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Degree of Approximationof Functionsby Newly Defined Polynomials onan unbounded interval

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ABSTRACT : *Chlodovsky(1937) has proved the theorem 1.1 & 1.2 for Bernstein Polynomials* $B_n(x)=B_n^f(x;b_n)=\sum_{k=0}^{n} f(\frac{b_n k}{n})\binom{n}{k}(\frac{x}{b_n})^k(1-\frac{x}{b_n})^{n-k}$ *on an unbounded interval.*
The object of this paper is to extend the above theorems for newly defined polynomials

$$A_n(x) = A_n^f(x,\alpha;b_n) = (n+1)\sum_{k=0}^{n}\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(tb)dt\}\, p_{n,k}\left(\frac{x}{b_n};\alpha\right)$$

on an unbounded interval .

Keywords:*Bernstein Polynomials, Lebsgue Integrable function, L_1 norm, generating function, Modified Polynomials*

I. Introduction & results

If f(x) is a function defined on [0 ,1] , the Bernstein polynomial $B_n^f(x)$ of f is $B_n^f(x)=\sum_{k=0}^{n} f(\frac{k}{n})p_{n,k}(x)$, where$p_{n,k}(x) = \binom{n}{k}(x)^k(1-x)^{n-k}$.

If the function f(x) defined in the interval (0,b),b>0 .The Bernstein polynomial$B_n^f(x;b)$ for this interval is given by

$B_n(x)=B_n^f(x;b)=\sum_{k=0}^{n} f(\frac{b}{n})\binom{n}{k}(\frac{x}{b})^k(1-\frac{x}{b})^{n-k}$-----------------(1.1)

Further a small modification of the Bernstein polynomial due toKantorovich [2] and Anwar&Umar [3] makes it possible to approximate Lebesgue integrable function in L_1 norm by a newly defined polynomial

$$A_n^{\alpha}(f,x) = (n+1)\sum_{k=0}^{n}\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt\}\, p_{n,k}(x;\alpha) \quad \text{-------------- (1.2)}$$

where

$$p_{n,k}(x;\alpha)=\binom{n}{k}\frac{x(x+k\alpha)^{k-1}(1-x+(n-k)\alpha)^{n-k}}{(1+n\alpha)^n} \quad \text{-------------------- (1.3)}$$

such that$\sum p_{n,k}(x;\alpha) = 1$.

Let the function f(x) be defined on the interval (0,b),b>0. To obtain a modified polynomial $A_n^f(x,\alpha;b)$ for this interval, we make the substitution
$y=xb^{-1}$ in the polynomial $A_n^{\Phi}(y)$ of the function $\Phi(y) = f(by)$,0≤y≤1
and obtain in this way

$$A_n(x) = A_n^f(x,\alpha;b) = (n+1)\sum_{k=0}^{n}\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(tb)dt\}\, p_{n,k}\left(\frac{x}{b};\alpha\right) \quad \text{---- (1.4)}$$

where

$$p_{n,k}\left(\frac{x}{b};\alpha\right)=\binom{n}{k}\frac{(\frac{x}{b})(\frac{x}{b}+k\alpha)^k(1-\frac{x}{b}+(n-k)\alpha)^{n-k}}{(1+n\alpha)^n} \quad \text{------------------------- (1.5)}$$

Chlodovsky (1937) has proved the theorem by assuming b=b_n is a function of n, which increases to +∞ with n and f(x) defined in the infinite interval 0≤x<∞.

Theorem 1.1:- If b_n=0(n) and the function f(x) is bounded in [0, +∞) , say $|f(x)|$≤M , then then $B_n(x) \to f(x)$ holdsat any point of continuity of the function f(x) .

Theorem 1.2:- If b_n=0(n)
and

$$M(b_n)e^{-\alpha n/bn} \to 0,$$

for each $\alpha > 0$, then $B_n(x) \to f(x)$ holds at each point of continuity of the function f(x).

In this paper our object is to improve the above results by taking the new polynomial $A_n(x)$ instead of $B_n(x)$ which may be stated as follows

Theorem 1.3:- If b_n=0(n) and the function f(x) is bounded lebesgue integrable in [0, +∞) , say $|f(x)|\leq M$, then $A_n(x) \to f(x)$ holdsat any point of continuity of the function f(x) .

Theorem 1.4:- If b_n=0(n)
and

$$M(b_n)e^{-\beta n/b_n} \to 0, \qquad \text{-------------(1.6)}$$

for each $\beta > 0$,then $A_n(x) \to f(x)$ holds at each point of continuity of the function f(x).

II. Lemmas

In order to proofour result we need the following Lemmas

Lemma2.1:[3] For all values of x ϵ [0,1] and for $\alpha=\alpha_n=0(\frac{1}{n})$

We have $(n+1)\sum_{k=0}^{n}\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^2dt\}\,p_{n,k}(x;\alpha)\leq\frac{x(1-x)}{n}$.

Lemma 2.2: If $0\leq x\leq 1$, the inequality**,**

$$0\leq z\leq\frac{3}{2}\left(\frac{x(1-x)}{n}\right)^{\frac{1}{2}} \quad \text{-------------------------(2.1)}$$

Implies

$$(n+1)\sum_{|t-x|\geq 2z\left(\frac{x(1-x)}{n}\right)^{\frac{1}{2}}}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}p_{n,k}(x)dt \leq 2e^{-z^2}. \text{------------ (2.2)}$$

Proof of lemma 2.2: Let Φ be the generating function of the polynomial

$$T= \sum_{k=0}^{n}(k-nx)p_{n,k}(x:\alpha),$$

which may be defined as

$$\Phi=\Phi_n(u,s) = \sum_{s=0}^{\infty}\frac{1}{s!}T_{ns}(x)u^s$$

$$=\sum_{k=0}^{n}p_{n,k}(x;\alpha)\sum_{s=0}^{\infty}\frac{1}{s!}(k-nx)^s u^s$$

$$=\sum_{k=0}^{n}e^{u(k-nx)}\binom{n}{k}\frac{x(x+k\alpha)^{k-1}(1-x+(n-k)\alpha)^{n-k}}{(1+n\alpha)^n}$$

$$=\frac{e^{-nxu}}{(1+n\alpha)^n}[\,(1-x+n\alpha)^n + nx(1-x+(n-1)\alpha)^{n-1}e^u$$

$$+\frac{n(n-1)}{2!}x(x+2\alpha)(1-x+(n-2)\alpha)^{n-2}e^{2u}+\cdots\ldots\ldots\ldots+\ x(x+n\alpha)^{n-1}e^{nu}$$

$$\Phi = e^{-nxu}(1-x+xe^u)^n \text{ , for } \alpha=\alpha_n=0(\tfrac{1}{n})$$

and therefore

$$\Phi = [e^{-xu}(1-x+xe^u)]^n \text{----------------------(2.3)}$$

To prove our result we first show that for $|u| \leq \frac{3}{2}$, the inequality

$$\Phi \leq \exp\{nx(1-x)u^2\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2.4)$$

holds.

For (2.3) can be written as

$$\Phi = [xe^{u(1-x)} + (1-x)e^{-ux}]^n$$

But since

$$xe^{u(1-x)} + (1-x)e^{-ux} = \sum_{\nu=0}^{\infty} \frac{u^{\nu}}{\nu!}[x(1-x) + (1-x)(-x)^{\nu}]$$

$$\leq 1+\sum_{\nu=2}^{\infty} \frac{u^{\nu}}{\nu!}[x(1-x) + (1-x)(-x)^{\nu}]$$

$$\leq 1+ x(1-x)\sum_{\nu=2}^{\infty} \frac{|u|^{\nu}}{\nu!}$$

$$\leq 1+ x(1-x)\frac{u^2}{2}(1 + \frac{|u|}{3} + \frac{|u^2|}{3^2} + \ldots\ldots\ldots\ldots)$$

$$= 1+ x(1-x)\frac{u^2}{2}(1 - \frac{1}{3}|u|)^{-1}$$

$$\leq 1+ x(1-x)u^2 \quad \text{for} \quad |u| \leq \frac{3}{2}$$

$$\leq e^{x(1-x)u^2} \text{as} e^k > k+1$$

and hence

$$\Phi \leq [e^{x(1-x)u^2}]^{n}$$

$$= \exp\{n\, x(1-x)u^2\} \qquad \text{which is (2.4)}.$$

Therefore if

$$\psi = \psi_n(u,x) = \sum_{k=0}^{n} e^{u|k-nx|} p_{n,k}(x;\alpha) \quad \text{------------------(2.5)}$$

then we obtain for $0 \leq u \leq \frac{3}{2}$

$$\psi \leq \psi_n(u,x) + \Psi_n(-u,x) \quad \text{and therefore , for } \alpha=\alpha_n=0(\frac{1}{n}) \text{ , we have}$$

$$\psi \leq \exp 2\{n\, x(1-x)u^2\}\ . \text{ ----------------- (2.6)}$$

now we get our required result, we note that for $c \geq 0$ and $u \geq 0$

$$(n+1)\sum_{\exp[u|k-nx|] \geq c\psi} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right) p_{n,k}(x;\alpha)$$

$$\leq \frac{1}{c\psi}(n+1)\sum_{k=0}^{n} \{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt\} e^{u|k-nx|} p_{n,k}(x;\alpha) \leq \frac{1}{c}$$

Now if we put $c=\frac{1}{2}z^2$, we obtain

$$(n+1)\sum_{\exp[u|k-nx|] \geq \frac{1}{2}e^{z^2}\psi} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right) p_{n,k}(x;\alpha) \leq 2e^{-z^2}$$

or

$$(n+1)\sum_{|k-nx| \geq z^2u^{-1} + nx(1-x)u} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right) p_{n,k}(x;\alpha) \leq 2e^{-z^2}$$

since for the given range of t $|k-nx| \sim |t-x|$,
we have

$$(n+1)\sum_{|t-x| \geq z^2u^{-1}n^{-1} + nx(1-x)u} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right) p_{n,k}(x;\alpha) \leq 2e^{-z^2} \quad \text{------- (2.7)}$$

Since 2.1 can be written as

$0 \leq z[nx(1-x)]^{-\frac{1}{2}} \leq \frac{3}{2}$

But (2.7)holds for $0 \leq u \leq \frac{3}{2}$ and therefore for $u = z[nx(1-x)]^{-\frac{1}{2}}$, we have

$$(n+1)\sum_{|t-x| \geq z\left[\frac{x(1-x)}{n}\right]^{\frac{1}{2}} + z\left[\frac{x(1-x)}{n}\right]^{\frac{1}{2}}} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right) p_{n,k}(x;\alpha) \leq 2e^{-z^2}$$

$(n+1)\sum_{|t-x|\geq 2z\left[\frac{x(1-x)}{n}\right]^{\frac{1}{2}}}\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\right)p_{n,k}(x;\alpha)\leq 2e^{-z^2}$

this completes the proof of lemma.

III. Proof of theorems

Proof of theorem 1.3:

We have

$|A_n(\mathrm{x})-f(x)|\leq(n+1)\sum_{k=0}^{n}\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}|f(tb)-f(x)|dt\}\,p_{n,k}\left(\frac{x}{b};\alpha\right)$

Let $\epsilon>0$ be arbitrary and choose infinitesimally small $\delta>0$ such that

$|f(\mathrm{x})-f(x')|<\epsilon$ for $|x-x'|<\delta$

then

$$|A_n(\mathrm{x})-f(x)|\leq(n+1)\sum_{|bnt-x|<\delta}\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}|f(tb)-f(x)|dt\}\,p_{n,k}\left(\frac{x}{b};\alpha\right)$$

$+(n+1)\sum_{|bnt-x|\geq\delta}\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}|f(tb)-f(x)|dt\}\,p_{n,k}\left(\frac{x}{b};\alpha\right)$

$=I_1+I_2$ -------------------------------------- (3.1)

$I_1=(n+1)\sum_{|bnt-x|<\delta}\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}|f(tb)-f(x)|dt\}\,p_{n,k}\left(\frac{x}{b};\alpha\right)$

$<\epsilon\ (n+1)\sum_{|bnt-x|<\delta}\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}dt\}\,p_{n,k}\left(\frac{x}{b};\alpha\right)$

$=\epsilon$ ----------------------------------- (3.2)

To calculate I_2 , we put $u=\frac{x}{b_n}$ and then we have

$I_2=(n+1)\sum_{|bnt-x|\geq\delta}\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}|f(tb)-f(x)|dt\}\,p_{n,k}\left(\frac{x}{b};\alpha\right)$

$\leq 2M\ (n+1)\sum_{|t-u|\geq\frac{\delta}{bn}}\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}dt\}\,p_{n,k}(u;\alpha)$

$\leq 2M\left(\frac{\delta}{bn}\right)^{-2}(n+1)\sum_{k=0}^{n}\{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-u)^2dt\}\,p_{n,k}(u;\alpha)$

$\leq 2M\left(\frac{\delta}{bn}\right)^{-2}\frac{u(1-u)}{n}$ for all $\alpha=0\left(\frac{1}{n}\right)$ byLemma(2.1),

$\leq 2M\,\frac{\frac{x}{bn}}{n\left(\frac{\delta}{bn}\right)^2}$ for all large n & $\alpha=0\left(\frac{1}{n}\right)$ since $b_n=\mathrm{o}(n)$,

$<\epsilon$ --- (3.3)

Hence

$|A_n(\mathrm{x})-f(x)|\leq\epsilon+\epsilon=2\epsilon$

this completes the proof of theorem 1.3.

Proof of theorem 1.4: Proceeding as in theorem 1.3 we obtain

$|A_n(\text{x}) - f(x)| \leq \epsilon + 2\text{M}(b_n)\,(n+1) \sum_{|t-u| \geq \frac{\delta}{b_n}} \{\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} dt\}\, p_{n,k}(u; \alpha)$

The second term can be easily estimated by means of lemma(2.2), if

$z = \delta_n (2b_n)^{-1} \left(\frac{u(1-u)}{n}\right)^{-\frac{1}{2}}$,

the condition (2.1) satisfied if we assume, for instance, δ<2x and that n is sufficiently large .

Hence by (1.6) we obtain

$|A_n(\text{x}) - f(x)| \leq \epsilon + 2\text{M}(b_n) \exp(-z^2)$

$= \epsilon + 2\text{M}(b_n) \exp\{-\delta^2.\text{n}[4\frac{x}{bn}\left(1 - \frac{x}{bn}\right)]^{-1}\}$

$= \epsilon + \epsilon = 2\epsilon$ for large n

this completes the proof of the theorem 1.4.

IV. Conclusion

In this paper we have improved the results of Chlodovsky by taking the new Modified Polynomials A_n(x) instead of Bernstein Polynomials B_n(x).

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