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On The n-Dimensional Generalized Weyl Fractional Calculus Associated With n-Dimensional $\overline{H}$-Transform

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Abstract: The main object of this paper is to obtain n-dimensional generalized Weyl fractional operators pertaining to multivariable $\overline{H}$-function. Here we get the results by using n-dimensional Laplace and H-transforms. The results of this paper are believed to be new and basic in nature. Some known results have been obtained by giving suitable values to the coefficients and parameters.

Key Words: Multivariable $\overline{H}$-function, Weyl fractional operator, $\overline{H}$-transform
Mathematics Subject Classification: 33C60, 33C65, 44-99

I. Introduction

In this paper the n-dimensional Weyl fractional operator associated with $\overline{H}$-transform is obtained. The results obtained in this paper are of manifold generality, basic in nature and include the results given earlier by Saigo, Saxena and Ram [15], Saxena and Ram [18], Chaurasia and Srivastava [3] etc.

The $\overline{H}$-function appearing here, introduced by Inayat-Hussain ([7], see also [2]) in terms of Mellin-Barnes type contour integral, is defined as

$$(1)\quad \overline{H}^{M,N}_{P,Q}\left[z \,\middle|\, \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right] = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \phi(s)\, z^s\, ds$$

where

$$(2)\quad \phi(s)=\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j s)\prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j s)\}^{A_j}}{\prod_{j=M+1}^{Q}\{\Gamma(1-b_j+\beta_j s)\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j s)}$$

which contains fractional powers of some of the Γ−functions. Here and throughout the paper A_j ($j = 1,\ldots,P$) and B_j ($j = 1,\ldots,Q$) are complex parameters, $\alpha_j \geq 0$ ($j = 1,\ldots,P$), $\beta_j \geq 0$ ($j = 1,\ldots,Q$), (not all zero simultaneously) and the exponents A_j ($j = 1,\ldots,N$) and B_j ($j = M+1,\ldots,Q$) can take on non-integer values. The contour in (1) is imaginary axis Re(s) = 0. It is suitably indented in order to avoid the singularities of the Γ−functions and to keep these singularities on appropriate sides. Again, for A_j ($j = 1,\ldots,N$) not an integer, the poles of the Γ−functions of the numerator in (2) are converted to branch points. However, as long as there is no coincidence of poles from any $\Gamma(b_j-\beta_j s),(j=1,\ldots,M)$ and $\Gamma(1-a_j+\alpha_j s)$, ($j = 1,\ldots,N$) pair the branch cuts can be chosen so that the path of integration can be distorted in the usual manner. For the sake of brevity

$$(3)\quad \sigma=\sum_{j=1}^{M}|\beta_j|+\sum_{j=1}^{N}A_j\alpha_j-\sum_{j=M+1}^{Q}|B_j\beta_j|-\sum_{j=N+1}^{P}\alpha_j>0$$

Some useful generalization of both Riemann-Liouville and Erdelyi-Kober fractional integration operators are introduced by Saigo [11], [12] in terms of Gauss's hypergeometric function as given below.

Let α, β and θ are complex numbers and let $x \in R_+(0,\infty)$. Following [11], [12] the fractional integral ($Re(\alpha) > 0$) and derivative ($Re(\alpha) < 0$) of the first kind of a function f(x) on R_+ are defined respectively in the forms

$$(4)\quad I^{\alpha,\beta,\theta}_{0,x} f=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\theta;\alpha;1-\frac{t}{x}\right)f(t)\,dt;Re(\alpha)>0$$

(5) $$= \frac{d^n}{dx^n} I_{0,x}^{\alpha+n,\beta-n,\theta-n} f,\ 0 < \mathrm{Re}(\alpha)+n \le 1 (n = 1,2,...)$$

where ${}_2F_1$ (a,b ; c ; x) is Gauss's hypergeometric function. The fractional integral (Re(α) > 0) and derivative (Re(α) < 0) of the second kind are given by

(6) $$I_{x,\infty}^{\alpha,\beta,\theta} f = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta}\ {}_2F_1\left(\alpha+\beta, -\theta; \alpha; 1-\frac{x}{t}\right) f(t)\, dt,\quad \mathrm{Re}\ \alpha > 0$$

(7) $$= (-1)^n \frac{d^n}{dx^n} I_{x,\infty}^{\alpha+n,\beta-n,\theta} f,\ 0 < \mathrm{Re}(\alpha)+n \le 1 (n = 1,2,...)$$

Following Miller [10, p.82], we denote by u_1 the class of functions $f(x_1)$ on R_+ which are infinitely differentiable with partial derivatives of any order behaving as $0(|x_1|^{-\xi_1}$ when $x_i \to \infty$ for all ξ_1. Similarly by u_2, we denote the class of function $f(x_1,x_2)$ on R_+ x R_+, which are infinitely differential with partial derivatives of any order behaving as $0(|x_1|^{-\xi_1} |x_2|^{-\xi_2})$ when $x_i \to \infty$ for all ξ_i (i = 1,2).

On the same pattern by u_n, we denote the class of function $f(x_1, x_2,\ldots,x_n)$ on $\underbrace{R_+ x\, R_+ x..xR_+}_{(n\ times)}$, which are infinitely differentiable with partial derivatives of any order behaving as

$$0\left(\prod_{i=1}^{n} |x_i|^{\xi_i}\right) \text{when } x_i \to \infty \text{ for all } \xi_i\ (i = 1,2,..., n).$$

The n-dimensional operator of Weyl type fractional integration of orders $\mathrm{Re}(a_i) > 0$, i = 1,2,…,n is defined in class u_n by

(8) $$J_{p_1,\infty}^{a_1,b_1,c_1} J_{p_2,\infty}^{a_2,b_2,c_2} \ldots J_{p_n,\infty}^{a_n,b_n,c_n} [f(p_1,p_2,...,p_n)]$$

$$= \frac{\prod_{i=1}^{n} (p_i)^{b_i}}{\prod_{i=1}^{n} \Gamma(a_i)} \int_{p_1}^{\infty} \int_{p_2}^{\infty} \ldots \int_{p_n}^{\infty} \prod_{i=1}^{n} \left[(u_i - p_i)^{a_i-1} u_i^{-a_i-b_i}\ {}_2F_1\left(a_i + b_i, -c_i; a_i; 1-\frac{p_i}{u_i}\right)\right]$$

$$.\ f(u_1,u_2,...,u_n)\, du_1\, du_2 \ldots du_n$$

where b_i and c_i (i = 1,2,…,n) are real numbers.

The n-dimensional Laplace transform L $(p_1, p_2,\ldots,p_n)$ of a function $F(x_1,x_2,\ldots,x_n) \in u_n$ is defined as

(9) $$L(p_1,p_2,...,p_n) = \sigma[F(x_1,x_2,...,x_n); p_1,p_2,...,p_n]$$

$$= \underbrace{\int_0^\infty \int_0^\infty \ldots \int_0^\infty}_{(n-times)} e^{-\sum_{i=1}^{n} (p_i x_i)} f(x_1,x_2,...,x_n)\, dx_1\, dx_2 \ldots dx_n$$

where $\mathrm{Re}(p_i) > 0$, i = 1,2,…,n.

Similarly, the Laplace transform of

$$f\left[u_1\sqrt{x_1^2-\lambda_1^2}\ H(x_1-\lambda_1), u_2\sqrt{x_2^2-\lambda_2^2}\ H(x_2-\lambda_2), ..., u_n\sqrt{x_n^2-\lambda_n}\ H(x_n-\lambda_n)\right]$$

is defined by the Laplace transform of $F(x_1, x_2,\ldots,x_n)$ where

(10) $$F(x_1,x_2,...,x_n) = f\left[u_1\sqrt{(x_1^2-\lambda_1^2)}\ H(x_1-\lambda_1), u_2\sqrt{(x_2^2-\lambda_2^2)}\ H(x_2-\lambda_2, \ldots,\right.$$

$$u_n \sqrt{(x_n^2 - \lambda_n^2)}\; H(x_n - \lambda_n) \Big],$$

$$x_i > \lambda_i > 0,\ i = 1,2,...,\ n$$

and H(t) denotes Heaviside's unit step function.

The n-dimensional $\overline{H}$ -transform ξ $(p_1,p_2,\ldots,p_n)$ of a function $F(x_1,x_2,\ldots,x_n)$ is defined as

(11) $$\xi(p_1,p_2,...,p_n) = \overline{H}^{M_1,N_1;M_2,N_2;...;M_n,N_n}_{P_1,Q_1;P_2,Q_2;...;P_n,Q_n}[F(x_1,x_2,...,x_n);\psi_1,\psi_2,...,\psi_n;p_1,p_2,...,p_n]$$

$$= \int_{\lambda_1}^{\infty}\int_{\lambda_2}^{\infty}\cdots\int_{\lambda_n}^{\infty}\prod_{i=1}^{n}\left\{(p_i x_i)^{\psi_i-1}\,\overline{H}^{M_i,N_i}_{P_i,Q_i}\left[(p_i x_i)^{k_i}\middle|\begin{matrix}(\{a_j\}_i,\{\alpha_j\}_i;\{A_j\}_i)_{1,N_i},\ (\{a_j\}_i,\{\alpha_j\}_i)_{N_i+1,P_i}\\ (\{b_j\}_i,\{\beta_j\}_i)_{1,M_i},\ (\{b_j\}_i,\{\beta_j\}_i;\{B_j\}_i)_{M_i+1,Q_i}\end{matrix}\right]\right\} f(x_1,x_2,...,x_n)\,dx_1\,dx_2...dx_n$$

where $\lambda_i > 0, k_i > 0 (i = 1,2,...,\ n); \xi(p_1,p_2,...,p_n)$ exists and belongs to u_n.

Also

(12) $$|\arg(p_i)^{k_i}| < \frac{1}{2}\sigma_i\,\pi,\ i = 1,2,...,\ n$$

where

$$\sigma_i = \sum_{j=1}^{M_i}|\beta_j| + \sum_{j=1}^{N_i}A_j\alpha_j - \sum_{j=M_i+1}^{Q_i}|B_j\beta_j| - \sum_{j=N_i+1}^{P_i}\alpha_j > 0,\ \text{for}\ i = 1,2,...,\ n\cdot$$

II. Main Results

In this section, following n-dimensional $\overline{H}$ -transform $\xi_1(p_1,p_2,...,p_n)$ of $F(x_1,x_2,...,x_n)$ is used and it is defined by

(13) $$\xi_1(p_1,p_2,...,p_n) = \overline{H}^{M_1+2,N_1;M_2+2,N_2;...;M_n+2,N_n}_{P_1+2,Q_1+2;P_2+2,Q_2+2;...;P_n+2,Q_n+2}$$

$$[F(x_1,x_2,...,x_n);\psi_1,\psi_2,...,\psi_n;p_1,p_2,...,p_n]$$

$$= \int_{\lambda_1}^{\infty}\int_{\lambda_2}^{\infty}\cdots\int_{\lambda_n}^{\infty}\prod_{i=1}^{n}\left\{(p_i x_i)^{\psi_i-1}\,\overline{H}^{M_i+2,N_i}_{P_i+2,Q_i+2}\left[(p_i x_i)^{k_i}\middle|\begin{matrix}(\{a_j\}_i,\{\alpha_j\}_i;(A_j\}_i)_{1,N_i},\ (\{a_j\}_i,\{\alpha_j\}_i)_{N_i+1,P_i},(1-\psi_i,k_i;1),(a_i+b_i+c_i-\psi_i+1,k_i;1)\\ (b_i-\psi_i+1,k_i;1),(c_i-\psi_i+1,k_i;1),\ (\{b_j\}_i,\{\beta_j\}_i)_{1,M_i},(\{b_j\}_i,\{\beta_j\}_i;\{B_j\}_i)_{M_i+1,Q_i}\end{matrix}\right]\right\} f(x_1,x_2,...,x_n)\,dx_1\,dx_2...dx_n\ ,$$

where it is assumed that $\xi_1(p_1,p_2,...,p_n)$ exists and belongs to u_n as well as $k_i > 0$, i = 1,2,…,n and other conditions on the parameters, in which additional parameters a_i, b_i and c_i, i = 1,2,…,n included correspond to those in (8).

THEOREM 1 Let $\xi(p_1,p_2,...,p_n)$ be given by definition (11), then for Re(a_i) > 0, $\lambda_i > 0$, $k_i > 0$, i = 1,2,…,n there holds the formula

$$J_{p_1,\infty}^{a_1,b_1,c_1} J_{p_2,\infty}^{a_2,b_2,c_2} \ldots J_{p_n,\infty}^{a_n,b_n,c_n} [\xi(p_1,p_2,\ldots,p_n)] = \xi_1(p_1,p_2,\ldots,p_n)$$

provided that $\xi_1(p_1,p_2,\ldots,p_n)$ exists and belongs to u_n.

Proof Let $Re(a_i) > 0$, i = 1,2,…,n then by using (8) and (11), we have

(14) $$J_{p_1,\infty}^{a_1,b_1,c_1} J_{p_2,\infty}^{a_2,b_2,c_2} \ldots J_{p_n,\infty}^{a_n,b_n,c_n} [\xi(p_1,p_2,\ldots,p_n)]$$

$$= \frac{\prod_{i=1}^{n} (p_i)^{b_i}}{\prod_{i=1}^{n} \Gamma(a_i)} \int_{p_1}^{\infty}\int_{p_2}^{\infty}\ldots\int_{p_n}^{\infty} \prod_{i=1}^{n} \left[(u_i - p_i)^{a_i-1} u_i^{-a_i-b_i} {}_2F_1\left(a_i + b_i, -c_i; a_i; 1-\frac{p_i}{u_i}\right)\right]$$

$$. \ \xi(u_1,u_2,\ldots,u_n)\, du_1\, du_2 \ldots du_n$$

$$= \frac{\prod_{i=1}^{n} (p_i)^{b_i}}{\prod_{i=1}^{n} \Gamma(a_i)} \int_{p_1}^{\infty}\int_{p_2}^{\infty}\ldots\int_{p_n}^{\infty} \prod_{i=1}^{n} \left[(u_i - p_i)^{a_i-1} u_i^{-a_i-b_i} {}_2F_1\left(a_i + b_i, -c_i; a_i; 1-\frac{p_i}{u_i}\right)\right]$$

$$\left[\int_{\lambda_1}^{\infty}\int_{\lambda_2}^{\infty}\ldots\int_{\lambda_n}^{\infty} \prod_{i=1}^{n} \left\{ (u_i x_i)^{\psi_i - 1} \overline{H}_{P_i,Q_i}^{M_i,N_i}\left[(u_i x_i)^{k_i} \left| \begin{matrix} (\{a_j\}_i,\{\alpha_j\}_i;(A_j\}_i)_{1,N_i}, (\{a_j\}_i,\{\alpha_j\}_i)_{N_i+1,P_i} \\ (\{b_j\}_i,\{\beta_j\}_i)_{1,M_i}, (\{b_j\}_i,\{\beta_j\}_i;\{B_j\}_i)_{M_i+1,Q_i} \end{matrix}\right.\right]\right\} f(x_1,x_2,\ldots,x_n)\, dx_1\, dx_2 \ldots dx_n \right] du_1 du_2 \ldots du_n$$

.

On interchanging the order of integration which is permissible and evaluating the $u_1,u_2,\ldots,u_n$ integrals using the integral formula

(15) $$\int_{p}^{\infty} u^{-\mu-\nu}(u-p)^{\nu-1} {}_2F_1\left(\tau,\omega;\nu;1-\frac{p}{u}\right)$$

$$. \ \overline{H}_{P,Q}^{M,N}\left[(pu)^k \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N}, (a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M}, (b_j,\beta_j;1)_{M+1,Q} \end{matrix}\right.\right] du$$

$$= \frac{\Gamma(\nu)}{p^{\mu}} \overline{H}_{P+2,Q+2}^{M+2,N}\left[(px)^k \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N}, (a_j,\alpha_j)_{N+1,P}, (\mu+\nu-\tau,k;1),(\mu+\nu-\omega,k;1) \\ (\mu,k;1),(\mu+\nu-\tau-\omega,k;1), (b_j,\beta_j)_{1,M}, (b_j,\beta_j;1)_{M+1,Q} \end{matrix}\right.\right]$$

where

$$Re(\nu) > 0,\ Re\left(\mu + \nu + \frac{k(1-a_j)}{\alpha_j}\right) > 0,$$

$$Re\left(\mu + \nu - \tau - \omega + \frac{k(1-a_j)}{a_j}\right) > 0, |\arg z| < \frac{T\pi}{2}$$

(Tisgivenin)

The right hand side of (14) reduces

$$=\int_{\lambda_1}^{\infty}\int_{\lambda_2}^{\infty}\cdots\int_{\lambda_n}^{\infty}\prod_{i=1}^{n}\left\{(p_i x_i)^{\psi_i-1}\,\overline{H}_{P_i+2,Q_i+2}^{M_i+2,N_i}\left[(p_i x_i)^{k_i}\left|\begin{matrix}(\{a_j\}_i,\{\alpha_j\}_i;(A_j)_i)_{1,N_i},(\{a_j\}_i,\{\alpha_j\}_i)_{N_i+1,P_i},(1-\psi_i,k_i;1),(a_i+b_i+c_i-\psi_i+1,k_i;1)\\(b_i-\psi_i+1,k_i;1),(c_i-\psi_i+1,k_i;1),(\{b_j\}_i,\{\beta_j\}_i)_{1,M_i},(\{b_j\}_i,\{\beta_j\}_i;(B_j)_i)_{M_i+1,Q_i}\end{matrix}\right.\right]\right\}f(x_1,x_2,\ldots,x_n)\,dx_1\,dx_2\ldots dx_n$$

$$=\overline{H}_{P_1+2,Q_1+2;P_2+2,Q_2+2;\ldots;P_n+2,Q_n+2}^{M_1+2,N_1;M_2+2,N_2;\ldots;M_n+2,N_n}\,[F_1(x_1,x_2,\ldots,x_n);\psi_1,\psi_2,\ldots,\psi_n;p_1,p_2,\ldots,p_n]$$

$$=\xi_1(p_1,p_2,\ldots,p_n)$$

As far as the n-dimensional Weyl type operators $J_{p_1,\infty}^{a_1,b_1,c_1}J_{p_2,\infty}^{a_2,b_2,c_2}\ldots J_{p_n,\infty}^{a_n,b_n,c_n}$ preserves the class u_n, it follows that $\xi_1(p_1,p_2,\ldots,p_n)$ also belongs to u_n.

III. Special Cases

(i) Setting n = 3 in result (13) the theorem 1 reduces to the result obtained by Chaurasia and Monika Jain [4, p.62].

(ii) By setting n = 2 and n = 1 in main result, we obtain two dimensional and one dimensional analogue respectively of theorem 1 given by Chaurasia and Monika Jain [4, p.66,67].

(iii) Taking n = 2, $A_j = B_j = 1$, the $\overline{H}$ -function in (1) converts to Fox's H-function and then (13) reduces to the result obtained by Saigo, Saxena and Ram [15, p.67].

(iv) Using n = 1, $A_j = B_j = 1$, (13) gives the result obtained by Saigo, Saxena and Ram [15, p.70].

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