Some approximation results on Otto Szász type positive linear operators

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ABSTRACT: Recently Deo N.et.al. (Appl. Maths. Comput., 201(2008), 604-612.) introduced a new Bernstein type special operators. Motivated by Deo N.et.al., in this paper we introduce generalization of positive linear operators (1.5) and (1.6) which is the particular case of positive linear operators (1.7) and (1.8). We shall study some approximation results on it.

I. INTRODUCTION

Recently Deo N.et.al. [1] introduced a new Bernstein type special operators \( \{ V_n f \} \) defined as,
\[
(V_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f \left( \frac{k}{n} \right)
\]
where
\[
p_{n,k}(x) = \left( 1 + \frac{1}{n} \right)^n \binom{n}{k} x^k \left( \frac{n}{n+1} - x \right)^{n-k}; \quad \text{for} \quad 0 \leq x \leq \frac{n}{n+1}
\]
Again Deo N.et.al. [1] gave the integral modification of the operators (1.1) which are defined as,
\[
(L_n f)(x) = n \left( 1 + \frac{1}{n} \right)^2 \sum_{k=0}^{n} p_{n,k}(x) \int_{0}^{\frac{n}{n+1}} p_{n,k}(t) f(t) dt
\]
and prove some approximation results on the operators (1.2).

Singh S.P. [4] studied some approximation results on a sequence of Szász type operators defined as,
\[
(S_n f)(x) = \sum_{k=0}^{n} b_{n,k}(t) f \left( x + \frac{k}{n} \right)
\]
where
\[
b_{n,k}(t) = e^{-nt} \frac{(nt)^k}{k!}; \quad x \in [0, \infty) \text{ is fixed.}
\]
which map the space of bounded continuous functions \( C [0, \infty] \) into itself following [3].

Kasana H.S. et al. [2] obtained a sequence of modified Szász operators for integrable function on \( [0, \infty) \) defined as,
\[
(M_n f)(x) = \sum_{k=0}^{n} b_{n,k}(t) \int_{0}^{\frac{n}{n+1}} b_{n,k}(y) f(x + y) dy
\]
where \( t, x \in [0, \infty) \) and \( x \) is fixed.

Motivated by Deo N.et.al.[1] we studied a sequence of positive linear operators \( \{ B_n f \} \) which are defined as,
\[
(B_n f)(x) = n \left( 1 + \frac{1}{n} \right)^2 e^{-(n+p)x} \sum_{k=0}^{n} \frac{(n+p)^k x^k}{k!} \int_{0}^{\frac{n}{n+1}} p_{n,k}(t) f(t) dt
\]
where
\[
p_{n,k}(t) = \left( 1 + \frac{1}{n} \right)^n \binom{n}{k} t^k \left( \frac{n}{n+1} - t \right)^{n-k}; \quad p > 0 \quad \text{and} \quad \text{for} \quad t \in \left[ 0, \frac{n}{n+1} \right]
\]
we study some approximation results on the operators (1.5).

Again following Kasana H.S. et al. [2] we introduce a sequence of positive linear operators \( \{ B_{n,x} f \} \) which are defined as,
\[
(B_{n,x} f)(t) = n \left( 1 + \frac{1}{n} \right)^2 e^{-(n+p)x} \sum_{k=0}^{n} \frac{(n+p)^k x^k}{k!} \int_{0}^{\frac{n}{n+1}} p_{n,k}(y) f(x + y) dy
\]
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where \( t, x \in \left[ 0, \frac{n}{n+1} \right] \) and \( x \) is fixed.

and studied some approximation results on the operators (1.6).

If we put \( p = 0 \) in the operators (1.5) and (1.6), it gives the new modified operators \( \{B_n^p f\} \) and \( \{B_{n,s}^p f\} \) as:

\[
(B_n^p f)(x) = n \left(1 + \frac{1}{n}\right)^2 e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \int_0^n p_{n,k}(t) f(t) dt
\]

and

\[
(B_{n,s}^p f)(t) = n \left(1 + \frac{1}{n}\right)^2 e^{-nt} \sum_{k=0}^{\infty} \frac{(nt)^k}{k!} \int_0^n p_{n,k}(y) f(x+y) dy
\]

where \( t, x \in \left[ 0, \frac{n}{n+1} \right] \) and \( x \) is fixed.

We shall study some approximation results on the operators (1.7) and (1.8).

### BASIC RESULTS-I

In order to prove our main result, the following basic results are needed.

1. \( e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} = nx \) \( \ldots \ldots \ldots \) (2.1)
2. \( e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} k^2 = n^2 x^2 + nx \) \( \ldots \ldots \ldots \) (2.2)
3. \( e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} k^3 = n^3 x^3 + 3n^2 x^2 + nx \) \( \ldots \ldots \ldots \) (2.3)
4. \( e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} k^4 = n^4 x^4 + 6n^3 x^3 + 7n^2 x^2 + nx \) \( \ldots \ldots \ldots \) (2.4)

We know that

\[
e^{nx} = \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \]

\( \ldots \ldots \ldots \) (2.5)

Differentiating with respect to \( x \), we get

\[
ne^{nx} = \sum_{k=0}^{\infty} \frac{n^k x^{k-1}}{k!} k!
\]

Multiplying \( x \) both sides, we get

\[
nx ne^{nx} = \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} k
\]

\( \ldots \ldots \ldots \) (2.6)

This completes the proof of (2.1).

Again differentiating (2.6) with respect to \( x \), we get

\[
n^2 xe^{nx} + ne^{nx} = \sum_{k=0}^{\infty} \frac{n^k x^{k-1}}{k!} k!
\]

Multiplying \( x \) both sides, we get

\[
n^2 xe^{nx} + nx e^{nx} = \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} k^2
\]

(2.7)

\[
[n^2 x^2 + nx] e^{nx} = \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} k^2
\]

This completes the proof of (2.2).

In the same way after differentiations and calculations, we get required result s (2.3) and (2.4).

### BASIC RESULTS-II

1. \( (B_1^0 f)(x) = 1 \) \( \ldots \ldots \ldots \) (2.8)

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2. \((B_n t)(x) \to x\) as \(n \to \infty\), \hspace{1cm} (2.9)

3. \((B_n t^2)(x) \to x^2\) as \(n \to \infty\), \hspace{1cm} (2.10)

4. \((B_n t^3)(x) = \frac{n^n x^n + 3n^2 x^3 + 96 n x^6}{(n+1)^3 (n+3)}\) \hspace{1cm} (2.11)

5. \((B_n t^4)(x) = \frac{n^n x^n + 16n^2 x^2 + 256 n^2 x^2 + 96 n^2 x^4 + 98 n^2 x^6}{(n+1)^4 (n+2) (n+3)}\) \hspace{1cm} (2.12)

6. \((B_n (t - x))(x) = \frac{n^n [1 - 3x - 2x^2]}{(n+1)(n+2)}\) \hspace{1cm} (2.13)

7. \((B_n (t - x)^2)(x) = \frac{n^n [2x - x^2] + n^2 [12x^2 - 6x^2] + 6n x^2}{(n+1)^2 (n+2) (n+3)}\) \hspace{1cm} (2.14)

8. \((B_n (t - x)^3)(x) = o\left(\frac{1}{n}\right)\) \hspace{1cm} (2.15)

9. \((B_n (t - x)^4)(x) = o\left(\frac{1}{n^2}\right)\) \hspace{1cm} (2.16)

Proof of Basic Results-II.

By putting \(f(t) = 1\) in equation (1.7), we get

\[
(B_n^1)(x) = n \left(1 + \frac{1}{n}\right)^2 e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} \int_0^n \frac{n}{k!} \left(\frac{n+1}{n}\right)^n (\frac{n}{k!}) t^k \left(\frac{n}{n+1} - t\right)^{n-k} dt
\]

\[
= n \left(1 + \frac{1}{n}\right)^2 e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} \frac{1}{n} \left(\frac{n}{n+1}\right)^2
\]

\[
= e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!}
\]

This completes the proof of (2.8).

By putting \(f(t) = t\) in equation (1.7), we get

\[
(B_n^t)(x) = n \left(1 + \frac{1}{n}\right)^2 e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} \int_0^n \frac{n}{k!} \left(\frac{n+1}{n}\right)^n (\frac{n}{k!}) t^{k+1} \left(\frac{n}{n+1} - t\right)^{n-k} dt
\]

\[
= n \left(1 + \frac{1}{n}\right)^2 e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} \frac{k+1}{n(n+2)} \frac{n}{k!} \left(\frac{n}{n+1}\right)^3
\]

\[
= \frac{n}{(n+1)(n+2)} \left\{ e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} k + e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} \frac{1}{k!} \right\}
\]

\[
= \frac{n}{(n+1)(n+2)} [nx + 1]
\]

\[
(B_n^t)(x) \to x\hspace{1cm} as \hspace{1cm} n \to \infty.
\]

This completes the proof of (2.9).

By putting \(f(t) = t^2\) in equation (1.7), we get

\[
(B_n^t)(x) = n \left(1 + \frac{1}{n}\right)^2 e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} \int_0^n \frac{n}{k!} \left(\frac{n+1}{n}\right)^n (\frac{n}{k!}) t^{k+2} \left(\frac{n}{n+1} - t\right)^{n-k} dt
\]

\[
= n \left(1 + \frac{1}{n}\right)^2 e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} \frac{(k+1)(k+2)}{n(n+2)(n+3)} \frac{n}{k!} \left(\frac{n}{n+1}\right)^4
\]

\[
= \frac{n^2}{(n+1)^2(n+2)(n+3)} \left\{ e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} k^2 + e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} 3k + e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} 2 \right\}
\]

\[
= \frac{n^2 [n^2 x^2 + 4nx + 2]}{(n+1)^2(n+2)(n+3)}
\]

\((B_n^t)(x) \to x^2\hspace{1cm} as \hspace{1cm} n \to \infty.
\]

This completes the proof of (2.10).
In the same way by taking  \( f(t) = t^3 \) \& \( f(t) = t^4 \) respectively in (1.7) and after little calculations we get required results (2.11) to (2.16).
This completes the proof.

**MAIN RESULTS**

In this section we shall give our main results

**LEMMA: Let** \( f \in C \left[ 0, \frac{n}{n+1} \right] \) then the sequence of positive linear operator defined by \( \{B_n^*f\} \) is converges uniformly to \( f \) as \( n \to \infty \).

**Proof:** Since from basic results (2.8), (2.9) and (2.10), we get

\[
\begin{align*}
(B_n^*1)(x) &\xrightarrow{\text{uniformly}} 1 \quad \text{as} \quad n \to \infty \\
(B_n^*t)(x) &\xrightarrow{\text{uniformly}} x \quad \text{as} \quad n \to \infty \\
(B_n^*t^2)(x) &\xrightarrow{\text{uniformly}} x^2 \quad \text{as} \quad n \to \infty
\end{align*}
\]

Then using Korokin theorem we can conclude that

\[
(B_n^*f)(x) \xrightarrow{\text{uniformly}} f \quad \text{as} \quad n \to \infty.
\]

This completes the proof.

**Theorem:** Let \( f \) be the integrable and bounded in the interval \( \left[ 0, \frac{n}{n+1} \right] \) and let if \( f'' \) exists at a point \( x \) in \( \left[ 0, \frac{n}{n+1} \right] \), then one gets that

\[
\lim_{n \to \infty} n \left[(B_n^*f)(x) - f(x) \right] = (1 - 3x)f''(x) + \frac{x(2 - x)}{2} f'''(x).
\]

where \( \{B_n^*f\} \) are defined in (1.7).

**Proof:** Since \( f''' \) exists at a point \( x \) in \( \left[ 0, \frac{n}{n+1} \right] \), then by using Taylor’s expansion, we write

\[
f(t) = f(x) + (t - x)f'(x) + \frac{(t - x)^2}{2} f''(x) + (t - x)^2 \lambda(t - x)
\]

where \( \lambda(t - x) \to 0 \) as \( t \to x \).

Now for each \( \varepsilon > 0 \), there corresponds \( \delta > 0 \) such that

\[
|\lambda(t - x)| \leq \varepsilon \quad \text{whenever} \quad |t - x| \leq \delta. 
\]

Again for \( |t - x| > \delta \), then there exist a positive number \( M \) such that

\[
|\lambda(t - x)| \leq M \leq \frac{M(t - x)^2}{\delta^2}.
\]

Thus for all \( t \) and \( x \in \left[ 0, \frac{n}{n+1} \right] \), we get

\[
|\lambda(t - x)| \leq \varepsilon + M \frac{(t - x)^2}{\delta^2}
\]

Applying \( \{B_n\} \) on (3.1), we get

\[
\begin{align*}
(B_n^*f)(x) &= f(x)(B_n^*1)(x) + f'(x)(B_n^*(t-x))(x) + \frac{f''(x)}{2} (B_n^*(t-x)^2)(x) \\
&\quad + (B_n^*(t-x^2)\lambda(t-x))(x)
\end{align*}
\]

where \( \lambda(t - x) \to 0 \) as \( t \to x \).

Multiplying \( n \) both side, we get

\[
n[(B_n^*f)(x) - f(x)] = f'(x) \left( \frac{n[1 - 3x] - 2x}{(n+1) (n+2)} \right) + \frac{n^2(x^2)}{2} \left[ \frac{n(1 + 1/n)(n + 1)}{n + 2} \right] \\
&\quad + n \left( 1 + \frac{1}{n} \right) e^{-na} \left( \sum_{k=0}^{n} \frac{n^k x^k}{k!} \int_0^x \right) (t-x)^2 \lambda(t-x) dt.
\]

Here we write,
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\[ nR_n(t, x) = n \left\{ \left( 1 + \frac{1}{n} \right)^2 e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} \int_0^{\frac{n}{n+1}} p_{n,k}(t)(t-x)^2 \lambda(t-x) \, dt \right\} \]

\[ |nR_n(t, x)| \leq n \left\{ \left( 1 + \frac{1}{n} \right)^2 e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} \int_0^{\frac{n}{n+1}} p_{n,k}(t)(t-x)^2 \lambda(t-x) \, dt \right\} \]

Using (3.2) in equation (3.4), we get

\[ |nR_n(t, x)| \leq n \varepsilon \left( B_n^*(t-x)^2 \right)(x) + \frac{nM}{\delta^2} \left( B_n^*(y-t)^4 \right)(x) \]

By choosing \( \delta = n^{-1/4} \), we get that

\[ |nR_n(t, x)| \leq \varepsilon + \frac{M}{n^{1/2}} \sqrt{\varepsilon} \]

\[ \text{Since } \varepsilon \text{ is arbitrary and small, we get} \]

\[ |nR_n(t, x)| \to 0 \text{ as } n \to \infty. \]          \[ \ldots \ldots \ldots \ldots \ldots (3.5) \]

Using (3.5) in equation (3.3), we get

\[ \lim_{n \to \infty} n \left[ (B_n^*f)(x) - f(x) \right] = (1 - 3x)f'(x) + \frac{x(2-x)}{2} f''(x). \]

This completes the proof.

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