On The n-Dimensional Generalized Weyl Fractional Calculus
Associated With n-Dimensional $H$-Transform

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Abstract: The main object of this paper is to obtain n-dimensional generalized Weyl fractional operators pertaining to multivariable $H$-function. Here we get the results by using n-dimensional Laplace and $H$-transforms. The results of this paper are believed to be new and basic in nature. Some known results have been obtained by giving suitable values to the coefficients and parameters.

Key Words: Multivariable $H$-function, Weyl fractional operator, $H$-transform

Mathematics Subject Classification: 33C60, 33C65, 44A10

I. Introduction

In this paper the n-dimensional Weyl fractional operator associated with $H$-transform is obtained. The results obtained in this paper are of manifold generality, basic in nature and include the results given earlier by Saigo, Saxena and Ram [15], Saxena and Ram [18], Chaurasia and Srivastava [3] etc.

The $H$-function appearing here, introduced by Inayat-Hussain ([7], see also [2]) in terms of Mellin-Barnes type contour integral, is defined as

$$H_{M,N}^{P,Q} \left[ \begin{array}{c}
(a_j;\alpha_j)_{N},(b_j;\beta_j)_{M} |_{N+1,P} \\
(b_j;\beta_j)_{M},(b_j;\beta_j)_{M+1,Q}
\end{array} \right] = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \phi(s) z^s ds$$

where

$$\phi(s) = \prod_{j=1}^{M} \Gamma(b_j - \beta_j s) \prod_{j=1}^{N} \{\Gamma(1-a_j + \alpha_j s)\}^{A_j} \prod_{j=M+1}^{Q} \{\Gamma(1-b_j + \beta_j s)\}^{B_j} \prod_{j=N+1}^{P} \Gamma(a_j - \alpha_j s)$$

which contains fractional powers of some of the $\Gamma$-functions. Here and throughout the paper $A_j (j = 1,\ldots,P)$ and $B_j (j = 1,\ldots,Q)$ are complex parameters, $\alpha_j \geq 0$ (not all zero simultaneously) and the exponents $A_j (j = 1,\ldots,N)$ and $B_j (j = M+1,\ldots,Q)$ can take on non-integer values. The contour in (1) is imaginary axis Re(s) = 0. It is suitably indented in order to avoid the singularities of the $\Gamma$-functions and to keep these singularities on appropriate sides. Again, for $A_j (j = 1,\ldots,N)$ not an integer, the poles of the $\Gamma$-functions of the numerator in (2) are converted to branch points. However, as long as there is no coincidence of poles from any $\Gamma(b_j - \beta_j s)$, (j = 1, M) and $\Gamma(1-a_j + \alpha_j s)$, (j = 1, N) pair the branch cuts can be chosen so that the path of integration can be distorted in the usual manner. For the sake of brevity

$$\sigma = \sum_{j=1}^{M} |\beta_j| + \sum_{j=1}^{N} A_j \alpha_j - \sum_{j=M+1}^{Q} B_j \beta_j - \sum_{j=N+1}^{P} \alpha_j > 0$$

Some useful generalization of both Riemann-Liouville and Erdelyi-Kober fractional integration operators are introduced by Saigo [11], [12] in terms of Gauss’s hypergeometric function as given below.

Let $\alpha$, $\beta$ and $\theta$ are complex numbers and let $x \in \mathbb{R}_{+}(0,\infty)$. Following [11], [12] the fractional integral (Re($\alpha$) > 0) and derivative (Re($\alpha$) < 0) of the first kind of a function $f(x)$ on $\mathbb{R}_{+}$ are defined respectively in the forms

$$f^{\alpha,0}_{0,x} = \frac{x^{-\alpha-\theta}}{\Gamma(\alpha)} \int_{0}^{x} (1-t)^{\alpha-1} \, _{2}F_{1}\left(\alpha + \beta, -\theta; \alpha; \frac{t}{x}\right) f(t) \, dt; \, \text{Re}(\alpha) > 0$$

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(5) \[ \frac{d^n}{dx^n} I_{0,x}^{\alpha+n,\beta-n} f, \quad 0 < \text{Re}(\alpha) + n \leq 1(n=1,2,...) \]

where \( {}_2F_1(a,b;c;x) \) is Gauss’s hypergeometric function. The fractional integral (Re(\alpha) > 0) and derivative (Re(\alpha) < 0) of the second kind are given by

(6) \[ I_{x,\infty}^{\alpha,\beta} f = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1 \left( \alpha + \beta, -\beta; \alpha; \frac{t-x}{t} \right) f(t) \, dt, \quad \text{Re} \, \alpha > 0 \]

(7) \[ (-1)^n \frac{d^n}{dx^n} I_{x,\infty}^{\alpha+n,\beta-n} f, \quad 0 < \text{Re}(\alpha) + n \leq 1(n=1,2,...) \]

Following Miller [10, p.82], we denote by \( u_1 \) the class of functions \( f(x_i) \) on \( \mathbb{R} \), which are infinitely differentiable with partial derivatives of any order behaving as \( 0(|x_i|^{-\xi_i}) \) when \( x_i \to \infty \) for all \( \xi_i \). Similarly by \( u_2 \), we denote the class of function \( f(x_i, x_j) \) on \( \mathbb{R} \times \mathbb{R} \), which are infinitely differentiable with partial derivatives of any order behaving as \( 0(|x_i|^{-\xi_i} |x_j|^{-\xi_j}) \) when \( x_i \to \infty \) for all \( \xi_i \) (\( i=1,2 \)).

On the same pattern by \( u_n \), we denote the class of function \( f(x_1, x_2, \ldots, x_n) \) on \( \mathbb{R}^n \), which are infinitely differentiable with partial derivatives of any order behaving as

\[ 0\left( \prod_{i=1}^{n} |x_i|^{-\xi_i} \right) \] when \( x_i \to \infty \) for all \( \xi_i \) (\( i=1,2,\ldots, n \)).

The n-dimensional operator of Weyl type fractional integration of orders Re(\( a_i \)) > 0, \( i = 1,2,\ldots,n \) is defined in class \( u_n \) by

(8) \[ \prod_{i=1}^{n} \left( \frac{1}{p_i} \right)^{a_i-b_i} \frac{1}{\Gamma(a_i)} \int_{0}^{\infty} \cdots \int_{0}^{\infty} \prod_{i=1}^{n} \left[ (u_i-p_i)^{a_i} u_i^{-a_i-b_i} {}_2F_1 \left( a_i+b_i-c_i-1, a_i; \frac{p_i}{u_i} \right) \right] f(u_1, u_2, \ldots, u_n) \, du_1 \, du_2 \cdots du_n \]

where \( b_i \) and \( c_i \) (\( i=1,2,\ldots,n \)) are real numbers.

The n-dimensional Laplace transform \( L(p_1, p_2, \ldots, p_n) \) of a function \( F(x_1, x_2, \ldots, x_n) \in u_n \) is defined as

(9) \[ L(p_1, p_2, \ldots, p_n) = \sigma[F(x_1, x_2, \ldots, x_n); p_1, p_2, \ldots, p_n] \]

\[ = \int_{0}^{\infty} \cdots \int_{0}^{\infty} e^{-\sum_{i=1}^{n} (p_i x_i)} f(x_1, x_2, \ldots, x_n) \, dx_1 \, dx_2 \cdots dx_n \quad \text{(n-times)} \]

where Re(\( p_i \)) > 0, \( i=1,2,\ldots,n \).

Similarly, the Laplace transform of

\[ f \left[ u_1 \sqrt{x_1^2 - \lambda_1^2} H(x_1 - \lambda_1), u_2 \sqrt{x_2^2 - \lambda_2^2} H(x_2 - \lambda_2), \ldots, u_n \sqrt{x_n^2 - \lambda_n^2} H(x_n - \lambda_n) \right] \]

is defined by the Laplace transform of \( F(x_1, x_2, \ldots, x_n) \) where

(10) \[ F(x_1, x_2, \ldots, x_n) = \frac{\sqrt{x_1^2 - \lambda_1^2} H(x_1 - \lambda_1), u_2 \sqrt{x_2^2 - \lambda_2^2} H(x_2 - \lambda_2), \ldots, \sqrt{x_n^2 - \lambda_n^2} H(x_n - \lambda_n)}{u_1} \]
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\[ u_n \sqrt{(x_n^2 - \lambda_n^2)} H(x_n - \lambda_n) \]

\[ x_i > \lambda_i > 0, \ i = 1, 2, ..., n \]

and H(t) denotes Heaviside’s unit step function.

The n-dimensional \( H \)-transform \( \xi(p_1, p_2, ..., p_n) \) of a function \( F(x_1, x_2, ..., x_n) \) is defined as

\[ (11) \quad \xi(p_1, p_2, ..., p_n) = \int_{\lambda_1}^{\infty} \int_{\lambda_2}^{\infty} \cdots \int_{\lambda_n}^{\infty} \prod_{i=1}^{n} (p_i x_i)^{\alpha_i - 1} H(p_i x_i)^{\alpha_i} \left[ (a_j; \beta_j; l_j; M_j; N_j), (b_j; \gamma_j; k_j; l_j; M_j; N_j) \right] f(x_1, x_2, ..., x_n) \ dx_1 dx_2 ... dx_n \]

where \( \lambda_i > 0, k_i > 0 (i = 1, 2, ..., n); \xi(p_1, p_2, ..., p_n) \) exists and belongs to \( u_n \).

Also

\[ (12) \quad |\arg(p_i)| < \frac{1}{2} \pi, \ i = 1, 2, ..., n \]

where

\[ \sigma_i = \sum_{j=1}^{M_i} |\beta_j| + \sum_{j=1}^{N_i} A_j - \sum_{j=M_i+1}^{Q_i} |B_j| \beta_j - \sum_{j=N_i+1}^{P_i} \alpha_j > 0, \text{ for } i = 1, 2, ..., n. \]

II. Main Results

In this section, following n-dimensional \( H \)-transform \( \xi_1(p_1, p_2, ..., p_n) \) of \( F(x_1, x_2, ..., x_n) \) is used and it is defined by

\[ (13) \quad \xi_1(p_1, p_2, ..., p_n) = \int_{\lambda_1}^{\infty} \int_{\lambda_2}^{\infty} \cdots \int_{\lambda_n}^{\infty} \prod_{i=1}^{n} (p_i x_i)^{\alpha_i - 1} H(p_i x_i)^{\alpha_i} \left[ (a_j; \beta_j; l_j; M_j; N_j), (b_j; \gamma_j; k_j; l_j; M_j; N_j) \right] f(x_1, x_2, ..., x_n) \ dx_1 dx_2 ... dx_n \]

where it is assumed that \( \xi_1(p_1, p_2, ..., p_n) \) exists and belongs to \( u_n \) as well as \( k_i > 0, \ i = 1, 2, ..., n \) and other conditions on the parameters, in which additional parameters \( a_i, b_i, c_i, \ i = 1, 2, ..., n \) included correspond to those in (8).

THEOREM 1 Let \( \xi(p_1, p_2, ..., p_n) \) be given by definition (11), then for \( \text{Re}(a_i) > 0, \lambda_i > 0, k_i > 0, i = 1, 2, ..., n \) there holds the formula
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\[
J_{p_1,\infty}^{a_1,b_1,c_1} J_{p_2,\infty}^{a_2,b_2,c_2} ... J_{p_n,\infty}^{a_n,b_n,c_n} [\xi(p_1,p_2,...,p_n)] = \xi(p_1,p_2,...,p_n)
\]

provided that \( \xi(p_1,p_2,...,p_n) \) exists and belongs to \( u_n \).

**Proof** Let \( \text{Re}(a_i) > 0 \), \( i = 1,2,...,n \) then by using (8) and (11), we have

\[
(14) \quad J_{p_1,\infty}^{a_1,b_1,c_1} J_{p_2,\infty}^{a_2,b_2,c_2} ... J_{p_n,\infty}^{a_n,b_n,c_n} [\xi(p_1,p_2,...,p_n)]
\]

\[
= \prod_{i=1}^{n} (p_i)^{b_i} \int_{p_i}^{\infty} ... \int_{p_n}^{\infty} \prod_{i=1}^{n} \Gamma(a_i) \left[ (u_i - p_i)^{a_i - 1} u_i^{-a_i - b_i} F_1 \left( a_i + b_i, -c_i; a_i, l - \frac{p_i}{u_i} \right) \right] \cdot \xi(u_1,u_2,...,u_n) \ du_1 du_2 ... du_n
\]

\[
= \prod_{i=1}^{n} (p_i)^{b_i} \int_{p_1}^{\infty} ... \int_{p_n}^{\infty} \prod_{i=1}^{n} \Gamma(a_i) \left[ (u_i - p_i)^{a_i - 1} u_i^{-a_i - b_i} F_1 \left( a_i + b_i, -c_i; a_i, l - \frac{p_i}{u_i} \right) \right] \left[ \int_{\lambda_1}^{\infty} ... \int_{\lambda_n}^{\infty} \prod_{i=1}^{n} \left( u_i x_i \right)^{\alpha_i - 1} H_{P_i,Q_i}^{M_i,N_i} \left[ \left( u_i x_i \right)^{k_i} \left( (a_i)_l; (a_i)_l \right)_{N_i} \right] \right] f(x_1,x_2,...,x_n) \ dx_1 dx_2 ... dx_n \ du_1 du_2 ... du_n
\]

On interchanging the order of integration which is permissible and evaluating the \( u_1,u_2,...,u_n \) integrals using the integral formula

\[
(15) \quad \int_{p}^{\infty} u^{-\mu \nu} (u-p)^{\nu-1} F_1 \left( \tau, \alpha, \nu; 1 - \frac{p}{u} \right) = \frac{\Gamma(\nu)}{p^\mu} \int_{P + Q + 2}^{P + Q + 2} (px)^{k} \left( (a_i)_l; (a_i)_l \right)_{N_i} \left( (\mu, k); (\mu + \nu - \tau - \alpha, k) \right)_{N_i} \left( (b_i)_l; (b_i)_l \right)_{M_i} \left( \mu + \nu - \tau, k \right)_{M_i + 1, Q} \ du
\]

where

\[
\text{Re}(\nu) > 0, \text{Re} \left( \frac{k(1-a_i)}{a_i} \right) > 0
\]

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The right hand side of (14) reduces
On the $n$-dimensional generalized Weyl fractional calculus associated with $n$-dimensional and one dimensional analogue respectively of theorem 1 given by Chaurasia and Monika Jain [4, p.66,67].

(iii) Taking $n = 2$, $A_i = B_i = 1$, the $\mathcal{H}$-function in (1) converts to Fox’s H-function and then (13) reduces to the result obtained by Saigo, Saxena and Ram [15, p.67].

(iv) Using $n = 1$, $A_i = B_i = 1$, (13) gives the result obtained by Saigo, Saxena and Ram [15, p.70].

As far as the $n$-dimensional Weyl type operators $\int_{p_1}^{a_1} \int_{p_2}^{a_2} \cdots \int_{p_n}^{a_n}$ preserves the class $u_\alpha$ it follows that $\xi \left( p_1, p_2, \ldots, p_n \right)$ also belongs to $u_\alpha$.

III. Special Cases

(i) Setting $n = 3$ in result (13) the theorem 1 reduces to the result obtained by Chaurasia and Monika Jain [4, p.62].

(ii) By setting $n = 2$ and $n = 1$ in main result, we obtain two dimensional and one dimensional analogue respectively of theorem 1 given by Chaurasia and Monika Jain [4, p.66,67].

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References


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