4 \text{ (say)} \quad \ldots \ldots (2.2)\]

Now first we evaluate \(I_1\):

\[
I_1 = (n + 1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} f(x) \, dx \right\} p_{n,k}(x; \alpha)
\]

\[= f(x) \quad \ldots \ldots (2.3)\]

and then

\[
I_2 = (n + 1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} f(t-x)f'(x) \, dx \right\} p_{n,k}(x; \alpha)
\]

\[= \sum_{k=0}^{n} \left( \frac{2k + 1}{2(n + 1)} - x \right) f(x)p_{n,k}(x; \alpha)
\]

\[\leq \frac{(1-2x)}{2n} f(x) \text{ for } \alpha = \alpha_n = o(1/n) \quad \ldots \ldots (2.4)\]

Now we evaluate \(I_3\):

\[
I_3 = \frac{1}{2}(n + 1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2f''(x) \, dx \right\} p_{n,k}(x; \alpha)
\]

\[\leq x(1-x)f''(x)/2n \quad \text{by lemma} \quad \ldots \ldots (2.5)\]

and then in the last we evaluate \(I_4\):

\[
I_4 = (n + 1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 \eta(t-x) \, dx \right\} p_{n,k}(x; \alpha)
\]

\(I_4\) can be estimated easily. Let \(\varepsilon > 0\) be arbitrary \(\delta > 0\) such that \(|\eta(h)| < \varepsilon\) for \(|h| < \delta\) thus breaking up the sum \(I_4\) into two parts corresponding to those values of \(t\) for which \(|t-x| < \delta\), and since in the given range of \(t\), \(\left[\frac{k}{n} - x \right] < |t - x|\), we have

\[
|I_4| \leq \sum_{\left\{\frac{k}{n} - x \right\} < \delta} (n + 1)p_{n,k}(x; \alpha) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)^2dt \quad |
\]

\[+ H \sum_{\left\{\frac{k}{n} - x \right\} \geq \delta} (n + 1)p_{n,k}(x; \alpha) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt \quad |
\]

\[= I_3 + I_6 \text{ (say)} \]

\[|I_3| \leq \frac{\varepsilon}{n} \left\{x(1-x)\right\}, \text{ for } \alpha = \alpha_n = o(1/n)
\]
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\[ I_6 = (n + 1) H \sum_{\left\lfloor \frac{k}{n} \right\rfloor - x \geq \delta} \left\{ \int_{\frac{k}{n}}^{\frac{k+1}{n}} dt \right\} p_{n,k}(x; \alpha) \]

\[ = (n + 1) \sum_{\left\lfloor \frac{k}{n} \right\rfloor - x \geq \delta} p_{n,k}(x; \alpha) \frac{1}{n+1} \]

But if \( n^{-\beta}, 0 < \beta < 1/2 \) (see also Kantorovich [4]), then for \( \alpha = \alpha_n = o(1/n) \)

\[ \sum_{\left\lfloor \frac{k}{n} \right\rfloor - x \geq \delta} p_{n,k}(x; \alpha) \leq Cn^{-\gamma} \]

For \( \gamma > 0 \), the constant \( C = C(\beta, \gamma) \).

whence \( I_6 < \frac{c}{n+1} \leq \epsilon/n \) for all \( n \) sufficiently large and therefore it follows

\[ I_k < \epsilon/n \], for all sufficiently large \( n \) \hspace{1cm} \text{.....(2.6)} \]

Hence from (2.2), (2.3), (2.4), (2.5) and (2.6), we have

\[ (n + 1) \sum_{k=0}^{n} \left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} f(t) dt \right\} p_{n,k}(x; \alpha) \]

\[ = f(x) + \left[ (1 - 2x)f'(x) + x(1 - x)f''(x) \right]/2n \] + \( \epsilon/n \)

and therefore, finally we get

\[ \lim_{n \to \infty} n \left[ A_n^\alpha(f, x) - f(x) \right] = \frac{1}{2} \left[ (1 - 2x)f'(x) - x(1 - x)f''(x) \right] \]

where \( \epsilon \to 0 \) as \( n \to \infty \), which completes the proof of the theorem.

IV. Conclusion

In this paper we have extended the result of Voronowskaja by taking a Generalized Polynomials

\[ A_n^\alpha(f, x) \] instead of Bernstein Polynomial \[ B_n^\alpha(f, x) \]

References