IOSR Journal of Mathematics (IOSRJM)
ISSN: 2278-5728 Volume 1, Issue 3 (July-Aug 2012), PP 04-11
www.iosrjournals.org

On Extensions of Green's Relations in Semi groups

[1]D.V.Vijay Kumar and [2]K.V.R.Srinivas

Abstract: *In this paper mainly we have obtained interesting and independent results using the equivalences* $\mathcal{L}^*$, $\mathcal{R}^*$ *and* $\mathcal{H}^*$.

Keywords: *Union of groups, Periodic semigroup and Cancellative semigroup.*

I. Introduction:

John M. Howie introduced relations $\mathcal{L}^*$, $\mathcal{R}^*$ and $\mathcal{H}^*$ in the book entitled "Fun- damentals of Semigroup Theory " [1]. The relation $\mathcal{L}^*$ defined on a semigroup S by the rule that $a\mathcal{L}^*b$ if and only if $ax = ay \iff bx = by, \forall x, y \in S^1$, and the relation $\mathcal{R}^*$ defined on a semigroup S by the rule that $a\mathcal{R}^*b$ if and only if $xa = ya \iff xb = yb, \forall x, y \in S^1$, and the relation $\mathcal{H}^*$ defined on a semigroup S by the rule that $a\mathcal{H}^*b$ if and only if $axa = aya \iff bxb = byb, \forall x, y \in S^1$. If S is a semigroup then $\mathcal{L} \subseteq \mathcal{L}^*$, and $\mathcal{L}^*$ is a right congruence on S, and for every idempotent e in S, $a\mathcal{L}^*e$ if and only if $ae = a$ and $ax = ay \Rightarrow ex = ey, \forall x, y \in S^1$. If S is regular then $\mathcal{L} = \mathcal{L}^*$. The containment $\mathcal{L} \subseteq \mathcal{L}^*$ well be proper. It is observed that in the cancellative semigroup S (see def 1.4) $\mathcal{L} = 1_S$, $\mathcal{L}^* = S \times S$. The equivalences $\mathcal{R}^*$ and $\mathcal{H}^*$ are defined by analogy with $\mathcal{L}^*$. Then every $\mathcal{H}^*$-class containing an idempotent is a subsemigroup of S and is a cancellative semigroup with identity element e.

In this paper we proved some interesting and independent results using the equiv- alences $\mathcal{L}^*$, $\mathcal{R}^*$ and $\mathcal{H}^*$.First we proved in theorem (2.4), that if S is a semi- group with zero, then $0\mathcal{L}^* = \{0\}$, $0\mathcal{R}^* = \{0\}$. Further it is obtained in theo- rem (2.5), that in a semigroup S, $\mathcal{L}^* \cap (RegS \times RegS) = \mathcal{L} \cap (RegS \times RegS)$ and $\mathcal{R}^* \cap (RegS \times RegS) = \mathcal{R} \cap (RegS \times RegS)$ where $RegS$ stands for semigroup S with regular elements. From Theorem (2.5) we obtained as a corollary (2.6), that if S is a regular semigroup then $\mathcal{L}^* = \mathcal{L}$ and $\mathcal{R}^* = \mathcal{R}$. It is interesting to observe that, if S is a periodic semigroup which is also cancellative, then S is a union of groups, which is obtained in lemma (2.7). It is observed in theorem (2.8), that in a periodic semigroup (see def 1.6) $H_e = H^*$. It is also observed in lemma (2.9), that H^* is a group if and only if $H^* = H_e$. Unlike the Green's relations $\mathcal{L}$ and $\mathcal{R}$ the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ do not permute, for this an example is obtained in (2.10). It is also very interesting to observe that on a semigroup S if a is a regular element of S, then every element of $a\mathcal{R}^*$ need not be regular. For this an example is obtained in (2.11).

AMS subject classification number: 20M18

First we start with following preliminaries.

Definition 1.0: A pair $(S, .)$ where S is a non empty set and $'.'$ is a binary operation defined on S is said to be a semigroup if $a.(b.c) = (a.b).c$ for all $a, b, c \in S$.

Definition 1.1: A semigroup $(S, .)$ is said to be a semigroup with zero if there exists an element $0 \in S$ such that $0.a = a.0 = 0$ for all $a \in S$.

A semigroup $(S, .)$ is said to be a semigroup with identity if there exists an element $1 \in S$ such that $1.a = a.1 = 1$ for all $a \in S$.

Definition 1.2: A semigroup $(S, .)$ is said to be commutative if $a.b = b.a$ for all $a, b \in S$.

Definition 1.3: Let S be a semigroup. An element $a \in S$ is said to be an idempotent if $aa = a$ and the set of all idempotents of S is denoted by $E(S)$ or E_S or E.

Definition 1.4: A semigroup S is called cancellative semigroup in which for all a, b, c, $ca = cb$ implies $a = b$ and $ac = bc$ implies $a = b$.

Definition 1.5: An element a of a semigroup S is said to be regular if there exists an $x \in S$ such that $axa = a$. If every element of S is regular, then we say that S is regular semigroup.

Definition 1.6: If every element of a semigroup S has finite order, then S is said to be periodic.

Definition 1.7: A semigroup S is called a union of groups if each of its elements is contained in some subgroup of S.

Definition 1.8: If S is a semigroup the equivalence relations L, R, H, I and D defined by $L = \{(a, b) \in S \times S \,/\, S^1a = S^1b\}$, $R = \{(a, b) \in S \times S \,/\, aS^1 = bS^1\}$, and $H = L \cap R$, $I = \{(a, b) \in S \times S \,/\, S^1 aS^1 = S^1 bS^1\}$, and $D = L o R$ are called Green's relations on S.

Definition 1.9: A relation R on a semigroup S is said to be right(left) compatible if for a, b in S, $(a, b) \in R$ implies $(ac, ba) \in R$ and $(ca, cb) \in R$ for every $c \in S$. A right(left) compatible equivalence relation on S is called right(left) congruence. By a congruence on S we mean an equivalence on S which is both right and left compatible.

First we start with the following theorem due to J.M.Howie [1], which is stated in exercise. But for the sake of definiteness we proved the following theorem.

Theorem 2.1: The relation L^* on a semigroup S defined by the rule that aL^*b if and only if $ax = ay \Longleftrightarrow bx = by$, $\forall x, y \in S^1$ is an equivalence relation. If S is a semigroup then $L \subseteq L^*$ and that L^* is a right congruence on S and for every idempotent e in S, aL^*e if and only if $ae = a$ and $ax = ay \Rightarrow ex = ey$, $\forall x, y \in S^1$. If S is regular then $L = L^*$.

Proof : First we observe that L^* is an equivalence relation.

aL^*a for every a as $ax = ay \Longleftrightarrow ax = ay$, so that L^* is reflexive.

Suppose aL^*b then $ax = ay \Longleftrightarrow bx = by$, $\forall x, y \in S^1$ and thus $bx = by \Longleftrightarrow ax = ay$ and hence bL^*a so that L^* is symmetric.

Suppose aL^*b and bL^*c so that $ax = ay \Longleftrightarrow bx = by$ and $bx = by \Longleftrightarrow cx = cy$, $\forall x, y \in S^1$ and thus $ax = ay \Longleftrightarrow cx = cy$, $\forall x, y \in S^1$ and hence aL^*c. Thus L^* is transtive. Hence L^* is an equivalence relation.

We claim that $L \subseteq L^*$.

Suppose $(a, b) \in L$ so that $S^1a = S^1b$ then $\exists u, v \in S^1$ such that $ua = b$ and $vb = a$. Assume that $ax = ay$ then $bx = uax = uay = by$ and also assume that $bx = by$ then $ax = vbx = vby = ay$ and hence $ax = ay \Longleftrightarrow bx = by$, $\forall x, y \in S^1$. Thus $(a, b) \in L^*$. Hence $L \subseteq L^*$.

Now we have to show that L^* is a right congruence on S.

Suppose let $(a, b) \in L^*$ so that $ax = ay \Longleftrightarrow bx = by$, $\forall x, y \in S^1$.

Now we claim that $(ac, bc) \in L^*$. i.e.$(ac)x = (ac)y \Longleftrightarrow (bc)x = (bc)y$, $\forall x, y \in S^1$.

Let $c \in S$. Assume that $(ac)x = (ac)y$ and thus $a(cx) = a(cy)$ and hence $b(cx) = b(cy)$ by the definition of L^* and therefore $(bc)x = (bc)y$.

Conversely, suppose that $(bc)x = (bc)y$ and thus $b(cx) = b(cy)$ and hence $a(cx) = a(cy)$ by the definition of L^* and therefore $(ac)x = (ac)y$.

Thus $(ac)x = (ac)y \Longleftrightarrow (bc)x = (bc)y,\ \forall x, y \in S^1$ and therefore $(ac, bc) \in L^*$. Hence L^* is a right congruence on S.

Now we show that, for every idempotent e in S, aL^*e if and only if $ae = a$ and $ax = ay \Longrightarrow ex = ey,\ \forall x, y \in S^1$. First we claim that $ae = a$. Assume that aL^*e so that $ax = ay \Longrightarrow ex = ey,\ \forall x, y \in S^1$. Put $x = e$ and $y = 1$, then $ex = e.e = e^2 = e$ and $ey = e.1 = e$ and thus $ex = ey$ and now $ax = ay$ so that $a.e = a.1$ and thus $ae = a$. Conversely suppose that $ae = a$ and $ax = ay \Longrightarrow ex = ey,\ \forall x, y \in S^1$. Assume that $ex = ey$ now $ax = aex = aey = ay$ and thus $ex = ey \Longrightarrow ax = ay$ and hence $ax = ay \Longleftrightarrow ex = ey,\ \forall x, y \in S^1$. Hence $(a, e) \in L^*$.

Now we prove that $L^* = L$, if S is regular.

Suppose, let $(a, b) \in L^*$ so that $ax = ay \Longleftrightarrow bx = by,\ \forall x, y \in S$.

Take $a' \in V(a)$ so that $a = aa'a$. Now $a = a.1 = a(a'a)$ so $b = b(1) = b(a'a)$ and thus $Sb = Sba'a \subseteq Sa$ and also take $b' \in V(b)$ so that $b = bb'b$. Now $b = b(1) = b(b'b)$ so $a = a(1) = a(b'b)$ and thus $Sa = Sab'b \subseteq S_b$ and hence $Sa = Sb$. Thus $(a, b) \in L$ so that $L^* \subseteq L$. Since $L \subseteq L^*$, so we have $L = L^*$.

Theorem 2.2: The containment $L \subseteq L^*$ may well be proper. If S is the cancellative semigroup then $L = 1_S$, $L^* = S \times S$.

Proof : Suppose S is the set of non negative integers, then S is a cancellative under addition. Suppose $(a, b) \in L$ so that $S^0 + a = S^0 + b$ (since S is additive and identity is 0) and thus $a = x + b$, $b = y + a$ for some $x, y \in S^0$ and hence $a = x + b \geq b$ and $b = y + a \geq a$. Hence $a = b$ and hence $L = 1_S$.

Suppose $(a, b) \in L^*$ so that $a + x = a + y \Longleftrightarrow b + x = b + y$, for some $x, y \in S^1$ and thus $x = y$ as S is cancellative and hence $L^* = S \times S$.

Theorem 2.3: The equivalences R^* and H^* are defined by analogy with L^*. Then every H^*-class containing an idempotent is a subsemigroup of S and is a cancellative semigroup with identity element 'e'.

Proof : Suppose 'e' is an idempotent and $a \in L^*e$ so that $(a, e) \in L^*$ if and only if $ae = a$ and $ax = ay \Longrightarrow ex = ey,\ \forall x, y \in S^1$ and $a \in R^*e$ so that $(a, e) \in R^*$ if and only if $ea = a$ and $xa = ya \Longrightarrow xe = ye,\ \forall x, y \in S^1$. Assume that $(a, e) \in L^*$ so that $ax = ay \Longrightarrow ex = ey,\ \forall x, y \in S^1$ put $x = e$, $y = 1$ then $ex = e.e = e^2 = e$, $ey = e.1 = e$ and thus $ex = ey$ and now $ax = ay$ so that $a.e = a.1$ and hence $ae = a$ and $ax = ay \Longrightarrow ex = ey$ (by definition).

Conversely, suppose that $ae = a$ and $ax = ay \Longrightarrow ex = ey$.

Now, we claim that $(a, e) \in L^*$. i.e. $ax = ay \Longleftrightarrow ex = ey$. Assume that $ex = ey$ and $ae = a$. Now $ax = aex = aey = ay$ and thus $(a, e) \in L^*$.

Now assume that $(a, e) \in R^*$ so that $xa = ya \Longleftrightarrow xe = ye,\ \forall x, y \in S^1$ put $x = e, y = 1$, then $xe = e.e = e^2 = e$, $ye = 1.e = e$ and thus $xe = ye$ and now $xa = ya \Longrightarrow e.a = 1.a$ and hence $ea = a$ and $xa = ya \Longrightarrow xe = ye$ (by definition).

Conversely, suppose that $ea = a$ and $xa = ya \Longrightarrow xe = ye$.

Now, we claim that $(a, e) \in R^*$. i.e. $xa = ya \Longleftrightarrow xe = ye$. Assume that $xe = ye$ and $ea = a$. Now $xa = xea = yea = ya$ and thus $(a, e) \in R^*$. Therefore $(a, e) \in R^* \cap L^* = H^*$ so that $a \in H^*e$.

Let $a \in H^*e$, so that $ea = a$, $ae = a$. Suppose $a, b \in H^*e$ so that $(a, e) \in H^*$ and $(b, e) \in H^*$. Now we claim that $(ab, e) \in H^* = R^* \cap L^*$. First we show that $(ab, e) \in R^*$. i.e. to show that $x(ab) = y(ab) \Longleftrightarrow xe = ye$, $\forall x, y \in S^1$. Now, let $x(ab) = y(ab)$ so that $(xa)b = (ya)b$ and thus $(xa)e = (ya)e$ (since $b \in R^*e$ then $xb = yb \Longleftrightarrow xe = ye$) and hence $x(ae) = y(ae)$ and therefore $xa = ya$ so that $xe = ye$ (since $(a, e) \in R^*$). Conversely, suppose that $xe = ye$. Now we claim that $x(ab) = y(ab)$ .We have $xe = ye$ so that $(xe)a = (ye)a$ and thus $x(ea) = y(ea)$ and hence $xa = ya$ and therefore $(xa)b = (ya)b$ and hence $x(ab) = y(ab)$. Hence $ab \in R^*e$.

Now, we show that $(ab, e) \in L^*$. i.e. to show that $(ab)x = (ab)y \Longleftrightarrow ex = ey$, $\forall x, y \in S^1$. Let $(ab)x = (ab)y$ so that $a(bx) = a(by)$ and thus $e(bx) = e(by)$ (since $(a, e) \in L^*e$ i.e. $ax = ay \Longleftrightarrow ex = ey$). Since e is the identity, so we have $bx = by$ and hence $ex = ey$ (since $(b, e) \in L^*e$ i.e. $bx = by \Longleftrightarrow ex = ey$).

Conversely, suppose that $ex = ey$. Now, we claim that $(ab)x = (ab)y$. We have $ex = ey$ so that $b(ex) = b(ey)$ and thus $bx = by$ and hence $a(bx) = a(by)$ and therefore $(ab)x = (ab)y$. Thus $ab \in L^*e$. Hence $ab \in R^*e \cap L^*e = H^*e$. Thus $ab \in H^*e$.

Let $a, x, y \in H^*e$. Now suppose that $ax = ay$ so that $ex = ey$ and hence $x = y$ (since $x \in H^*e$ and e is the identity of H^*e) and also suppose that $xa = ya$ so that $xe = ye$ and hence $x = y$. Thus H^*e is a cancellative semigroup with identity.

Now we prove the following interesting and independent results.

Theorem 2.4: If S is a semigroup with zero, then $0L^* = \{0\}$, $0R^* = \{0\}$.

proof : Let $a \in 0L^*$ so that $(a, 0) \in L^*$ and hence $ax = ay \Longleftrightarrow 0x = 0y$, $\forall x, y \in S^1$, putting $x = 1$ and $y = 0$ we have $0.x = 0.1 = 0$ and $0.y = 0.0 = 0$ and hence $ax = ay$ so that $ax = a.1 = a$ and $ay = a.0 = 0$ and thus $a = 0$. Hence $L^* = \{0\}$.

Let $a \in 0R^*$ so that $(a, 0) \in R^*$ and hence $xa = ya \Longleftrightarrow x0 = y0$, $\forall x, y \in S^1$, putting $x = 1$ and $y = 0$ we have $x.0 = 1.0 = 0$ and $y0 = 0.0 = 0$ and hence $xa = 1.a = a$ and $ya = 0.a = 0$ and thus $a = 0$. Hence $0R^* = \{0\}$.

Theorem 2.5: In a semigroup S, $L^* \cap (RegS \times RegS) = L \cap (RegS \times RegS)$ where $RegS$ stands for semigroup S with regular elements.

Proof : Let $(a, b) \in L^* \cap (RegS \times RegS)$ so that $(a, b) \in L^*$ and a, b are regular elements of S and thus $ax = ay \Longleftrightarrow bx = by$, $\forall x, y \in S^1$ and $a = aa'a$, $b = bb'b$ (since a, b are regular). As a is regular, there exists $a' \in V(a)$ such that $a = aa'a$. Now $a = a.1 = a(a'a)$, so $b = b.1 = b(a'a)$ and thus $S^1b = S^1ba'a \subseteq S^1a \longrightarrow (1)$ and also b is regular, so there exists $b' \in V(b)$ such that $b = bb'b$. Now $b = b.1 = b(b'b)$, so $a = a.1 = a(b'b)$ and thus $S^1a = S^1ab'b \subseteq S^1 \longrightarrow (2)$. From (1) and (2) $S^1a = S^1b$ and thus $(a, b) \in L$. Hence $(a, b) \in L \cap (RegS \times RegS)$.

Since $L \subseteq L^*$, so $L \cap (RegS \times Regs) \subseteq L^* \cap (RegS \times RegS)$. Hence $L^* \cap (RegS \times Regs) = L \cap (RegS \times RegS)$.

Remark : By the above theorem (2.5), similarly we can prove that $R^* \cap (RegS \times RegS) = R \cap (RegS \times RegS)$.

Corollary 2.6: If S is a regular semigroup, then $\mathcal{L}^* = \mathcal{L}$ and $\mathcal{R}^* = \mathcal{R}$.

Lemma 2.7: If S is a periodic semigroup which is also cancellative then S is a union of groups.
Proof : Suppose S is a periodic semigroup. Let $a \in S$, such that index of a is m and period of a is r. Then $(a) = \{a, a^2, \ldots, a^m, a^{m+1}, \ldots, a^{m+r-1}\}$. Since S is periodic, so (a) is finite and also S is cancellative, so (a) is cancellative. Thus (a) is finite
cancellative group. Thus S is a union of groups.

Theorem 2.8: In a periodic semigroup $He = H^*e$.

Proof : Suppose $a \in H^*e$. Since S be periodic, so by Lemma (2.7) S is a union of groups. So $a \in Hf$, where Hf is a subgroup of S. Also we have H^*e is cancellative semigroup with identity $'e'$ so we have $ea = a$and $fa = a$ and thus $ea = fa$ and hence $e = f$. Thus $a \in He$ and therefore $H^*e \subseteq He$, we have $He \subseteq H^*e$ so that $H^*e = He$.

Lemma 2.9: H^*e is a group if and only if $H^*e = He$.

Proof : Assume that $He = H^*e$, then H^*e is a group (since He is a group). Con- versely, suppose that H^*e is group. Let $a \in H^*e$ so that $\exists a' \ni aa' = e$, $a'a = e$ and $ae = ea = a$.

Now we claim that $a \in He$, i.e. $a \in Le \bigcap Re$. We have $S^1a = S^1 ae \subseteq S^1e = S^1a'a \subseteq S^1a$ and thus $S^1a = S^1e$ and hence $(a, e) \in \mathcal{L}$. Let $aS^1 = eaS^1 \subseteq eS^1 = aa'S^1 \subseteq aS^1$ and thus $aS^1 = eS^1$ and hence $(a, e) \in \mathcal{R}$. Thus $(a, e) \in \mathcal{L} \bigcap \mathcal{R} = \mathcal{H}$. Hence $a \in He$. Thus $H^*e \subseteq He$. We have $He \subseteq H^*e$ and Hence $H^*e = He$.

Example 2.10: Unlike the Green's relations $\mathcal{L}$ and $\mathcal{R}$ the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ do not permute an example is given to show that the inequality $\mathcal{R}^* o \mathcal{L}^* = \mathcal{L}^* o \mathcal{R}^*$ does not hold in general.

Let S be the set of all 2×2 matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, where a, b, d are all real numbers.

Let $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \begin{pmatrix} c & e \\ 0 & f \end{pmatrix} \in S$.

Now $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} c & e \\ 0 & f \end{pmatrix} = \begin{pmatrix} ac & ae+bf \\ 0 & df \end{pmatrix} \in S$

Let $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = e$ is an idempotent,

$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = f$ is an idempotent

and $ef = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = a$.

Thus product of two idempotents is not an idempotent.

Also $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = a^2$.

Suppose $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is regular,

then $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Thus $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not regular.

Let $S = \{0, e, f, a\}$ be a semigroup.

.	0	e	f	a
0	0	0	0	0
e	0	e	e	e
f	0	0	f	0
a	0	0	a	0

$$fa = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$ea = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = a$$

$$fe = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$ae = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$af = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = a.$$

We have $(a, e) \in R^*$ since $xa = ya \Longleftrightarrow xe = ya$, for $x, y \in S^1$.

Now, we have $ea = a = 1.a$, putting $x = e, y = 1$ and $e.e = e = 1.e$ also $a.a = a^2 = 0$, $0.a = 0$ and $a.e = 0$, $0.e = 0$ and also $f.a = 0 = a.a$ and $f.e = 0 = a.e$.

If $(a, f) \in R^*$ then $xa = ya \Longleftrightarrow xf = yf$. Now $ea = a = 1.a$, $ef = a$, $1.f = f$, $fa = 0 = 0a$, $ff = f$, $0f = 0$. Thus $(a, f) \notin R^*$. Thus $R^* = 1_S \cup \{(a, e), (e, a)\}$.

We have $(a, f) \in L^*$ since $ax = ay \Longleftrightarrow fx = fy$. Now, we have $a.a = a^2 = 0$, $a.0 = 0$, $f.a = 0 = f.0$ and $a.e = 0 = a.a$, $f.e = 0 = f.a$.

If $(a, e) \in L^*$ then $ax = ay \Longleftrightarrow ex = ey$. Now $a.e = 0 = a.0$, $e.e = e$, $e.0 = 0$, $a.f = a = a.f$, $e.f = a$, $e.1 = e$. Thus $(a, e) \notin L^*$. Thus $L^* = 1_S \cup \{(a, f), (f, a)\}$.

Now $(e, a) \in R^*$ and $(a, f) \in L^*$, then $(e, f) \in R^* o L^*$ and suppose that $(e, z) \in L^*$ and $(z, f) \in R^*$ but $(e, f) \notin L^* o R^*$ (since $(e, a) \notin L^*, (a, f) \in R^*$). Thus $R^* o L^* = L^* o R^*$.

Example 2.11: This is an example to show that, if a is a regular element of S, then every element of aR^* need not be regular.

Suppose G is a group with more than two elements and let $S = \{x, 0\}$ be a null semigroup. Let $((a, x), (b, x)) \in (G \times S) \times (G \times S)$. Now, we claim that $((a, x), (b, x)) \in R$ so that $(a, x)(G \times S)^1 = (b, x)(G \times S)^1$ and thus $(a, x) = (b, x).1$, where $1 \in (G \times S)^1$, if $a = b$, then (a, x) is not R-equivalent to (b, x). Now $((a, x), (b, 0)) \in R$ so that $(a, x)(G \times S)^1 = (b, 0)(G \times S)^1$ so that $(a, x) = (b, 0).(c, x) = (bc, 0)$ which is not true (since $x = 0$). Now it can be verified that $((a, 0), (b, 0)) \in R$ so that

$(a,0)(G\times S)^1 = (b,0)(G\times S)^1$ and thus $(a,0) = (b,0).(c,x) = (bc,0)$.

Thus, $\mathcal{R} = 1_{(G\times S)} \bigcup \{(a,0),(b,0)\}$, for $a,b \in G$.

Now, we claim that $((a,x),(b,x)) \in \mathcal{R}^*$. We have $(u,v)(a,x) = (m,n)(a,x) \Longleftrightarrow (u,v)(b,x) = (m,n)(b,x)$ and thus $(ua,vx) = (ma,nx) \Longleftrightarrow (ub,vx) = (mb,nx)$ and hence $(ua,0) = (ma,0) \Longleftrightarrow (ub,0) = (mb,0)$ (since $vx = 0, nx = 0$).

Thus $ua = ma \Longrightarrow u = m$ (since $a \in G$, G has cancellative property) so thus $ub = mb$ and suppose that $ub = mb \Longrightarrow ua = ma$. Thus $((a,x),(b,x)) \in \mathcal{R}^*$.

Now, we claim that $(a,x) \in \mathcal{R}^*(b,0)$. We have $(u,v)(a,x) = (m,n)(a,x) \Longleftrightarrow (u,v)(b,0) = (m,n)(b,0)$ and thus $(ua,vx) = (ma,nx) \Longleftrightarrow (ub,0) = (mb,0)$. Hence $(ua,0) = (ma,0) \Longleftrightarrow (ub,0) = (mb,0)$. Thus $((a,x),(b,0)) \in \mathcal{R}^*$.

Hence $\mathcal{R}^* = \{(a,x),(b,x),(a,0),(b,0)\}$. Thus $\mathcal{R}^* = (G\times S)\times(G\times S)$.

Let $(e,0)(e,0) = (e^2,0) = (e,0)$. Thus $(e,0)$ is an idempotent in $\mathcal{R}^*$.

Now, we claim that (e,x) is not regular. Suppose (e,x) is regular, then there exists (a,u) such that $(e,x)(a,u)(e,x) = (e,x)$. Now $(e,x)(a,u)(e,x) = (ea,xu)(e,x) = (ea,0)(e,x) = (eae,0) = (e,x)$ (since $x = 0$). Thus particular (e,x) is in $(e,0)\mathcal{R}^*$ which is not regular.

References:

[1] Howie, J. M. *Fundamentals of semigroup theory*, Oxford University Press Inc., New York.

(1) *D.V. VijayKumar*
Assistant Professor
T.S.R and T.B.K PG Centre
Visakhapatnam − 530026.
INDIA.

(2) *Dr.K.V.R. Srinivas*
Associate Professor
Regency Institute of Technology
Yanam − 533464.
INDIA.