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A note on unsteady flow of a dusty viscous fluid through hexagonal duct

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ABSTRACT : *This note presents the discussion regarding the flow of a dusty viscous fluid through hexagonal duct, i.e. , the cross-section of the long rectilinear duct is a regular hexagon. Developed integral transform and trilinear co-ordinates have been employed to solve the problem. The pressure gradient has been taken any function of time. A few particular cases, i.e. , flow under an impulsive pressure gradient and flow under constant pressure gradient have also been discussed.*

I. INTRODUCTION

Unsteady laminar viscous incompressible flow through large rectilinear duct has been studied by Chien Fan (1965) when the axial pressure gradient is any arbitrary function of time. Saffman (1962) studied the stability of the laminar flow of a dusty gas where in dust particles were uniformly distributed and their size and shape were also uniform and neglected the bulk concentration. Michael (1965) , Michael and Norway (1968) and Rao (1969) have studied unsteady flow of a dusty viscous fluid for viscous geometries.

In this chapter we consider the unsteady flow of dusty viscous fluids through a rectilinear duct having cross-section as a regular hexagon and pressure gradient is assumed to be any function of time. For the suitability of boundaries a special not-orthogonal co-ordinate system, a system of trilinear co-ordinates, has been used and the corresponding integral transform has been considered to solve the differential equation. Two particular cases have also been discussed.

II. GOVERNING EQUATIONS OF MOTION

Using Cartesian co-ordinates (x, y, z) , let z −axis be along the axis of the duct. For the present geometry, the components of the fluid velocity u_x, u_y, u_z and those of duct particle v_x, v_y, v_z are assumed as:

$$u_x = 0, \qquad u_y = 0, \qquad u_z = u_z(x, y, t)$$

and

$$v_x = 0, \qquad v_y = 0, \qquad v_z = v_z(x, y, t).$$

It is further assumed that the number density of the duct particle $N = N_0$, a constant.

The governing equation of motion (Saffman, 1962) are

$$\frac{\partial u_z}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2}\right) + \frac{KN_0}{\rho}(v_z - u_z) \qquad \ldots\ldots\ldots\ldots (1)$$

and

$$m\frac{\partial v_z}{\partial t} = K(u_z - v_z) \qquad \ldots\ldots\ldots\ldots (2)$$

where K is the Stoke's resistance coefficient, ν is the kinematic viscosity, ρ is the fluid density, m is the mass of the dust particle and p is the fluid pressure.

We further assume $-\frac{1}{\rho}\frac{\partial p}{\partial z} = any\ function\ of\ time = f(t) \qquad \ldots\ldots\ldots (3)$

Eliminating v_z between (1) and (2) and using (3), we get

$$\frac{\partial^2 u_z}{\partial t^2}+K\left(\frac{N_0}{\rho}+\frac{1}{m}\right)\frac{\partial u_z}{\partial t}=\left(\frac{K}{m}+\frac{\partial}{\partial t}\right)f(t)+\nu\left(\frac{K}{m}+\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}\right) \quad \ldots\ldots\ldots\ldots\ldots(4)$$

Let $f(t)=P+F(t)$ ……………….(5)

where P is a constant and $F(t)$ is a function of time. Also let

$$u_z=u_1(x,y)+u_2(x,y,t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

where u_1 is steady component and u_2 is unsteady component of velocity of the fluid.

Substituting (5) and (6) in (4) and separating the steady and unsteady parts, we get

$$\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_2}{\partial y^2}+\frac{P}{\nu}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

$$\frac{\partial^2 u_2}{\partial t^2}+K\left(\frac{N_0}{\rho}+\frac{1}{m}\right)\frac{\partial u_2}{\partial t}=\left(\frac{K}{m}+\frac{\partial}{\partial t}\right)F(t)+\nu\left(\frac{K}{m}+\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial^2 u_2}{\partial y^2}\right) \quad \ldots\ldots\ldots\ldots\ldots(8)$$

Equations (7) and (8) are main governing equation of motion.

III. TRANSFORMATION OF THE GOVERNING EQUATION IN TRILINEAR CO-ORDINATES AND INITIAL AND BOUNDARY CONDITIONS

Let us consider a reference equilateral triangle of side '3a' , so that the hexagonal cross-section has three alternative sides as the sides of the reference triangle and remaining three alternative sides will be given by constant perpendicular distances from the reference triangle sides. Hence the regular hexagon will have the side of length 'a'. Expressing the equations of motion and boundary conditions in terms of trilinear co-ordinates, we have

$$\nabla^2_{\alpha,\beta,\gamma}u_1+\frac{P}{\nu}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

$$\frac{\partial^2 u_2}{\partial t^2}+K\left(\frac{N_0}{\rho}+\frac{1}{m}\right)\frac{\partial u_2}{\partial t}=\left(\frac{K}{m}+\frac{\partial}{\partial t}\right)F(t)+\nu\left(\frac{K}{m}+\frac{\partial}{\partial t}\right)\nabla^2_{\alpha,\beta,\gamma}u_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

and

$$\left.\begin{aligned}u_1(0,\beta,\gamma)&=w_1=u_1\left(\frac{2K}{3},\beta,\gamma\right)\\ u_1(\alpha,0,\gamma)&=w_1=u_1\left(\alpha,\frac{2K}{3},\gamma\right)\\ u_1(\alpha,\beta,0)&=w_1=u_1\left(\alpha,\beta,\frac{2K}{3}\right)\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

$$\left.\begin{aligned}&u_2(\alpha,\beta,\gamma,0)=0\\ &u_2(0,\beta,\gamma,t)=0=u_2\left(\frac{2K}{3},\beta,\gamma,t\right)\\ &u_2(\alpha,0,\gamma,t)=0=u_2\left(\alpha,\frac{2K}{3},\gamma,t\right)\\ &u_2(\alpha,\beta,0,t)=0=u_2\left(\alpha,\beta,\frac{2K}{3},t\right)\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

where K is the perpendicular distance from a vertex of the reference triangle to its opposite side and the trilinear co-ordinates α, β, γ of any point are related by

$$\alpha + \beta + \gamma = K \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..(13)$$

and

$$\nabla^2_{\alpha,\beta,\gamma} \equiv \left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2} + \frac{\partial^2}{\partial \gamma^2} - \frac{\partial^2}{\partial \alpha \partial \beta} - \frac{\partial^2}{\partial \beta \partial \gamma} - \frac{\partial^2}{\partial \gamma \partial \alpha} \right)$$

The other physical requirement is that w must be finite.

IV. SOLUTION OF THE PROBLEM

To solve this problem we shall use the following developed integral transform of function of trilinear co-ordinates :

$$T[f(\alpha,\beta,\gamma)] = f^*(m) = \int_0^{K_1}\int_0^{K_2}\int_0^{K_3} f(\alpha,\beta,\gamma)\left(\sin\frac{2\pi m\alpha}{K} + \sin\frac{2\pi m\beta}{K} + \sin\frac{2\pi m\gamma}{K}\right) d\alpha\, d\beta\, d\gamma$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(14)$$

where $K_1 = K_2 = K_3 = K_0 = \frac{q}{p}K$, p and q being integers and K being constant such that $\alpha + \beta + \gamma = K$. Also m is an integer.

The transform has the inverse formula

$$f(\alpha,\beta,\gamma) = \sum_m f^*(m)\, c_m \left(\sin\frac{2\pi m\alpha}{K} + \sin\frac{2\pi m\beta}{K} + \sin\frac{2\pi m\gamma}{K}\right) \qquad \dots\dots\dots..(15)$$

where $\frac{1}{c_m} = \frac{3}{2}\left(\frac{q}{p}K\right)^3$

With the help of operational property

$$T[(L)\, f(\alpha,\beta,\gamma)] = -\sum_i \int_0^{K_s}\int_0^{K_t} \left[\frac{\partial u_m}{\partial n_i} f(\alpha,\beta,\gamma)\right]_0^{K_i} dx_s\, dx_t - \frac{4\pi^2 m^2}{K^2} f^*(m) \qquad \dots..(16)$$

where $i \neq s \; and \; s \neq t \; and \; dn_i$ are the perpendicular distances to the sides of the reference triangle in the trilinear system of co-ordinates and the function u_m is

$$u_m = \left(\sin\frac{2\pi m\alpha}{K} + \sin\frac{2\pi m\beta}{K} + \sin\frac{2\pi m\gamma}{K}\right) \qquad \dots\dots\dots\dots\dots\dots\dots\dots.(17)$$

Now solution (9) satisfying the boundary condition (ii) is

$$u_1 = w_1 + \sum_{m=1}^{\infty} \frac{2}{m\pi}\left(\frac{K}{2\pi m}\right)^2 \frac{P}{\nu}\left(\sin\frac{2\pi m\alpha}{K} + \sin\frac{2\pi m\beta}{K} + \sin\frac{2\pi m\gamma}{K}\right) \qquad \dots\dots\dots\dots\dots\dots.(18)$$

This solution can be verified by substituting in (9) and knowing that (by Fourier Series Law)

$$K - 2\alpha = \sum_{m=1}^{\infty} \frac{2K}{m\pi} \; sin\frac{2\pi m\alpha}{K}$$

$$K-2\beta=\sum_{m=1}^{\infty}\frac{2K}{m\pi}\quad sin\frac{2\pi m\beta}{K}$$

$$K-2\gamma=\sum_{m=1}^{\infty}\frac{2K}{m\pi}\quad sin\frac{2\pi m\gamma}{K}$$

$$l=\frac{1}{K}[3K-2(\alpha+\beta+\gamma)]=\sum_{m=1}^{\infty}\frac{2}{m\pi}\left(\sin\frac{2\pi m\alpha}{K}+\sin\frac{2\pi m\beta}{K}+\sin\frac{2\pi m\gamma}{K}\right)$$

Again for the solution of (10) with respect to boundary and initial conditions (12) we use Laplace transform for the time variable so that (10) becomes

$$\left[s^2+K\left(\frac{N_0}{\rho}+\frac{1}{m}\right)s-\nu\left(\frac{K}{m}+s\right)\nabla^2_{\alpha,\beta,\gamma}\right]\overline{u_2}=\left(\frac{K}{m}+s\right)\bar{f}(s)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19)$$

Applying the integral transform of trilinear co-ordinates as defined by (14) to (19), we eliminate $\nabla^2_{\alpha,\beta,\gamma}$ operator from it so as to obtain

$$\left[s^2+K\left(\frac{N_0}{\rho}+\frac{1}{m}\right)s+\nu\left(\frac{K}{m}+s\right)\left(\frac{6\pi m}{K}\right)^2\right]\overline{u_2{}^*}=\frac{4K^3}{27\pi m}\left(\frac{K}{m}+s\right)\bar{f}(s)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20)$$

Whence we have

$$\overline{u_2{}^*}=\frac{4K^3}{27\pi m}\ \frac{\bar{f}(s)}{(\theta-\phi)}\left[\frac{\frac{K}{m}+\theta}{s-\theta}-\frac{\frac{K}{m}+\phi}{s-\phi}\right]\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(21)$$

where $\theta,\emptyset$ are the roots of the equation

$$s^2+\left\{\frac{KN_0}{\rho}+\frac{K}{m}+\nu\left(\frac{6\pi m}{K}\right)^2\right\}s+\frac{\nu K}{m}\left(\frac{6\pi m}{K}\right)^2=0\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(22)$$

To get the solution in terms of the original variable α,β,γ *and* t we make use of inversion theorem (15) and inverse Laplace transform, we get

$$u_2(\alpha,\beta,\gamma,t)=\sum_{m=1}^{\infty}\frac{2}{3}\left(\frac{3}{2K}\right)^3\frac{4K^3}{27\pi m(\theta-\phi)}\left(\sin\frac{2\pi m\alpha}{K}+\sin\frac{2\pi m\beta}{K}+\sin\frac{2\pi m\gamma}{K}\right)\left[\left(\frac{K}{m}+\theta\right)\int_0^t e^{\theta\xi}\ f(t-\xi)\ d\xi-\left(\frac{K}{m}+\phi\right)\int_0^t e^{\phi\xi}\ f(t-\xi)\ d\xi\right]$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(23)$$

which obviously satisfies the initial and boundary conditions.

Substituting (18) and (23) in (6), we get

$$u_z=w_1+\sum_{m=1}^{\infty}\frac{2}{m\pi}\left(\frac{K}{2\pi m}\right)^2\frac{P}{\nu}\left(\sin\frac{2\pi m\alpha}{K}+\sin\frac{2\pi m\beta}{K}+\sin\frac{2\pi m\gamma}{K}\right)+\sum_{m=1}^{\infty}\frac{1}{3\pi m(\theta-\phi)}\left(\sin\frac{2\pi m\alpha}{K}+\sin\frac{2\pi m\beta}{K}+\sin\frac{2\pi m\gamma}{K}\right)T_m\quad\ldots\ldots\ldots..(24)$$

where $T_m = \left[\left(\frac{K}{m}+\theta\right)\int_0^t e^{\theta\xi}\, f(t-\xi)\ d\xi - \left(\frac{K}{m}+\phi\right)\int_0^t e^{\phi\xi}\, f(t-\xi)\ d\xi\right]$ ……………(25)

Substituting the value of u_z in equation (1), we have

(taking trilinear co-ordinates into consideration)

$$v_z = w_1 + \sum_{m=1}^{\infty}\frac{2}{m\pi}\left(\frac{K}{2\pi m}\right)^2\frac{P}{\nu}\left(\sin\frac{2\pi m\alpha}{K}+\sin\frac{2\pi m\beta}{K}+\sin\frac{2\pi m\gamma}{K}\right)$$
$$+\sum_{m=1}^{\infty}\frac{1}{3\pi m(\theta-\phi)}\left(\sin\frac{2\pi m\alpha}{K}+\sin\frac{2\pi m\beta}{K}+\sin\frac{2\pi m\gamma}{K}\right)T_m$$
$$+\frac{\rho}{KN_0}\left[\frac{\partial}{\partial t}\left\{\sum_{m=1}^{\infty}\frac{1}{3\pi m(\theta-\phi)}\left(\sin\frac{2\pi m\alpha}{K}+\sin\frac{2\pi m\beta}{K}+\sin\frac{2\pi m\gamma}{K}\right)T_m\right\}+F(t)\right.$$
$$\left.-\nu\sum_{m=1}^{\infty}\left(\frac{2P}{m\pi\nu}+\frac{4\pi m T_m}{3K^2(\theta-\phi)}\right)\left(\sin\frac{2\pi m\alpha}{K}+\sin\frac{2\pi m\beta}{K}+\sin\frac{2\pi m\gamma}{K}\right)T_m\right]$$

……………………………..…(26)

These equations (24) and (26) give the general solutions for u_z *and* v_z satisfying all the boundary and initial conditions.

V. FLOW UNDER AN IMPULSIVE PRESSURE GRADIENT

Let $f(t) = -A\,\delta(t)$,

where A is a positive constant and $\delta(t)$ is the Dirac delta function. Physically this represents the case when an impulsive pressure gradient of magnitude ρA is suddenly impressed on the fluid at $t = 0^+$. Without loss of generality we can take $A = 1$ for convenience.

Hence equations (24) and (25) give

$$u_z = w_1 + \sum_{m=1}^{\infty}\frac{2}{m\pi}\left(\frac{K}{2\pi m}\right)^2\frac{P}{\nu}\left(\sin\frac{2\pi m\alpha}{K}+\sin\frac{2\pi m\beta}{K}+\sin\frac{2\pi m\gamma}{K}\right)$$
$$+\sum_{m=1}^{\infty}\frac{1}{3\pi m(\theta-\phi)}\left(\sin\frac{2\pi m\alpha}{K}+\sin\frac{2\pi m\beta}{K}+\sin\frac{2\pi m\gamma}{K}\right)\left[\left(\frac{K}{m}+\theta\right)e^{\theta t}-\left(\frac{K}{m}+\phi\right)e^{\phi t}\right]$$

$$v_z = w_1 + \sum_{m=1}^{\infty}\frac{2}{m\pi}\left(\frac{K}{2\pi m}\right)^2\frac{P}{\nu}\left(\sin\frac{2\pi m\alpha}{K}+\sin\frac{2\pi m\beta}{K}+\sin\frac{2\pi m\gamma}{K}\right)$$
$$+\sum_{m=1}^{\infty}\frac{1}{3\pi m(\theta-\phi)}\left(\sin\frac{2\pi m\alpha}{K}+\sin\frac{2\pi m\beta}{K}+\sin\frac{2\pi m\gamma}{K}\right)R_m$$
$$+\frac{\rho}{KN_0}\left[\frac{\partial}{\partial t}\left\{\sum_{m=1}^{\infty}\frac{1}{3\pi m(\theta-\phi)}\left(\sin\frac{2\pi m\alpha}{K}+\sin\frac{2\pi m\beta}{K}+\sin\frac{2\pi m\gamma}{K}\right)R_m\right\}+F(t)\right.$$
$$\left.-\nu\sum_{m=1}^{\infty}\left(\frac{2P}{m\pi\nu}+\frac{4\pi m R_m}{3K^2(\theta-\phi)}\right)\left(\sin\frac{2\pi m\alpha}{K}+\sin\frac{2\pi m\beta}{K}+\sin\frac{2\pi m\gamma}{K}\right)\right]$$

where

$$R_m = \left[\left(\frac{K}{m}+\theta\right)e^{\theta t}-\left(\frac{K}{m}+\phi\right)e^{\phi t}\right]$$

VI. FLOW UNDER CONSTANT PRESSURE GARDIENT

Let $f(t) = -A\,H(t)$,
where A is a positive constant and $H(t)$ is the Heaviside unit step function. Then

$$u_z = w_1 + \sum_{m=1}^{\infty} \frac{2}{m\pi}\left(\frac{K}{2\pi m}\right)^2 \frac{P}{\nu}\left(\sin\frac{2\pi m\alpha}{K} + \sin\frac{2\pi m\beta}{K} + \sin\frac{2\pi m\gamma}{K}\right) + \sum_{m=1}^{\infty} \frac{A}{3\pi m(\theta-\phi)}\left(\sin\frac{2\pi m\alpha}{K} + \sin\frac{2\pi m\beta}{K} + \sin\frac{2\pi m\gamma}{K}\right) S_m$$

and

$$v_z = w_1 + \sum_{m=1}^{\infty} \frac{2}{m\pi}\left(\frac{K}{2\pi m}\right)^2 \frac{P}{\nu}\left(\sin\frac{2\pi m\alpha}{K} + \sin\frac{2\pi m\beta}{K} + \sin\frac{2\pi m\gamma}{K}\right) + \sum_{m=1}^{\infty} \frac{A}{3\pi m(\theta-\phi)}\left(\sin\frac{2\pi m\alpha}{K} + \sin\frac{2\pi m\beta}{K} + \sin\frac{2\pi m\gamma}{K}\right) S_m + \frac{\rho}{KN_0}\left[\frac{\partial}{\partial t}\left\{\sum_{m=1}^{\infty} \frac{1}{3\pi m(\theta-\phi)}\left(\sin\frac{2\pi m\alpha}{K} + \sin\frac{2\pi m\beta}{K} + \sin\frac{2\pi m\gamma}{K}\right) S_m\right\} + F(t) - \nu\sum_{m=1}^{\infty}\left(\frac{2P}{m\pi\nu} + \frac{4\pi m S_m}{3K^2(\theta-\phi)}\right)\left(\sin\frac{2\pi m\alpha}{K} + \sin\frac{2\pi m\beta}{K} + \sin\frac{2\pi m\gamma}{K}\right)\right]$$

where

$$S_m = \left[\left(\frac{K}{m} + \theta\right)\left(\frac{e^{\theta t} - 1}{\theta}\right) - \left(\frac{K}{m} + \phi\right)\left(\frac{e^{\phi t} - 1}{\phi}\right)\right]$$

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