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Asymptotic Behavior of a Generalized Polynomial

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Abstract: *We have extended the corresponding result of Voronowskaja for Lebesgue integrable function in L_1-norm by our newly defined Generalized Polynomial.*

$$A_n^{\alpha}(f,x) = (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt\right\} p_{n,k}(x;\alpha)$$

where $p_{n,k}(x;\alpha) = \binom{n}{k}\frac{x(x+k\alpha)^{k-1}(1-x+(n-k)\alpha)^{n-k}}{(1+n\alpha)^n}$

***Keywords**: Bernstein Polynomials, Convergence, Generalized Polynomials, Integrable function, L_1 norm,*

I. Introduction and Results

If $f(x)$ is a function defined [0,1], the Bernstein polynomial $B_n^f(x)$ of f is given as

$B_n^f(x) = \sum_{k=0}^{n} f(k/n)\, p_{n,k}(x)$ ……(1.1)

where

$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ ……….(1.2)

One question arises about the rapidity of convergence of $B_n^f(x)$ to $f(x)$. An answer to this question has been given in different directons. One direction is that in which $f(x)$ is supposed to be at least twice differentiable in a point x of [0,1].

Voronowskaja [6] proved that

$\lim_{n\to\infty} n\left|f(x) - B_n^f(x)\right| = -\frac{1}{2}x(1-x)f''(x).$ …(1.3)

In particular , if $f''(x) \neq 0$, difference $f(x) - B_n^f(x)$ is exactly of order n^{-1}

A small modification of Bernstein polynomial due to Kantorovitch [4] makes it possible to approximate Lebesgue integrable function in L_1-norm by the modified polynomials

$P_n^f(x) = (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt\right\} p_{n,k}(x)$ …(1.4)

where $p_{n,k}(x)$ is defined by (1.2)

By Abel's formula(see Jensen [3])

$(x+y+n\alpha)^n = \sum_{k=0}^{n}\binom{n}{k} x(x+k\alpha)^{k-1}(y+(n-k)\alpha)^{n-k}$ ….(1.5)

If we put $= 1-x$, we obtain (see Cheney and Sharma [2])

$(1+n\alpha)^n = \sum_{k=0}^{n}\binom{n}{k} x(x+k\alpha)^{k-1}(1-x+(n-k)\alpha)^{n-k}$

Or

$1 = \sum_{k=0}^{n}\binom{n}{k}\frac{x(x+k\alpha)^{k-1}(1-x+(n-k)\alpha)^{n-k}}{(1+n\alpha)^n}$ …..(1.6)

Thus defining

$p_{n,k}(x;\alpha) = \binom{n}{k}\frac{x(x+k\alpha)^{k-1}(1-x+(n-k)\alpha)^{n-k}}{(1+n\alpha)^n}$ …..(1.7)

we have

$\sum_{k=0}^{n} p_{n,k}(x;\alpha) = 1$ ….(1.8)

and (see Anwar Habib [1])

$$\sum_{k=0}^{n} kp_{n,k}(x;\alpha) \leq nx/(1+\alpha) \text{------------------} 1.9$$

$\sum_{k=0}^{n} k^2 p_{n,k}(x;\alpha) \leq) \frac{nx(1-x)+n^2x(x+\alpha)}{1+\alpha}$ -------------1.10

we now define the Generalized Polynomial

$$A_n^{\alpha}(f,x) = (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt\right\} p_{n,k}(x;\alpha) \quad \ldots(1.11)$$

where $p_{n,k}(x;\alpha)$ is defined in (1.7) and moreover when $\alpha = 0$, (1.7) and (1.11) reduces to (1.2) and (1.4) respectively.

In this paper, we shall prove the corresponding results of approximation due to Voronowskja[6] for Lebesgue integrable function in L_1-norm by the our polynomial (1.11). In fact we state our result is as follows of

Theorem: let $f(x)$ be bounded Lebesgue integrable function with its first derivative in [0,1] and suppose that the second derivative $f''(x)$ exists at a certain point x of [0,1],

then for $\alpha = \alpha_n = o(1/n)$

$$\lim_{n\to\infty} n\left[A_n^{\alpha}(f,x) - f(x)\right] = \frac{1}{2}[(1-2x)f'(x) - x(1-x)f''(x)]$$

II. Lemma

we first like to prove the lemma which would be useful for the proof of our theorem

Lemma - For all values of $x \in |0,1|$ and for $\alpha = \alpha_n = o(1/n)$, we have

$$(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 dt\right\} p_{n,k}(x;\alpha) \le x(1-x)/n$$

Proof of Lemma:

$$(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 dt\right\} p_{n,k}(x;\alpha)$$

$$= \sum_{k=0}^{n}[x^2 - \frac{2kx+x}{n+1} + \frac{k^2+k}{(n+1)^2} + \frac{1}{3(n+1)^2}]p_{n,k}(x;\alpha)$$

$$\le x^2 - \frac{1}{(n+1)}[\frac{2nx^2}{1+\alpha} + x] + \frac{1}{(n+1)^2}\ [n(n-1)x\frac{x+2\alpha}{(1+2\alpha)^2} + \frac{(n-2)\alpha^2}{(1+3\alpha)^3} + \frac{2nx}{1+\alpha}] + \frac{1}{3(n+1)^2} \qquad (by\ 1.9\ \&\ 1.10)$$

$$\le \frac{1}{n(1-\alpha)(1+2\alpha)^2(1+3\alpha)^3}\ [x(1-x) + \alpha x(1-x)(2n+9) + x$$
$$+\alpha^2 x(1-x)(17n+23) + 9x$$
$$+\alpha^3 x(1-x)(57n-13) + 7nx^2 + x(5n^2+35)$$
$$+\alpha^4 x(1-x)(96n-144) + 86nx^2 + x(65n^2+12n)$$
$$+\alpha^5 x(1-x)(54n-216) + x(4n-12n+46) + 162nx^2$$
$$+\alpha^6 108x(1-x) + 108nx^2] + 1/3n^2$$
$$\le \frac{x(1-x)}{n} \text{ for } \alpha = \alpha_n = o\left(\frac{1}{n}\right) \text{ and for large n}$$

which completes the proof of Lemma.

III. Proof of the Theorem

Proof of Theorem

We write(in view of Taylor's Theorem)

$$f(t) = f(x) + (t-x)f'(x) + (t-x)^2[\frac{1}{2}f''(x) + \eta(t-x)\ ] \quad \ldots..(2.1)$$

where $\eta(h)$ is bounded $|\eta(h)| \le H$ for all h and converges to zero with h.

Multiplying eqn. (2.1) by $(n+1)p_{n,k}(x;\alpha)$and integrating it from *k/(n+1)* to *(k+1)/(n+1),* then on summing ,we get

$$(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt\right\} p_{n,k}(x;\alpha)$$

$$= (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(x)dt\right\} p_{n,k}(x;\alpha)$$

$$+(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(t-x)f'(x)dt\right\} p_{n,k}(x;\alpha)$$

$$+\frac{1}{2}(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 f''(x)dt\right\} p_{n,k}(x;\alpha)$$

$$+(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 \eta(t-x)dt\right\} p_{n,k}(x;\alpha)$$

$$= I_1 + I_2 + I_3 + I_4 \text{ (say)} \qquad \ldots\ldots(2.2)$$

Now first we evaluate I_1:

$$I_1 = (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(x)dt\right\} p_{n,k}(x;\alpha)$$

$= f(x)$ ….(2.3)

and then

$$I_2 = (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(t-x)f'(x)dt\right\} p_{n,k}(x;\alpha)$$

$$= \sum_{k=0}^{n}\left(\frac{2k+1}{2(n+1)} - x\right) f'(x) p_{n,k}(x;\alpha)$$

$\leq \frac{(1-2x)}{2n} f'(x)$ for $\alpha = \alpha_n = o(1/n)$ ……(2.4)

Now we evaluate I_3:

$$I_3 = \frac{1}{2}(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 f''(x)dt\right\} p_{n,k}(x;\alpha)$$

$$\leq x(1-x)f''(x)/2n \text{ (by lemma)} \qquad \ldots..(2.5)$$

and then in the last we evaluate I_4 :

$$I_4 = (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 \eta(t-x)dt\right\} p_{n,k}(x;\alpha)$$

I_4 can be estimated easily. Let $\in > 0$ be arbitrary $\delta > 0$ such that $|\eta(h)| < \epsilon$ for $|h| < \delta$ thus breaking up the sum I_9 into two parts corresponding to those values of t for which | t - x |< δ, and since in the given range of t, $\left|\frac{k}{n} - x\right| \sim |t-x|$, we have

$$|I_4| \leq \in \sum_{\left|\left(\frac{k}{n}\right)-x\right|<\delta} (n+1)p_{n,k}(x;\alpha)\left|\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)^2 dt\right|$$

$$+ H \sum_{\left|\left(\frac{k}{n}\right)-x\right|\geq\delta} (n+1)p_{n,k}(x;\alpha)\left|\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right|$$

$= I_5 + I_6$ (say)

$|I_5| \leq \frac{\in}{n} |\{x(1-x)\}|$, for $\alpha = \alpha_n = o(1/n)$

$$I_6 = (n+1)\, H \sum_{|\left(\frac{k}{n}\right)-x|\geq\delta} \left\{ \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt \right\} p_{n,k}(x;\alpha)$$

$$= (n+1) \sum_{\left|\left(\frac{k}{n}\right)-x\right|\geq\delta} p_{n,k}\,(x;\alpha)\frac{1}{n+1}$$

But if $= n^{-\beta}$, 0 < β < 1/2 (see also Kantorovitch [4]),
then for $\alpha = \alpha_n \ = o(1/n)$

$$\sum_{\left|\left(\frac{k}{n}\right)-x\right|\geq n^{-\beta}} p_{n,k}\,(x;\alpha) \leq Cn^{-\nu}$$

For ν>0, the constant C= C(β,ν).
whence $I_6 < \frac{\epsilon}{n+1} < \epsilon/n$ for all *n* sufficiently large and therefore it follows
$I_4 < \epsilon/n$, for all sufficiently large n …..(2.6)
Hence from (2.2), (2.3), (2.4), (2.5) and (2.6), we have

$$(n+1)\sum_{k=0}^{n}\left\{ \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt \right\} p_{n,k}(x;\alpha)$$
$$= f(x) + [\{(1-2x)f^{'}(x) + x(1-x)f"(x)\}/2n] + (\epsilon/n)$$

and therefore, finally we get

$$\lim_{n\to\infty} n\,\left[A_n^{\alpha}(f,x) - f(x)\right] = \frac{1}{2}[(1-2x)f^{'}(x) - x(1-x)f^{"}(x)]$$

where $\epsilon \to 0 \ as\ n \to \infty$
which completes the proof of the theorem.

IV. Conclusion

In this paper we have extended the result of Voronowskaja by taking a Generalized Polynomials $A_n^f(x)$ instead of Bernstein Polynomial $B_n^f(x)$

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