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A Strong Form of Lindelof Spaces

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Abstract: In this paper, we introduce and investigate a new class of set called ω-λ-open set which is weaker *than both ω-open and λ-open set. Moreover, we obtain the characterization of λ-Lindelof spaces.*
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I. Introduction And Preliminaries

Throughout this paper (X, τ) and (Y, σ) stand for topological spaces with no separation axioms assumed, unless otherwise stated. Maki [3] introduced the notion of Λ-sets in topological spaces. A Λ-set is a set A which is equal to its kernel, that is, to the intersection of all open supersets of A. Arenas et.al. [1] Introduced and investigated the notion of λ-closed sets by involving Λ-sets and closed sets. Let A be a subset of a topological space (X,τ). The closure and the interior of a set A is denoted by Cl(A), Int(A) respectively. A subset A of a topological space (X,τ) is said to be λ-closed [1] if $A = B\cap C$, where B is a Λ-set and C is a closed set of X. The complement of λ-closed set is called λ-open [1]. A point $x\in X$ in a topological space (X,τ) is said to be λ-cluster point of A [2] if for every λ-open set U of X containing x, $A \cap U \neq \phi$. The set of all λ-cluster points of A is called the λ-closure of A and is denoted by $Cl_\lambda(A)$ [2]. A point $x\in X$ is said to be the λ-interior point of A if there exists a λ-open set U of X containing x such that $U\subset A$. The set of all λ-interior points of A is said to be the λ-interior of A and is denoted by $Int_\lambda(A)$. A set A is λ-closed (resp.λ-open) if and only if $Cl_\lambda(A) = A$ (resp. $Int_\lambda(A) = A$) [2].

The family of all λ-open (resp. λ-closed) sets of X is denoted by $\lambda O(X)$ (resp. $\lambda C(X)$). The family of all λ-open (resp. λ-closed) sets of a space (X,τ) containing the point $x\in X$ is denoted by $\lambda O(X, x)$ (resp. $\lambda C(X, x)$).

II. ω-λ-OPEN SETS

Definition 2.1: A subset A of a topological space X is said to be ω-λ-open if for every $x \in A$, there exists a λ-open subset $U_x \in X$ containing x such that $U_x - A$ is countable. The complement of an ω-λ-open subset is said to be ω-λ-closed.

Proposition 2.2: Every λ-open set is ω-λ-open. Converse not true.

Corollary 2.3: Every open set is ω-λ-open, but not conversely.

Proof: Follows from the fact that every open set is λ-open.

Lemma 2.4: For a subset of a topological space, both ω-openness and λ-openness imply ω-λ-openness.

Proof: (i) Assume A is ω-open then, for each $x \in A$, there is an open set containing x such that $U_x - A$ is countable. Since every open set is λ-open, A is ω-λ-open. (ii) Let A be ω-λ-open. For each $x \in A$, there exists a λ-open set $U_x = A$ such that $x \in U_x$ and $U_x - A = \phi$.Therefore, A is ω-λ-open.

Example 2.5: Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, X\}$, then $\{b\}$ is ω-open but not λ-open (since X is a countable set).

Example 2.6: Let X = R with the usual topology. Let A = Q be the set of all rational numbers. Then A is λ-open but it is not ω-open.

Lemma 2.7: A subset A of a topological space X is ω-λ-open if and only if for every $x \in A$, there exists a λ-open subset U containing x and a countable subset C such that $U - C \subset A$.

Proof: Let A be ω-λ-open and $x \in \mathrm{A}$, then there exists a λ-open subset U_x containing x such that $n(\mathrm{U}_x - \mathrm{A})$ is countable. Let $\mathrm{C} = \mathrm{U}_x - \mathrm{A} = \mathrm{U}_x \cap (\mathrm{X} - \mathrm{A})$. Then $\mathrm{U}_x - \mathrm{C} \subset \mathrm{A}$. Conversely, let $x \in \mathrm{A}$. Then there exists a λ-open subset U_x containing x and a countable subset C such that $\mathrm{U}_x - \mathrm{C} \subset \mathrm{A}$. Thus, $\mathrm{U}_x - \mathrm{A} \subset \mathrm{C}$ and $\mathrm{U}_x - \mathrm{A}$ is countable.

Theorem 2.8: Let X be a topological space and $\mathrm{C} \subset \mathrm{X}$ If C is λ-closed, then $\mathrm{C} \subset \mathrm{K} \cup \mathrm{B}$ for some λ-closed subset K and a countable subset B.

Proof: If C is λ-closed, then $\mathrm{X} - \mathrm{C}$ is λ-open and hence for every $x \in \mathrm{X} - \mathrm{C}$, there exists a λ-open set U containing x and a countable set B such that $\mathrm{U} - \mathrm{B} \subset \mathrm{X} - \mathrm{C}$. Thus $\mathrm{C} \subset \mathrm{X} - (\mathrm{U} - \mathrm{B}) = \mathrm{X} - (\mathrm{U} \cap (\mathrm{X} - \mathrm{B})) = \mathrm{X} - (\mathrm{U} \cup \mathrm{B})$. Let $\mathrm{K} = \mathrm{X} - \mathrm{U}$. Then K is λ-closed such that $\mathrm{C} \subset \mathrm{K} \cup \mathrm{B}$.

Corollary 2.9 The intersection of an ω-λ-open set with an open set is ω-λ-open.

Question: Does there exist an example for the intersection of ω-λ-open sets is ω-λ-open?

Proposition 2.10: The union of any family of ω-λ-open sets is ω-λ-open.

Proof: If $\{\mathrm{A}_\alpha : \alpha \in \Lambda\}$ is a collection of ω-λ-open subsets of X, then for every $x \in \bigcup_{\alpha\in\Lambda} \mathrm{A}_\alpha$, $x \in \mathrm{A}_\gamma$ for some $\gamma \in \Lambda$. Hence there exists a λ-open subset U of X containing x such that $\mathrm{U} - \mathrm{A}_\gamma$ is countable. Now $\mathrm{U} - \bigcup_{\alpha\in\Lambda} \mathrm{A}_\alpha \subset \mathrm{U} - \mathrm{A}_\gamma$ and thus $\mathrm{U} - \bigcup_{\alpha\in\Lambda} \mathrm{A}_\alpha$ is countable. Therefore, $\bigcup_{\alpha\in\Lambda} \mathrm{A}_\alpha$ is ω-λ-open.

Definition 2.11: The union of all ω-λ-open sets contained in $\mathrm{A} \subset \mathrm{X}$ is called the ω-λ-interior of A, and is denoted by $\omega\text{-}\mathrm{Int}_\lambda(\mathrm{A})$. The intersection of all ω-λ-closed sets of X containing A is called the ω-λ-closure of A, and is denoted by $\omega\text{-}\mathrm{Cl}_\lambda(\mathrm{A})$.

The proof of the following Theorems follows from the Definitions hence they are omitted.

Theorem 2.12: Let A and B be subsets of (X, τ). Then the following properties hold:

(i) $\omega\text{-}\mathrm{Int}_\lambda(\mathrm{A})$ is the largest ω-λ-open subset of X contained in A

(ii) A is ω-λ-open if and only if $\mathrm{A} = \omega\text{-}\mathrm{Int}_\lambda(\mathrm{A})$

(iii) $\omega\text{-}\mathrm{Int}_\lambda(\omega\text{-}\mathrm{Int}_\lambda(\mathrm{A})) = \omega\text{-}\mathrm{Int}_\lambda(\mathrm{A})$

(iv) If $\mathrm{A} \subset \mathrm{B}$, then $\omega\text{-}\mathrm{Int}_\lambda(\mathrm{A}) \subset \omega\text{-}\mathrm{Int}_\lambda(\mathrm{B})$

(v) $\omega\text{-}\mathrm{Int}_\lambda(\mathrm{A}) \cup \omega\text{-}\mathrm{Int}_\lambda(\mathrm{B}) \subset \omega\text{-}\mathrm{Int}_\lambda(\mathrm{A} \cup \mathrm{B})$

(vi) $\omega\text{-}\mathrm{Int}_\lambda(\mathrm{A} \cap \mathrm{B}) \subset \omega\text{-}\mathrm{Int}_\lambda(\mathrm{A}) \cap \omega\text{-}\mathrm{Int}_\lambda(\mathrm{B})$.

Theorem 2.13: Let A and B be subsets of (X, τ). Then the following properties hold:

(i) $\omega\text{-}\mathrm{Cl}_\lambda(\mathrm{A})$ is the smallest ω-λ-closed subset of X contained in A;

(ii) A is ω-λ-closed if and only if $\mathrm{A} = \omega\text{-}\mathrm{Cl}_\lambda(\mathrm{A})$;

(iii) $\omega\text{-}\mathrm{Cl}_\lambda(\omega\text{-}\mathrm{Cl}_\lambda(\mathrm{A})) = \omega\text{-}\mathrm{Cl}_\lambda(\mathrm{A})$;

(iv) If $\mathrm{A} \subset \mathrm{B}$, then $\omega\text{-}\mathrm{Cl}_\lambda(\mathrm{A}) \subset \omega\text{-}\mathrm{Cl}_\lambda(\mathrm{B})$;

(v) $\omega\text{-}\mathrm{Cl}_\lambda(\mathrm{A} \cup \mathrm{B}) = \omega\text{-}\mathrm{Cl}_\lambda(\mathrm{A}) \cup \omega\text{-}\mathrm{Cl}_\lambda(\mathrm{B})$;

(vi) $\omega\text{-}\mathrm{Cl}_\lambda(\mathrm{A} \cap \mathrm{B}) \subset \omega\text{-}\mathrm{Cl}_\lambda(\mathrm{A}) \cap \omega\text{-}\mathrm{Cl}_\lambda(\mathrm{B})$.

Theorem 2.14: Let (X, τ) be a topological space and $\mathrm{A} \subset \mathrm{X}$. A point $x \in \omega\text{-}\mathrm{Cl}_\lambda(\mathrm{A})$ if and only if $\mathrm{U} \cap \mathrm{A} \neq \phi$ for every $\mathrm{U} \in \omega\lambda\mathrm{O}(\mathrm{X}, x)$.

Theorem 2.15: Let (X, τ) be a topological space and $\mathrm{A} \subset \mathrm{X}$. Then the following properties hold:

(i) $\omega\text{-}\mathrm{Int}_\lambda(\mathrm{X} - \mathrm{A}) = \mathrm{X} - (\omega\text{-}\mathrm{Cl}_\lambda(\mathrm{A}))$

(ii) $\omega\text{-}\mathrm{Cl}_\lambda(\mathrm{X} - \mathrm{A}) = \mathrm{X} - (\omega\text{-}\mathrm{Int}_\lambda(\mathrm{A}))$

Theorem 2.16: If each nonempty ω-λ-open set of a topological space X is uncountable, then $\omega\text{-}\mathrm{Int}_{\lambda}(\mathrm{A}) \subset \mathrm{Int}_{\lambda}(\mathrm{A})$ for each open set A of X.

Proof: Clearly $\omega\text{-}\mathrm{Cl}_{\lambda}(\mathrm{A}) \subset \mathrm{Cl}_{\lambda}(\mathrm{A})$. On the other hand, let $x \in \mathrm{Cl}_{\lambda}(\mathrm{A})$ and B be an ω-λ-open subset containing *x*. Then by Lemma 2.7, there exists a λ-open set V containing *x* and a countable set C such that $\mathrm{V}-\mathrm{C} \subset \mathrm{B}$. Thus $(\mathrm{V}-\mathrm{C}) \cap \mathrm{A} \subset \mathrm{B} \cap \mathrm{A}$ and so $(\mathrm{V} \cap \mathrm{A})-\mathrm{C} \subset \mathrm{B} \cap \mathrm{A}$.

Since $x \in \mathrm{V}$ and $x \in \mathrm{Cl}_{\lambda}(\mathrm{A})$, $\mathrm{V} \cap \mathrm{A} \neq \phi$ and $\mathrm{V} \cap \mathrm{A}$ is λ-open since V is λ-open and A is open. By the hypothesis each nonempty λ-open set of a topological space X is uncountable and so is $(\mathrm{V} \cap \mathrm{A})-\mathrm{C}$. Thus $\mathrm{B} \cap \mathrm{A}$ is uncountable. Therefore, $\mathrm{B} \cap \mathrm{A} \neq \phi$ which means that $x \in \omega\text{-}\mathrm{Cl}_{\lambda}(\mathrm{A})$.

Corollary 2.17: If each nonempty λ-open set of a topological space X is uncountable, then $\omega\text{-}\mathrm{Int}_{\lambda}(\mathrm{A}) = \mathrm{Int}_{\lambda}(\mathrm{A})$ for each closed set A of X.

Definition 2.18: A function $f : \mathrm{X} \to \mathrm{Y}$ is said to be quasi λ-open if the image of each λ-open set in X is open in Y.

Theorem 2.19: If $f : \mathrm{X} \to \mathrm{Y}$ is quasi λ-open, then the image of an ω-λ-open set of X is ω-open in Y.

Proof: Let $f : \mathrm{X} \to \mathrm{Y}$ be quasi λ-open and W an ω-λ-open subset of X. Let $y \in f(\mathrm{W})$, there exists $x \in \mathrm{W}$ such that $f(x) = y$. Since W is ω-λ-open, there exists a λ-open set U such that $x \in \mathrm{U}$ and is countable. Since f is quasi λ-open, $f(\mathrm{U})$ is open in Y such that $y = f(x) \in f(\mathrm{U})$ and $f(\mathrm{U}) - f(\mathrm{W}) \subset f(\mathrm{U}-\mathrm{W}) = f(\mathrm{C})$ is countable. Therefore, $f(\mathrm{W})$ is λ-open in Y.

Definition 2.20: A collection $\{\mathrm{U}_{\alpha} : \alpha \in \Delta\}$ of λ-open sets in a topological space X is called a λ-open cover of a subset B of X if $\mathrm{B} \subset \{\mathrm{U}_{\alpha} : \alpha \in \Delta\}$ holds.

Definition 2.21: A topological space X is said to be λ- Lindelof if every λ-open cover of X has a countable subcover.

A subset A of a topological space X is said to be λ- Lindelof relative to X if every cover of A by λ-open sets of X has a countable subcover.

Theorem 2.22: Every λ- Lindelolf space is Lindelof.

Theorem 2.23: If X is a topological space such that every λ-open subset is λ-Lindelof relative to X, then every subset is λ-Lindelof relative to X.

Proof: Let B be an arbitrary subset of X and let $\{\mathrm{U}_{\alpha} : \alpha \in \Delta\}$ be λ-open cover of B. Then the family $\{\mathrm{U}_{\alpha} : \alpha \in \Delta\}$ is a λ-open cover of the λ-open set $\cup\{\mathrm{U}_{\alpha} : \alpha \in \Delta\}$. Hence by hypothesis there is a countable subfamily $\{\mathrm{U}_{\alpha_i} : \alpha_i \in \mathrm{N}\}$ which covers $\cup\{\mathrm{U}_{\alpha} : \alpha \in \Delta\}$. This subfamily is also a cover of the set B.

Theorem 2.24: Every λ-closed subset of a λ- Lindelof space X is λ-Lindelof relative to X.

Proof: Let A be a λ-closed subset of X and $\widetilde{\mathrm{U}}$ be a cover of A by λ-open subsets in X. Then $\widetilde{\mathrm{U}}^{*} = \widetilde{\mathrm{U}} \cup \{\mathrm{X}-\mathrm{A}\}$ is a λ-open cover of X. Since X is λ- Lindelof, $\widetilde{\mathrm{U}}^{*}$ has a countable subcover $\widetilde{\mathrm{U}}^{**}$ for X. Now $\widetilde{\mathrm{U}}^{**} - \{\mathrm{X}-\mathrm{A}\}$ is a countable subcover of $\widetilde{\mathrm{U}}$ for A, so A is λ- Lindelof relative to X.

Theorem 2.25: For any topological space X, the following properties are equivalent:

(i) X is λ-Lindelof.

(ii) Every λ-open cover of X has a countable subcover.

Proof: (i) $\Rightarrow$ (ii) : Let $\{\mathrm{U}_{\alpha} : \alpha \in \Delta\}$ be any cover of X by ω-λ-open sets of X. For each $x \in \mathrm{X}$, there exists $\alpha(x) \in \Delta$ such that $x \in \mathrm{U}_{\alpha(x)}$. Since $\mathrm{U}_{\alpha}(x)$ is ω-λ-open, there exists a λ-open set $\mathrm{V}_{\alpha(x)}$ such that $x \in \mathrm{V}_{\alpha(x)}$ and $\mathrm{U}_{\alpha(x)} - \mathrm{V}_{\alpha(x)}$ is countable. The family $\{\mathrm{V}_{\alpha(x)} : x \in \mathrm{X}\}$ is a λ-open cover of X and X is λ-Lindelof. There exists a countable subset, say $\alpha_1(x), \alpha_2(x) \ldots\ldots \alpha_n(x), \ldots$ such that

$X=\cup\{V_{\alpha(x_i)} : i\in N\}$ Now, we have $X=\bigcup_{i\in N}\{(V_{\alpha(x_i)}-U_{\alpha(x_i)})\cup U_{\alpha(x_i)}\}$ $=\left(\bigcup_{i\in N}V_{\alpha(x_i)}-U_{\alpha(x_i)}\right)\cup\left(\bigcup_{i\in N}U_{\alpha(x_i)}\right)$. For each $\alpha(x_i)$, $V_{\alpha(x_i)}-U_{\alpha(x_i)}$ is a countable set and there exists a countable subset $\Delta_{\alpha(x_i)}$ of Δ such that $V_{\alpha(x_i)}-U_{\alpha(x_i)}\subset\cup\{U_\alpha : \alpha\in\Delta_{\alpha(x_i)}\}$. Therefore, we have $X\subset\cup\{U_\alpha : \alpha\in\Delta_{\alpha(x_i)}\}$.

$(ii)\Rightarrow(i)$: Since every λ-open is ω-λ-open, the proof is obvious.

III. ω-λ-CONTINUOUS FUNCTION

Definition 3.1: A function $f:(X,\tau)\to(Y,\sigma)$ is said to be ω-λ-continuous if the inverse image of every open subset of Y is ω-λ-open in X.

It is clear that every λ-continuous function is ω-λ-continuous but not conversely.

Example 3.2: Let $X=\{a,b,c\}$, $\tau=\{\phi,\{a\},X\}$ and $\sigma=\{\phi,\{a,c\},X\}$. Clearly the identity function $f:(X,\tau)\to(X,\sigma)$ is ω-λ-continuous but not λ-continuous.

Theorem 3.3: For a function $f:(X,\tau)\to(Y,\sigma)$, the following statements are equivalent:

(i) f is ω-λ-continuous;

(ii) For each point x in X and each open set F of Y such that $f(x)\in F$, there is an ω-λ-open set A in X such that $x\in A$, $f(A)\subset F$;

(iii) The inverse image of each closed set of Y is ω-λ-closed in X;

(iv) For each subset A of X, $f(\omega\text{-}Cl_\lambda(A))\subset Cl(f(A))$;

(v) For each subset B of Y, $\omega\text{-}Cl_\lambda(f^{-1}(B))\subset f^{-1}(Cl_\lambda(B))$;

(vi) For each subset C of Y, $f^{-1}(Int(C))\subset\omega\text{-}Int_\lambda(f^{-1}(C))$;

Proof: $(i)\Rightarrow(ii)$: Let $x\in X$ and F be an open set of Y containing $f(x)$. By (i), $f^{-1}(F)$ is ω-λ open in X. Let $A=f^{-1}(F)$. Then $x\in A$ and $f(A)\subset F$.

$(ii)\Rightarrow(i)$: Let F be an open in Y and let $x\in f^{-1}(F)$. Then $f(x)\in F$. By (ii), there is an ω-λ-open set U_x in X such that $x\in U_x$ and $f(U_x)\subset F$. Then $x\in U_x\subset f^{-1}(F)$. Hence $f^{-1}(F)$ is ω-λ-open in X.

$(i)\Leftrightarrow(iii)$: This follows due to the fact that for any subset B of Y, $f^{-1}(Y-B)=X-f^{-1}(B)$.

$(iii)\Rightarrow(iv)$: Let A be a subset of X. Since $A\subset f^{-1}(f(A))$ we have $A\subset f^{-1}(\sigma_i\text{-}Cl(f(A)))$ Now, $Cl(f(A))$ is closed in Y and hence $\omega\text{-}Cl_\lambda(A)\subset f^{-1}(Cl(f(A)))$, for $\omega\text{-}Cl_\lambda(A)$ is the smallest ω-λ-closed set containing A. Then $f(\omega\text{-}Cl_\lambda(A))\subset Cl(f(A))$.

$(iv)\Rightarrow(iii)$: Let F be any closed subset of Y. Then $f(\omega\text{-}Cl_\lambda(f^{-1}(F)))\subset Cl(f(f^{-1}(F)))$ $\subset Cl(F)=F$. Therefore, $\omega\text{-}Cl_\lambda(f^{-1}(F))\subset f^{-1}(F)$ consequently, $f^{-1}(F)$ is ω-λ-closed in X.

$(iv)\Rightarrow(v)$: Let B be any subset of Y. Now, $f(\omega\text{-}Cl_\lambda(f^{-1}(B)))\subset Cl(f(f^{-1}(B)))\subset Cl(B)$. Consequently, $\omega\text{-}Cl_\lambda(f^{-1}(B))\subset f^{-1}(Cl(B))$.

$(v)\Rightarrow(iv)$: Let $B=f(A)$ where A is a subset of X. Then, $\omega\text{-}Cl_\lambda(A)\subset\omega\text{-}Cl_\lambda(f^{-1}(B))$ $\subset f^{-1}(Cl(B))=f^{-1}(Cl(f(A)))$. This shows that $f(\omega\text{-}Cl_\lambda(A))\subset Cl(f(A))$.

$(i)\Rightarrow(iv)$: Let C be any subset of Y. Clearly, $f^{-1}(Int(C))$ is ω-λ-open and we have $f^{-1}(Int(C))\subset\omega\text{-}Int_\lambda(f^{-1}(Int(C)))\subset\omega\text{-}Int_\lambda(f^{-1}(C))$.

$(vi) \Rightarrow (i)$: Let B be an open set in Y. Then $\mathrm{Int}(B) = B$ and $f^{-1}(B) \subset f^{-1}(\mathrm{Int}(B)) \subset \omega\text{-}\mathrm{Int}_{\lambda}\left(f^{-1}(B)\right)$. Hence we have $f^{-1}(B) = \omega\text{-}\mathrm{Int}_{\lambda}\left(f^{-1}(B)\right)$ .This shows that $f^{-1}(B)$ is ω-λ-open in X.

Theorem 3.4: Let $f : (X, \tau) \to (Y, \sigma)$ be a ω-λ-continuous surjective function. If X is λ-Lindelof, then Y is Lindelof.

Proof: Let $\{V_{\alpha} : \alpha \in \Delta\}$ be an open cover of Y. Then, $\{f^{-1}(V_{\alpha}) : \alpha \in \Delta\}$ is a ω-λ-cover of X. Since X is λ-Lindelof, X has a countable subcover, say $\left\{f^{-1}\left(V_{\alpha_i}\right)\right\}_{i=1}^{\infty}$ and $V_{\alpha_i} \in \{V_{\alpha} : \alpha \in \Delta\}$ Hence $\left\{f^{-1}\left(V_{\alpha_i}\right) : \alpha \in \Delta\right\}$ is a countable subcover of Y. Hence, Y is Lindelof.

Definition 3.5:[1] A function $f : X \to Y$ is said to be λ-continuous if the inverse image of each open subset of Y is λ-open in X.

Corollary 3.6: Let $f : (X, \tau) \to (Y, \sigma)$ be a ω-λ-continuous (or λ-continuous) surjective function. If X is λ- Lindelof, then Y is Lindelof.

Definition 3.7: A function $f : X \to Y$ is said to be ω-λ^*-continuous if the inverse image of each λ-open subset of Y is ω-λ-open in X.

The proof of the following Theorem is similar to Theorem 3.4 and hence omitted.

Theorem 3.8: Let $f : (X, \tau) \to (Y, \sigma)$ be a ω-λ^*-continuous surjective function. If X is λ-Lindelof, then Y is λ-Lindelof.

Theorem 3.9: A ω-λ-closed subset of a λ-Lindelof space X is λ-Lindelof relative to X.

Proof: Let A be an ω-λ-closed subset of X. Let $\{U_{\alpha} : \alpha \in \Delta\}$ be a cover of A by λ-open sets of X. Now for each $x \in X - A$, there is a λ-open set V_x such that $V_x \cap A$ is countable. Since $\{U_{\alpha} : \alpha \in \Delta\} \cup \{V_x : x \in X - A\}$ is a λ-open cover of X and X is λ-Lindelof, there exists a countable subcover $\{U_{\alpha_i} : i \in N\} \cup \{V_{x_i} : i \in N\}$. Since $\bigcup_{i \in N}\left(V_{x_i} \cap A\right)$ is countable, so for each $x_j \in \cup\left(V_{x_i} \cap A\right)$, there is $U_{\alpha(x_j)} \in \{U_{\alpha} : \alpha \in \Delta\}$ such that $x_j \in U_{\alpha(x_j)}$ and $j \in N$. Hence $\{U_{\alpha_i} : i \in N\} \cup \{U_{\alpha(x_j)} : j \in N\}$ is a countable subcover of $\{U_{\alpha} : \alpha \in \Delta\}$ and it covers A. Therefore, A is λ-Lindelof relative to X.

Corollary 3.10: If a topological space X is λ-Lindelof and A is ω-closed (or λ-closed), then A is λ- Lindelof relative to X.

Definition 3.11: A function $f : X \to Y$ is said to be λ-closed if $f(A)$ is ω-λ-closed in Y for each λ-closed set A of X.

Theorem 3.12: If $f : X \to Y$ is an ω-λ-closed surjection such that $f^{-1}(y)$ is λ-Lindelof relative to X and Y is λ-Lindelof, then X is λ-Lindelof.

Proof: Let $\{U_{\alpha} : \alpha \in \Delta\}$ be any λ-open cover of X. For each $y \in Y$, $f^{-1}(y)$ is λ-Lindelof relative to X and there exists a countable subset $\Delta_1(y)$ of Δ such that $f^{-1}(y) \subset \cup\{U_{\alpha} : \alpha \in \Delta_1(y)\}$. Now, we put $V(y) = Y - f(X - V(y))$. Then, since *f* is ω-λ-closed, $V(y)$ is an ω-λ-open set in Y containing *y* such that $f^{-1}(V(y)) \subset U(y)$. Since $V(y)$ is ω-λ-open, there exists a λ-open set $W(y)$ containing *y* such that $W(y) - V(y)$ is a countable set. For each $y \in Y$, we have $W(y) \subset (W(y) - V(y)) \cup V(y)$ and hence $f^{-1}(W(y)) \subset f^{-1}(W(y) - V(y)) \cup f^{-1}(V(y)) \subset f^{-1}(W(y) - V(y)) \cup U(y)$.

Since $W(y) - V(y)$ is a countable set and $f^{-1}(y)$ is λ- Lindelof relative to X, there exists a countable set $\Delta_1(y)$ of Δ such that $f^{-1}(W(y) - V(y)) \subset \cup\{U_{\alpha} : \alpha \in \Delta_2(y)\}$ and hence $f^{-1}(W(y)) \subset (\cup\{U_{\alpha} : \alpha \in \Delta_2(y)\}) \cup U(y)$. Since $\{W(y) : y \in Y\}$ is a λ-open cover of the λ-Lindelof space Y, there exist countable points of Y, say $y_1, y_2, ... y_n ...$ such that $Y = \cup\{W(y_i) : i \in N\}$. Therefore, we obtain

$$X = \bigcup_{i \in N} f^{-1}(W(y_i)) = \bigcup_{i \in N}\left(\left(\bigcup_{\alpha \in \Delta_2(y_i)} U_\alpha\right) \cup \left(\bigcup_{\alpha \in \Delta_1(y_i)} U_\alpha\right)\right) = \{U_\alpha : \alpha \in \Delta_1(y_i) \cup \alpha \in \Delta_2(y_i), i \in N\}.$$

This shows that X is λ -Lindelof.

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